Robustness of Epistemic Gossip Protocols Against Data Loss Supplementary Material: Lemmas 5.1-5.9

Yoshikatsu Kobayashi
Department of Computer Science
Univesity of Tsukuba
Tsukuba, Japan
kobayashi@mas.cs.tsukuba.ac.jp

Koji Hasebe
Department of Computer Science
University of Tsukuba
Tsukuba, Japan
hasebe@cs.tsukuba.ac.jp

In the following, we provide Lemmas 5.1 to 5.9 and their full proofs, which are not included in the main paper. There lemmas are required to prove Theorem 5.1.

Lemma 5.1. Let σ_1, σ_2 be sequences, and let x and a be agents, with an initial gossip graph G. If $a \notin \sigma_1$ and neither σ_1 nor σ_2 contains failures, then $G^{\sigma_1;[a];\sigma_2} \sim_x G^{\sigma_1;\sigma_2}$.

Proof. The proof proceeds by induction on the length of σ_2.

(B. C.) Since $\sigma_2 = \epsilon$, the statement of this lemma holds.

(I. S.) Assume $\sigma_2 = \sigma_2'; kl$. The result follows from the inductive hypothesis for both cases, $x \in \{k, l\}$ and $x \notin \{k, l\}$. □

Lemma 5.2. Let σ_1, σ_2 be sequences, and let x and a be agents ($x \neq a$), with an initial gossip graph G. If either $a \notin \sigma_1$ or $a \notin \sigma_2$, and neither σ_1 nor σ_2 contains failures, then $G^{\sigma_1;[a];\sigma_2} \sim_x G^{\sigma_1;\sigma_2}$.

Proof. The proof proceeds by induction on the length of σ_2.

(B. C.) Since $mtree(x, \sigma_1; [a]) = mtree(x, \sigma_1)$, the result holds.

(I. S.) Assume $\sigma_2 = \sigma_2'; kl$. The result follows from the inductive hypothesis, considering the cases where $x \in \{k, l\}$ and $x \notin \{k, l\}$. □

Next, we prove Lemmas 5.3 and 5.4. These lemmas show that if an agent fails, but the failed agent never made a call before the failure, or has not made any calls after the failure, the failure remains undetected.

Lemma 5.3. Let σ_1, σ_2 be sequences, and let a and x be agents, with an initial gossip graph G. If $\langle a\rangle \notin leaves(mtree(x, \sigma_1; [a]; \sigma_2))$, and neither σ_1 nor σ_2 contains failures, then $G^{\sigma_1;[a];\sigma_2} \sim_x G^{\sigma_1^{-a};[a];\sigma_2}$.

Proof. The proof proceeds by induction on the combined length of σ_1 and σ_2.

(B. C.) Since $\sigma_1, \sigma_2 = \epsilon$, the result holds.

(I. S.) We distinguish cases depending on whether $\sigma_2 = \epsilon$.

(Case 1) If $\sigma_2 = \epsilon$, and $x \neq a$, it suffices to show that $G^{\sigma_1} \sim_x G^{\sigma_1^{-a}}$. Let $\sigma_1 = \sigma_1'; kl$. By the inductive hypothesis, $G^{\sigma_1'} \sim_x G^{\sigma_1'^{-a}}$. We further distinguish cases based on the type of the last call kl.

(Case 1.1) If $x \in \{k, l\}$ and $a \notin \{k, l\}$, let $k = x$. By the inductive hypothesis, $G^{\sigma_1';[a];\sigma_2} \sim_l G^{\sigma_1'^{-a};[a];\sigma_2}$, so $G^{\sigma_1'} \sim_l G^{\sigma_1'^{-a}}$. Hence, $mtree(x, \sigma_1') = mtree(x, \sigma_1'^{-a})$, and $mtree(l, \sigma_1') = mtree(l, \sigma_1'^{-a})$. Therefore, $mtree(x, \sigma_1'; xl) = \langle mtree(x, \sigma_1'), xl, mtree(l, \sigma_1')\rangle$, and $mtree(x, \sigma_1'^{-a}; xl) = \langle mtree(x, \sigma_1'^{-a}), xl, mtree(l, \sigma_1'^{-a})\rangle$. Hence, $G^{\sigma_1} \sim_x G^{\sigma_1^{-a}}$.

