行列式の展開 — 準備 1

系 4.12

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

行列式の展開 — 準備 1

系 4.12

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

注意) $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ の第 1 列は次のように表せる:

$$\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = \begin{bmatrix} a_{11} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ a_{21} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ a_{31} \end{bmatrix}$$

行列式の展開 — 準備 2

$$|\mathbf{A}| = \left| \begin{bmatrix} a_{11} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ a_{21} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ a_{31} \end{bmatrix} \quad \begin{matrix} a_{12} & a_{13} \\ a_{22} & a_{23} \\ a_{32} & a_{33} \end{matrix} \right|$$

行列式の展開 — 準備 2

$$
\begin{aligned}
|\mathbf{A}| &= \left| \begin{bmatrix} a_{11} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ a_{21} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ a_{31} \end{bmatrix} \quad \begin{matrix} a_{12} & a_{13} \\ a_{22} & a_{23} \\ a_{32} & a_{33} \end{matrix} \right| \\
&= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad \textbf{(A-1)}
\end{aligned}
$$

行列式の展開 — 準備 2

$$
\begin{aligned}
|\mathbf{A}| &= \left| \begin{bmatrix} a_{11} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ a_{21} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ a_{31} \end{bmatrix} \quad a_{12} \quad a_{13} \right| \\
&= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad \text{(A-1)}
\end{aligned}
$$

行列式の展開 — 準備 2

$$
\begin{aligned}
|\mathbf{A}| &= \left| \begin{bmatrix} a_{11} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ a_{21} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ a_{31} \end{bmatrix} \quad a_{12} \quad a_{13} \right| \\
&= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad \text{(A-1)} \\
&= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} - \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ 0 & a_{12} & a_{13} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{31} & a_{32} & a_{33} \\ 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \end{vmatrix} \qquad \text{(P-2)}
\end{aligned}
$$

行列式の展開 — 準備 2

$$
\begin{aligned}
|\mathbf{A}| &= \left| \begin{bmatrix} a_{11} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ a_{21} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ a_{31} \end{bmatrix} \quad a_{12} \quad a_{13} \right| \\
&= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad \text{(A-1)} \\
&= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} - \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ 0 & a_{12} & a_{13} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{31} & a_{32} & a_{33} \\ 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \end{vmatrix} \qquad \text{(P-2)}
\end{aligned}
$$

行列式の展開 — 準備 2

$$
\begin{aligned}
|\mathbf{A}| &= \left| \begin{bmatrix} a_{11} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ a_{21} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ a_{31} \end{bmatrix} \quad a_{12} \quad a_{13} \right| \\
&= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad \text{(A-1)} \\
&= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} - \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ 0 & a_{12} & a_{13} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{31} & a_{32} & a_{33} \\ 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \end{vmatrix} \qquad \text{(P-2)} \\
&= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \qquad \text{(系 4.12)}
\end{aligned}
$$

行列式の展開 — 準備 2

$$
\begin{aligned}
|\mathbf{A}| &= \left|\begin{bmatrix} a_{11} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ a_{21} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ a_{31} \end{bmatrix} \quad a_{12} \quad a_{13} \right. \\
&\qquad \left. \begin{matrix} \\ a_{22} \\ a_{32} \end{matrix} \quad \begin{matrix} \\ a_{23} \\ a_{33} \end{matrix} \right| \\
&= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad \textbf{(A-1)} \\
&= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} - \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ 0 & a_{12} & a_{13} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{31} & a_{32} & a_{33} \\ 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \end{vmatrix} \qquad \textbf{(P-2)} \\
&= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \qquad \text{(系 4.12)} \\
&= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{31}(a_{12}a_{23} - a_{13}a_{22})
\end{aligned}
$$

行列式の展開 — 準備 3

補題 4.13

$$\begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & 0 & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & 0 & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i1} & \cdots & a_{i,j-1} & a_{ij} & a_{i,j+1} & \cdots & a_{in} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & 0 & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & 0 & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix} = (-1)^{i+j} a_{ij} \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