(Case 1.2) If $a \in \{k, l\}$ and $x \notin \{k, l\}$, let $k = a$. Since $G^{\sigma_1'} \sim_x G^{\sigma_1'^{-a}}$, $mtree(x, \sigma_1') = mtree(x, \sigma_1'^{-a})$. Since al does not involve x, $mtree(x, \sigma_1') = mtree(x, \sigma_1)$. Therefore, $G^{\sigma_1} \sim_x G^{\sigma_1^{-a}}$.

(Case 2) Let $\sigma_2 = \sigma_2'; kl$, and consider the case $x \neq a$. By the inductive hypothesis, $G^{\sigma_1;[a];\sigma_2'} \sim_x G^{\sigma_1^{-a};[a];\sigma_2'}$.

(Case 2.1) If $x \in \{k, l\}$, let $k = x$. By the assumption, $\langle a\rangle \notin leaves(\langle mtree(x, \sigma_1; [a]; \sigma_2'), xl, mtree(l, \sigma_1; [a]; \sigma_2')\rangle)$. By the inductive hypothesis, $G^{\sigma_1;[a];\sigma_2'} \sim_l G^{\sigma_1^{-a};[a];\sigma_2'}$. Therefore, $G^{\sigma_1;[a];\sigma_2} \sim_x G^{\sigma_1^{-a};[a];\sigma_2}$.

(Case 2.2) If $x \notin \{k, l\}$, then $mtree(x, \sigma_1; [a]; \sigma_2') = mtree(x, \sigma_1; [a]; \sigma_2)$, and similarly, $mtree(x, \sigma_1^{-a}; [a]; \sigma_2') = mtree(x, \sigma_1^{-a}; [a]; \sigma_2)$. Therefore, since $G^{\sigma_1;[a];\sigma_2'} \sim_x G^{\sigma_1^{-a};[a];\sigma_2'}$, it follows that $G^{\sigma_1;[a];\sigma_2} \sim_x G^{\sigma_1^{-a};[a];\sigma_2}$. □

Lemma 5.4. Let σ_1, σ_2 be sequences, and let a and x be agents, with an initial gossip graph G. If $\langle a\rangle \notin leaves(mtree(x, \sigma_1; [a]; \sigma_2))$, and neither σ_1 nor σ_2 contains failures, then $G^{\sigma_1;[a];\sigma_2} \sim_x G^{\sigma_1;[a];\sigma_2^{-a}}$.

Proof. The proof proceeds by induction on the length of σ_2.

(B. C.) Since $\sigma_2 = \epsilon$, the statement of this lemma holds.

(I. S.) Let $\sigma_2 = \sigma_2'; kl$. By the inductive hypothesis, $G^{\sigma_1;[a];\sigma_2'} \sim_x G^{\sigma_1;[a]\sigma_2'^{-a}}$. We now distinguish cases based on the type of kl.

(Case 1) If $x \in \{k, l\}$ and $a \notin \{k, l\}$, let $k = x$. By the inductive hypothesis, $mtree(l, \sigma_1; [a]; \sigma_2') = mtree(l, \sigma_1; [a]; \sigma_2'^{-a})$. Since $mtree(x, \sigma_1; [a]; \sigma_2') = mtree(x, \sigma_1; [a]; \sigma_2'^{-a})$, it follows that $G^{\sigma_1;[a];\sigma_2} \sim_x G^{\sigma_1;[a];\sigma_2^{-a}}$.

(Case 2) If $a \in \{k, l\}$ and $x \notin \{k, l\}$, then $mtree(x, \sigma_1; [a]; \sigma_2) = mtree(x, \sigma_1; [a]; \sigma_2')$. Similarly, $mtree(x, \sigma_1^{-a}; [a]; \sigma_2') = mtree(x, \sigma_1^{-a}; [a]; \sigma_2)$. Therefore, since $G^{\sigma_1;[a];\sigma_2'} \sim_x G^{\sigma_1^{-a};[a];\sigma_2'}$, it follows that $G^{\sigma_1;[a];\sigma_2} \sim_x G^{\sigma_1^{-a};[a];\sigma_2}$. □

Next, we prove Lemmas 5.5, 5.6, and 5.7. These lemmas describe basic properties of the structure of memory trees.

Lemma 5.5. For any sequence σ, agent x, and memory tree t, if $[a] \notin \sigma$ and $t \in sub_a(mtree(x, \sigma))$, then there exists some $\tau \sqsubseteq \sigma$ such that $t = mtree(a, \tau)$.

Proof. The proof proceeds by induction on the length of σ.

(B. C.) Since $\sigma = \epsilon$, $sub_a(mtree(x, \epsilon))$ is empty, which leads to a contradiction.

(I. S.) Let $\sigma = \sigma'; \mathrm{e}$, and distinguish cases based on the type of event e. Here, we only show the case when $\mathrm{e} = xy$. Then, $mtree(x, \sigma) = \langle mtree(x, \sigma'), xy, mtree(y, \sigma')\rangle$. By the assumption, either

$t \in sub_a(mtree(x, \sigma'))$ or $t \in sub_a(mtree(y, \sigma'))$, or $t = mtree(x, \sigma)$. In the first two cases, by the inductive hypothesis, there exists $\tau \sqsubseteq \sigma$ such that $t = mtree(a, \tau)$. In the third case, we can take σ as τ. □

Lemma 5.6. For any sequence σ and agent x, either $mtree(x, \sigma)$ contains no non-leaf nodes, or all non-leaf nodes in $mtree(x, \sigma)$ are contained in σ.

Proof. The proof proceeds by induction on the length of σ.

(B. C.) Since $\sigma = \epsilon$, we have $mtree(x, \sigma) = \langle x \rangle$.

(I. S.) Let $\sigma = \sigma'$; e. By the inductive hypothesis, either $mtree(x, \sigma')$ contains no non-leaf nodes, or all non-leaf nodes in $mtree(x, \sigma')$ are contained in σ'. We now distinguish cases based on the type of event e. Here, we show the case when e $= xy$. By the inductive hypothesis, either $mtree(y, \sigma')$ contains no non-leaf nodes, or all non-leaf nodes in $mtree(y, \sigma')$ are contained in σ'. In either case, the statement of this lemma holds. □

Lemma 5.7. For any agent x and sequence σ, $mtree(x, \sigma)$ either contains a memory tree with a single memory of x, or $mtree(x, \sigma) = \langle x \rangle$.

Proof. The proof proceeds by induction on the length of σ.

(B. C.) Since $\sigma = \epsilon$, we have $mtree(x, \sigma) = \langle x \rangle$.

(I. S.) Let $\sigma = \sigma'$; e. By the inductive hypothesis, either $mtree(x, \sigma')$ contains a memory tree with a single memory of x, or $mtree(x, \sigma') = \langle x \rangle$. We now distinguish cases based on whether this holds and on the type of event e.

(Case 1) If $mtree(x, \sigma')$ contains a memory tree with a single memory of x, we show the case where e $= xy$ involves x. Then, $mtree(x, \sigma) = mtree(x, \sigma'; xy) = \langle mtree(x, \sigma'), xy, mtree(y, \sigma') \rangle$, so $mtree(x, \sigma)$ contains a memory tree with a single memory of x.

(Case 2) If $mtree(x, \sigma') = \langle x \rangle$, we similarly show the case where e $= xy$ involves x. Then, $mtree(x, \sigma) = mtree(x, \sigma'; xy) = \langle mtree(x, \sigma'), xy, mtree(y, \sigma') \rangle$, and since $mtree(x, \sigma') = \langle x \rangle$, $mtree(x, \sigma)$ contains a memory tree with a single memory of x. □

Lemmas 5.8 and 5.9 play a central role in proving Theorem 5.1. When agent x calls agent a, x also receives the history of calls that a has made. In this theorem, agent x knows only one person whom a has called since the initial state, so x must not have called a after a failed. Therefore, removing calls involving a from the sequence does not affect the secrets held by x.

Lemma 5.8. Let σ_1 and σ_2 be sequences, and let x and a be agents, with an initial gossip graph G. If the following conditions hold, then $G^{\sigma_1;[a];\sigma_2} \sim_x G^{\sigma_1^{-a};[a];\sigma_2}$:

- Neither σ_1 nor σ_2 contains failures.
- $|sub_a(mtree(x, \sigma_1; [a]\sigma_2))| = 1$.
- For the memory tree $t \in sub_a(mtree(x, \sigma_1; [a]\sigma_2))$, $r(t) \notin \sigma_1$.

Proof. The proof proceeds by induction on the combined length of σ_1 and σ_2.

(B. C.) Since $\sigma_1 = \sigma_2 = \epsilon$, the result holds.

(I. S.) We distinguish cases based on whether $\sigma_2 = \epsilon$.