行列式の展開 — 準備 3

補題 4.13

$$\begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & 0 & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & 0 & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i1} & \cdots & a_{i,j-1} & a_{ij} & a_{i,j+1} & \cdots & a_{in} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & 0 & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & 0 & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix} = (-1)^{i+j} a_{ij} \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

行列式の展開 — 準備 3

補題 4.13

$$\begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & 0 & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & 0 & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i1} & \cdots & a_{i,j-1} & a_{ij} & a_{i,j+1} & \cdots & a_{in} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & 0 & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & 0 & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix} = (-1)^{i+j} a_{ij} \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

$\|$ **(P-2)**

$$(-1)^{i-1} \begin{vmatrix} a_{i1} & \cdots & a_{i,j-1} & a_{ij} & a_{i,j+1} & \cdots & a_{in} \\ a_{11} & \cdots & a_{1,j-1} & 0 & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & 0 & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & 0 & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & 0 & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

行列式の展開 — 準備 3

補題 4.13

$$\begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & 0 & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & 0 & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i1} & \cdots & a_{i,j-1} & a_{ij} & a_{i,j+1} & \cdots & a_{in} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & 0 & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & 0 & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix} = (-1)^{i+j} a_{ij} \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

$$\| $$

$$(-1)^{i-1} \begin{vmatrix} a_{i1} & \cdots & a_{i,j-1} & a_{ij} & a_{i,j+1} & \cdots & a_{in} \\ a_{11} & \cdots & a_{1,j-1} & 0 & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & 0 & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & 0 & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & 0 & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

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$$\begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & 0 & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & 0 & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i1} & \cdots & a_{i,j-1} & a_{ij} & a_{i,j+1} & \cdots & a_{in} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & 0 & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & 0 & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix} = (-1)^{i+j} a_{ij} \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

$$\| $$

$$(-1)^{i-1} \begin{vmatrix} a_{i1} & \cdots & a_{i,j-1} & a_{ij} & a_{i,j+1} & \cdots & a_{in} \\ a_{11} & \cdots & a_{1,j-1} & 0 & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & 0 & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & 0 & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & 0 & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix} = (-1)^{i+j-2} \begin{vmatrix} a_{ij} & a_{i1} & \cdots & a_{i,j-1} & a_{i,j+1} & \cdots & a_{in} \\ 0 & a_{11} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ 0 & a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & a_{n1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

(P-2)

行列式の展開 — 準備 3

補題 4.13

$$\begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & 0 & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & 0 & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i1} & \cdots & a_{i,j-1} & a_{ij} & a_{i,j+1} & \cdots & a_{in} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & 0 & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & 0 & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix} = (-1)^{i+j} a_{ij} \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

$$\| \quad (-1)^{i-1} \begin{vmatrix} a_{i1} & \cdots & a_{i,j-1} & a_{ij} & a_{i,j+1} & \cdots & a_{in} \\ a_{11} & \cdots & a_{1,j-1} & 0 & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & 0 & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & 0 & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & 0 & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix} = (-1)^{i+j-2} \begin{vmatrix} a_{ij} & a_{i1} & \cdots & a_{i,j-1} & a_{i,j+1} & \cdots & a_{in} \\ 0 & a_{11} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ 0 & a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & a_{n1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

行列式の展開 — 準備 3

補題 4.13

$$\begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & 0 & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & 0 & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i1} & \cdots & a_{i,j-1} & a_{ij} & a_{i,j+1} & \cdots & a_{in} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & 0 & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & 0 & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix} = (-1)^{i+j} a_{ij} \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

$$\| \qquad\qquad \| \quad \text{(系 4.12)}$$

$$(-1)^{i-1} \begin{vmatrix} a_{i1} & \cdots & a_{i,j-1} & a_{ij} & a_{i,j+1} & \cdots & a_{in} \\ a_{11} & \cdots & a_{1,j-1} & 0 & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & 0 & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & 0 & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & 0 & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix} = (-1)^{i+j-2} \begin{vmatrix} a_{ij} & a_{i1} & \cdots & a_{i,j-1} & a_{i,j+1} & \cdots & a_{in} \\ 0 & a_{11} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ 0 & a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & a_{n1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