(Case 1) If $\sigma_2 = \epsilon$, let $\sigma_1 = \sigma_1'; kl$.

(Case 1.1) First, we consider when $G^{\sigma_1'} \sim_x G^{\sigma_1'^{-a}}$ holds. We then further distinguish cases based on the type of the last call kl.

(Case 1.1.1) Assume $x, a \in \{k, l\}$, and consider the case when $x \neq a$. In this case, $kl = xa$, and $mtree(x, \sigma_1'; xa) = mtree(a, \sigma_1'; xa)$ contains a memory tree with a single memory of a by Lemma 5.7. Moreover, by Lemma 5.6, all non-leaf nodes in $mtree(x, \sigma_1)$ are contained in σ_1. Thus, if we let t_1 be the memory tree in $mtree(x, \sigma_1)$ that contains a's single memory, the root of t_1 must be in σ_1, which contradicts the assumption.

(Case 1.1.2) Assume $x \in \{k, l\}$ and $a \notin \{k, l\}$, and let $kl = xy$. We distinguish cases depending on whether $|sub_a(mtree(y, \sigma_1'))| = 1$ or $|sub_a(mtree(y, \sigma_1'))| = 0$.

(Case 1.1.2.1) If $|sub_a(mtree(y, \sigma_1'))| = 1$, then since the last event of σ_1 is the call xy, $mtree(x, \sigma_1)$ also contains the memory tree t_2 with a's single memory. By Lemma 5.6, all non-leaf nodes in $mtree(x, \sigma_1)$ are contained in σ_1. Therefore, the root of t_2 must be in σ_1, which contradicts the assumption.

(Case 1.1.2.2) If $|sub_a(mtree(y, \sigma_1'))| = 0$, then $\langle a \rangle \notin$ $leaves(mtree(y, \sigma_1'; [a]; \sigma_2))$. By Lemma 5.3, $G^{\sigma_1'} \sim_y G^{\sigma_1'^{-a}}$. Together with $G^{\sigma_1'} \sim_x G^{\sigma_1'^{-a}}$, this implies that $G^{\sigma_1;[a];\sigma_2} \sim_x G^{\sigma_1^{-a};[a];\sigma_2}$.

(Case 1.2) Next, consider the case when $G^{\sigma_1'} \sim_x G^{\sigma_1'^{-a}}$ does not hold. We distinguish cases based on whether $mtree(x, \sigma_1')$ contains a memory tree with a single memory of a. If $mtree(x, \sigma_1')$ does not contain a memory tree with a single memory of a, and $x \neq a$, then $a \notin leaves(mtree(x, \sigma_1'; [a]; \sigma_2))$. Therefore, by Lemma 5.3, $G^{\sigma_1'} \sim_x G^{\sigma_1'^{-a}}$, which leads to a contradiction. If $mtree(x, \sigma_1')$ contains a memory tree with a single memory of a, let t_3 be this tree, and assume $x \neq a$. By Lemma 5.6, the root of t_3 must be in σ_1, which contradicts the assumption.

(Case 2) Let $\sigma_2 = \sigma_2'; kl$. We now distinguish cases where $|sub_a(mtree(x, \sigma_1; [a]; \sigma_2'))| = 0$ and $|sub_a(mtree(x, \sigma_1; [a]; \sigma_2'))|$ $= 1$, showing in both cases that $G^{\sigma_1;[a];\sigma_2'} \sim_x G^{\sigma_1^{-a};[a];\sigma_2'}$. If $|sub_a(mtree(x, \sigma_1; [a]; \sigma_2'))| = 1$, let t_4 be the memory tree in $mtree(x, \sigma_1; [a]; \sigma_2')$ that contains a's single memory. By assumption, $r(t_4) \notin \sigma_1$. Hence, by the inductive hypothesis, $G^{\sigma_1;[a];\sigma_2'} \sim_x G^{\sigma_1^{-a};[a];\sigma_2'}$. Finally, we distinguish cases based on the last call kl. If $x \in \{k, l\}$, let $kl = xy$. We then distinguish cases depending on whether $|sub_a(mtree(y, \sigma_1; [a]; \sigma_2'))| = 0$, and in both cases, we show that $G^{\sigma_1;[a];\sigma_2'} \sim_y G^{\sigma_1^{-a};[a];\sigma_2'}$. Together with $G^{\sigma_1;[a];\sigma_2'} \sim_x G^{\sigma_1^{-a};[a];\sigma_2'}$, we conclude that $G^{\sigma_1;[a];\sigma_2} \sim_x G^{\tau_1^{-a};[a];\tau_2}$. □

Lemma 5.9. Let σ_1 and σ_2 be sequences, and let x and a be agents ($x \neq a$), with an initial gossip graph G. If the following conditions hold, then $G^{\sigma_1;[a];\sigma_2} \sim_y G^{\sigma_1;[a];\sigma_2^{-a}}$:

- Neither σ_1 nor σ_2 contains failures.
- $|sub_a(mtree(x, \sigma_1; [a]\sigma_2))| = 1$.
- For the memory tree $t \in sub_a(mtree(x, \sigma_1; [a]\sigma_2))$, $r(t) \in \sigma_1$.

Proof. The proof proceeds by induction on the length of σ_2.

(B. C.) Since $\sigma_2 = \epsilon$, the result holds.

(I. S.) Let $\sigma_2 = \sigma_2'; kl$. We distinguish cases based on whether $G^{\sigma_1;[a];\sigma_2'} \sim_x G^{\sigma_1;[a];\sigma_2'^{-a}}$ holds.

(Case 1) If $G^{\sigma_1;[a];\sigma_2'} \sim_x G^{\sigma_1;[a];\sigma_2'^{-a}}$ holds, we further distinguish cases based on whether the last call kl involves both x and a.

(Case 1.1) If $x, a \in \{k, l\}$, then the last call in σ_2, kl, involves both x and a, so $\sigma_1; [a]; \sigma_2 = \sigma_1; [a]; \sigma_2'; xa$. Since $a \in \sigma_2$, there must exist a call involving a that is the earliest in σ_2. Let this call be ab, and write $\sigma_2 = \tau_1; ab; \tau_2$. Then, $mtree(a, \sigma_1; [a]; \tau_1; ab) = \langle\langle a\rangle, ab, mtree(b, \sigma_1; [a]; \tau_1)\rangle$. Therefore, $mtree(x, \sigma_1; [a]; \tau_1; ab; \tau_2)$ contains a memory tree with a's single memory, and if $ab \in \sigma_1$, we can write $\sigma_1 = \upsilon_1; ab; \upsilon_2$. Hence, $mtree(b, \upsilon_1; ab)$ contains a memory tree with a's single memory. As a result, $mtree(a, \sigma_1; [a]; \tau_1; ab)$ contains multiple single memories of a, which is a contradiction.

(Case 1.2) If $x \in \{k, l\}$ and $a \notin \{k, l\}$, let $kl = xz$. We now distinguish cases depending on whether $mtree(z, \sigma_1; [a]; \sigma_2')$ contains a memory tree with a's single memory. If $mtree(z, \sigma_1; [a]; \sigma_2')$ contains such a subtree t_1, then by assumption, $r(t_1) \in \sigma_1$. By the inductive hypothesis, $G^{\sigma_1;[a];\sigma_2'} \sim_z G^{\sigma_1;[a];\sigma_2'^{-a}}$. Together with $G^{\sigma_1;[a];\sigma_2'} \sim_x G^{\sigma_1;[a];\sigma_2'^{-a}}$, this implies that $G^{\sigma_1;[a];\sigma_2} \sim_x G^{\sigma_1;[a];\sigma_2^{-a}}$.

(Case 2) Assume that $G^{\sigma_1;[a];\sigma_2'} \sim_x G^{\sigma_1;[a];\sigma_2'^{-a}}$ does not hold. We distinguish cases based on whether $mtree(x, \sigma_1; [a]; \sigma_2')$ contains a memory tree with a's single memory. If $mtree(x, \sigma_1; [a]; \sigma_2')$ contains such a tree, let t_2 be this tree. If $r(t_2) \in \sigma_1$, then by the inductive hypothesis, $G^{\sigma_1;[a];\sigma_2'} \sim_x G^{\sigma_1;[a];\sigma_2'^{-a}}$, which leads to a contradiction. On the other hand, if $r(t_2) \notin \sigma_1$, this contradicts the assumption that $r(t_2) \in \sigma_1$. □

We now present the necessary and sufficient conditions for identifying a single failure.