第 j 列に関する展開

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$ の第 j 列は次のように表せる:

$$\begin{bmatrix} a_{1j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{nj} \end{bmatrix} = \begin{bmatrix} a_{1j} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} \vdots \\ 0 \\ a_{ij} \\ 0 \\ \vdots \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_{nj} \end{bmatrix}$$

第 j 列に関する展開

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

$$|\mathbf{A}| = \begin{vmatrix} & a_{1j} & \\ & 0 & \\ * & \vdots & * \\ & 0 & \end{vmatrix} + \cdots + \begin{vmatrix} & \vdots & \\ & 0 & \\ * & a_{ij} & * \\ & 0 & \\ & \vdots & \end{vmatrix} + \cdots + \begin{vmatrix} & 0 & \\ & \vdots & \\ * & & * \\ & 0 & \\ & a_{nj} & \end{vmatrix}$$

$$= (-1)^{1+j} a_{1j} |\mathbf{A}_{1j}| + \cdots + (-1)^{i+j} a_{ij} |\mathbf{A}_{ij}| + \cdots + (-1)^{n+j} a_{nj} |\mathbf{A}_{nj}|$$

第 j 列に関する展開

$$\mathbf{A}_{1j} = \begin{bmatrix} a_{11} & & a_{1j} & & a_{1n} \\ \vdots & & & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

$$|\mathbf{A}| = \begin{vmatrix} & a_{1j} & \\ & 0 & \\ * & \vdots & * \\ & 0 & \end{vmatrix} + \cdots + \begin{vmatrix} & \vdots & \\ & 0 & \\ * & a_{ij} & * \\ & 0 & \\ & \vdots & \end{vmatrix} + \cdots + \begin{vmatrix} & 0 & \\ & \vdots & \\ * & & * \\ & 0 & \\ & a_{nj} & \end{vmatrix}$$

$$= (-1)^{1+j} a_{1j} |\mathbf{A}_{1j}| + \cdots + (-1)^{i+j} a_{ij} |\mathbf{A}_{ij}| + \cdots + (-1)^{n+j} a_{nj} |\mathbf{A}_{nj}|$$

第 j 列に関する展開

$$\mathbf{A}_{ij} = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

$$|\mathbf{A}| = \begin{vmatrix} & a_{1j} & \\ & 0 & \\ * & \vdots & * \\ & 0 & \end{vmatrix} + \cdots + \begin{vmatrix} & \vdots & \\ & 0 & \\ * & a_{ij} & * \\ & 0 & \\ & \vdots & \end{vmatrix} + \cdots + \begin{vmatrix} & 0 & \\ & \vdots & \\ * & & * \\ & 0 & \\ & a_{nj} & \end{vmatrix}$$

$$= (-1)^{1+j} a_{1j} |\mathbf{A}_{1j}| + \cdots + (-1)^{i+j} a_{ij} |\mathbf{A}_{ij}| + \cdots + (-1)^{n+j} a_{nj} |\mathbf{A}_{nj}|$$

第 j 列に関する展開

$$\mathbf{A}_{nj} = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

$$|\mathbf{A}| = \begin{vmatrix} & a_{1j} & \\ & 0 & \\ * & \vdots & * \\ & 0 & \end{vmatrix} + \cdots + \begin{vmatrix} & \vdots & \\ & 0 & \\ * & a_{ij} & * \\ & 0 & \\ & \vdots & \end{vmatrix} + \cdots + \begin{vmatrix} & 0 & \\ & \vdots & \\ * & & * \\ & 0 & \\ & a_{nj} & \end{vmatrix}$$

$$= (-1)^{1+j} a_{1j} |\mathbf{A}_{1j}| + \cdots + (-1)^{i+j} a_{ij} |\mathbf{A}_{ij}| + \cdots + (-1)^{n+j} a_{nj} |\mathbf{A}_{nj}|$$

第 i 行に関する展開

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$ の第 i 行は次のように表せる:

$$[a_{i1} \cdots a_{ij} \cdots a_{in}]$$

$$= [a_{i1}\ 0 \cdots 0] + \cdots + [\cdots 0\ a_{ij}\ 0 \cdots] + \cdots + [0 \cdots 0\ a_{in}]$$

第i行に関する展開

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

$$|\mathbf{A}| = \begin{vmatrix} & * & \\ a_{i1} & 0 \cdots 0 & \\ & * & \end{vmatrix} + \cdots + \begin{vmatrix} & * & \\ \cdots 0 & a_{ij} & 0 \cdots \\ & * & \end{vmatrix} + \cdots + \begin{vmatrix} & * & \\ 0 \cdots 0 & a_{in} & \\ & * & \end{vmatrix}$$

$$= (-1)^{i+1} a_{i1} |\mathbf{A}_{i1}| + \cdots + (-1)^{i+j} a_{ij} |\mathbf{A}_{ij}| + \cdots + (-1)^{i+n} a_{in} |\mathbf{A}_{in}|$$

第i行に関する展開

$$\mathbf{A}_{i1} = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & & a_{ij} & & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

$$|\mathbf{A}| = \begin{vmatrix} & * & \\ a_{i1} & 0 \cdots 0 & \\ & * & \end{vmatrix} + \cdots + \begin{vmatrix} & * & \\ \cdots 0 & a_{ij} & 0 \cdots \\ & * & \end{vmatrix} + \cdots + \begin{vmatrix} & * & \\ 0 \cdots 0 & a_{in} & \\ & * & \end{vmatrix}$$

$$= (-1)^{i+1} a_{i1} |\mathbf{A}_{i1}| + \cdots + (-1)^{i+j} a_{ij} |\mathbf{A}_{ij}| + \cdots + (-1)^{i+n} a_{in} |\mathbf{A}_{in}|$$

第i行に関する展開

$$\mathbf{A}_{ij} = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

$$|\mathbf{A}| = \begin{vmatrix} & * & \\ a_{i1} & 0 \cdots 0 & \\ & * & \end{vmatrix} + \cdots + \begin{vmatrix} & * & \\ \cdots 0 & a_{ij} & 0 \cdots \\ & * & \end{vmatrix} + \cdots + \begin{vmatrix} & * & \\ 0 \cdots 0 & a_{in} & \\ & * & \end{vmatrix}$$

$$= (-1)^{i+1} a_{i1} |\mathbf{A}_{i1}| + \cdots + (-1)^{i+j} a_{ij} |\mathbf{A}_{ij}| + \cdots + (-1)^{i+n} a_{in} |\mathbf{A}_{in}|$$

第i行に関する展開

$$\mathbf{A}_{in} = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & & a_{ij} & & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

$$|\mathbf{A}| = \begin{vmatrix} & * & \\ a_{i1} & 0 \cdots 0 & \\ & * & \end{vmatrix} + \cdots + \begin{vmatrix} & * & \\ \cdots 0 & a_{ij} & 0 \cdots \\ & * & \end{vmatrix} + \cdots + \begin{vmatrix} & * & \\ 0 \cdots 0 & a_{in} & \\ & * & \end{vmatrix}$$

$$= (-1)^{i+1} a_{i1} |\mathbf{A}_{i1}| + \cdots + (-1)^{i+j} a_{ij} |\mathbf{A}_{ij}| + \cdots + (-1)^{i+n} a_{in} |\mathbf{A}_{in}|$$