Theorem 5.1. *Let σ be a sequence containing at most one failure, and let x, a be agents, with an asynchronous gossip model $\mathcal{G}^\sim$ and an initial gossip graph G. Then $\mathcal{G}^\sim, G^\sigma \models K_x\mathsf{F}(a)$ if and only if $mtree(x, \sigma)$ contains multiple single memories of a.*

Proof. ($\Rightarrow$) Assume $\mathcal{G}^\sim, G^\sigma \models K_x\mathsf{F}(a)$. Write $\sigma = \sigma_1; [a]; \sigma_2$. Suppose $mtree(x, \sigma)$ does not contain multiple single memories of a.

(Case 1) If $|sub_a(mtree(x, \sigma_1; [a]; \sigma_2))| = 0$, and $x \neq a$, then $\langle a\rangle \notin leaves(mtree(x, \sigma_1; [a]; \sigma_2))$. By Lemma 5.3, $G^{\sigma_1;[a];\sigma_2} \sim_x G^{\sigma_1^{-a};[a];\sigma_2}$. Therefore, by Lemma 5.1, $G^{\sigma_1^{-a};[a];\sigma_2} \sim_x G^{\sigma_1^{-a};\sigma_2}$, which leads to a contradiction.

(Case 2) If $|sub_a(mtree(x, \sigma_1; [a]; \sigma_2))| = 1$, let t be the memory tree containing a's single memory in $mtree(x, \sigma)$. By Lemma 5.6, $r(t) \in \sigma_1; [a]; \sigma_2$.

(Case 2.1) If $r(t) \in \sigma_1$, and $x = a$, then $a \in \sigma_2$. Let the first call involving a be ab, and write $\sigma_2 = \tau_1; ab; \tau_2$. Then, $mtree(a, \sigma_1; [a]; \tau_1; ab)$ contains a single memory of a. Therefore, since $t = mtree(a, \sigma_1; [a]; \tau_1; ab)$, it follows that $ab \in \sigma_1$. Hence, we can write $\sigma_1 = \upsilon_1; ab; \upsilon_2$, and $mtree(b, \upsilon_1; ab)$ contains a memory tree with a's single memory. Therefore, $mtree(a, \sigma_1; [a]; \tau_1; ab)$ contains multiple single memories of a, which is a contradiction. Next, assume $x \neq a$. By Lemmas 5.2 and 5.9, $G^{\sigma_1;[a];\sigma_2} \sim_x G^{\sigma_1;[a];\sigma_2^{-a}}$, which leads to a contradiction.

(Case 2.2) Assume that $r(t) \notin \sigma_1$. By Lemma 5.8, $G^{\sigma_1;[a];\sigma_2} \sim_x G^{\sigma_1^{-a};[a];\sigma_2}$. Therefore, by Lemma 5.1, $G^{\sigma_1^{-a};[a];\sigma_2} \sim_x G^{\sigma_1^{-a};\sigma_2}$, which leads to a contradiction.

($\Leftarrow$) Assume $|sub_a(mtree(x, \sigma))| \geq 2$. If $\mathcal{G}^\sim, G^\sim \models \neg K_x\mathsf{F}(a)$, then there exists some G^τ such that $G^\sigma \sim_x G^\tau$ and $\mathcal{G}^\sim, G^\tau \models \neg\mathsf{F}(a)$. In $mtree(x, \tau)$, there exist two distinct memory trees, t_1 and t_2, each containing a single memory of a. By Lemma 5.5, there exist prefixes τ_1 and τ_2 of τ such that $t_1 = mtree(a, \tau_1)$ and $t_2 = mtree(a, \tau_2)$. Next, we distinguish cases based on the inclusion relation between τ_1 and τ_2. Here, we show only the case where $\tau_1 \sqsubset \tau_2$.

If $\tau_1 \sqsubset \tau_2$, then $mtree(a, \tau_1) \subseteq mtree(a, \tau_2)$. In this case, since $t_2 = mtree(a, \tau_2)$ contains a single memory of a, we must have $a \in \tau_2$. Let the last call in τ_2 involving a be ab, and write $\tau_2 = \upsilon_1; ab; \upsilon_2$. Then, $mtree(a, \tau_2) = \langle\langle a\rangle, ab, mtree(b, \upsilon_1)\rangle$, thus $mtree(a, \tau_1) \subseteq mtree(b, \upsilon_1)$. Thus, there exists some $\tau_3 \subseteq \upsilon_1$ such that $t_1 = mtree(a, \tau_3)$. Therefore, $t_1 = mtree(a, \tau_3) \subseteq mtree(a, \upsilon_1) = \langle a\rangle$, which is a contradiction. □