INFERIORITY BASED MEASURING

TECHNICAL INEFFICIENCY OF PRODUCTION

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Abstract

Various system states' inefficiencies for different systems exist, such as technical and profit inefficiencies for production systems and social and Pareto inefficiencies for games. A state is inefficient if it is inferior to some other state. We have a general procedure to obtain each inefficiency measure based on the corresponding inferiority. The procedure brings as each inefficiency measure, the maximum degree of corresponding inferiority of the state to some other. In production theory, sophisticated technical-inefficiency measures exist, but they reflect no corresponding inferiority explicitly. Firstly, we present a united Farrel-type (radial but not hyperbolic) technical-inefficiency measure (UTE) that integrates input- and output-oriented famous Farrel measures. Then we follow the procedure mentioned above and obtain simple technical-inefficiency measures based on the corresponding inferiority. Our proposed measures generalize and solidify the inferiority-based foundation of UTE. Our measures behave identically with UTE for technically-efficient technology-frontiers. We propose another more complicated measure that solves the slacks problem for weakly technically-efficient frontiers, like the additive measures. We show a fixed relation between the values of the examined measures for an arbitrary state. We discuss the measures by using illustrative examples.

Keywords: production theory, degree of inferiority, profit inefficiency, radial expansion, Farrel inefficiency measure

1 Introduction

When considering a system state, a significant concern is whether the state is *inefficient*; that is, whether there exists another state to which the state in question is *inferior* (less favorable). If inefficient, then to what degree? Various system states' inefficiencies for different systems exist, such as technical and profit inefficiencies for production systems and social and Pareto inefficiencies for games. For games, we have presented a general procedure to obtain each inefficiency measure based on the corresponding inferiority degree; this procedure gives as each inefficiency measure of a state the maximum degree of the corresponding inferiority of the state to another (Kameda, 2020).[1] This article follows the procedure to obtain inefficiency measures in production activity.

There have already been proposals for sophisticated technical-inefficiency measures in production theory. Sickles and Zelenyuk (2019) present a comprehensive review. However, these proposed measures do not seem to reflect the corresponding inferiority degrees explicitly. This article follows the procedure mentioned above and obtains simple technical-inefficiency measures based on the corresponding inferiority. We expect that our inferiority-based approach may shed light on the already proposed of technical inefficiency measures from another viewpoint. Moreover, we require that our measures be as simple as possible for broad applicability.[2]

Following Sickles and Zelenyuk (2019), we consider that firms make production activity. A firm (or decision making unit (DMU)) produces output $\boldsymbol{y} = (y_1, y_2, \ldots, y_M) \in \Re_+^M$ from input $\boldsymbol{x} = (x_1, x_2 \ldots, x_N) \in \Re_+^N$. We regard a firm's instance with a value of $(\boldsymbol{x}, \boldsymbol{y})$ as a *state* S. We assume that we can characterize the technology of a particular firm by the *technology set*, $\mathcal{T}$, as follows:

$$\mathcal{T} \equiv \{(\boldsymbol{x}, \boldsymbol{y}) \in \Re_+^N \times \Re_+^M : \boldsymbol{y} \text{ is producible from } \boldsymbol{x}\}.$$

Denote by $\boldsymbol{x}^{-1}$ the vector $(x_1^{-1}, x_2^{-1}, \ldots x_N^{-1})$. We think of the integrated input-output technology space of $\boldsymbol{z} \equiv (\boldsymbol{x}^{-1}, \boldsymbol{y})$.[3] We consider such x_i that $x_i > 0$. We use $S(\boldsymbol{z})$ and $\boldsymbol{z}(S)$ interchangeably to denote the same state. We can regard the production as an important economic activity. Each firm's decision results in the firm's profit $\Pi(\boldsymbol{x}, \boldsymbol{y})$, given the prices $\boldsymbol{w}$ and $\boldsymbol{p}$ of inputs and outputs: $\Pi(\boldsymbol{x}, \boldsymbol{y}) \equiv \boldsymbol{py} - \boldsymbol{wx}$. We can think of certain inefficiencies of a state and inferiority relations between two states: profit inferiority and inefficiency,[4] strict technical inferiority and inefficiency,[5] and technical in-

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[1] Given the definition of (particular) inferiority degree $Q(S, S')$ of state S to S', we obtain the (corresponding) inefficiency measure of S as $M_Q(S) \triangleq \max_{S' \in \mathcal{S}} Q(S, S')$ (5) as shown later. We prefer this procedure to possible candidates based on other relations with the efficient set. We have justified it in Kameda (2020) in the context of game theory.

[2] In game theory, to measure the inefficiency of Nash equilibria (NE) (Dubey (1986) et al.), many people use the simple social-inefficiency measure, the price of anarchy (PoA) (Koutsoupias and Papadimitriou, 1999; Roughgarden, 2006). PoA of an NE is the social utility of the social optimum divided by that of the NE. The social utility of a state is the sum of the players' utilities in the state. We find vastly many papers that apply PoA to various situations. It appears that the PoA's simplicity allows the spread of its application domains. *This trend looks horizontal*.

In contrast, Pareto inefficiency measures are yet to establish, with only a few researchers pursuing Pareto-inefficiency measures more sophisticated than PoA (Legrand and Touati, 2007; Aumann and Dombb, 2010; Kameda, 2019; Kameda, 2020). It looks contrastive to the situation of production theory, where many researchers accumulate and deepen technical-inefficiency measures to higher levels (Sickles and Zelenyuk, 2019; Ando, Kato, Maeda and Sekitani., 2012). *This trend looks vertical*.

[3] If one minds the dimensions of the variables, we may replace $(z_1, z_2) \equiv (1/x, y)$ by $(z_1, z_2) \equiv (x^0/x, y/y^0)$ for some constant (x^0, y^0) to make (z_1, z_2) dimensionless. Subsequent discussions would not change by the replacement.

[4] if a state S's profit is less than that of another state S', S is profit inferior to S' ($S \prec_\Pi S$). A state S is profit *inefficient* if there exists another state $S'$to which S is profit *inferior*. More details are in Section 3.1.

[5] If each input of a state S is greater than that of another S'. If each output of S is less than that of S', then S is strictly technically inferior to S' ($S \prec S$).

feriority and inefficiency.[6]

At the primary step, we present a united Farrel-type (radial) technical-inefficiency measure (UTE) in integrated input-output spaces (z). UTE integrates both the input- and output-oriented famous Farrel inefficiency measures (Debreu, 1951; Farrell, 1957) to one radial-type inefficiency measure. This integration seems similar to the 'hyperbolic' inefficiency measure by Färe et al. (2016), but ours remains radial.

Following the procedure mentioned above, as the inefficiency measures, we obtain the magnitudes of profit inefficiency (MoΠ), strict technical inefficiency (MoST), technical inefficiency (MoT), and technical profit inefficiency (MoTΠ). $Mo\Pi(S)$ gives the factor by which the profit increases due to moving from a state S to a profit optimum. (However, some players may decrease their profits by the movement.) Contrarily, every player's utility can increase simultaneously at least by the factor $MoST(S)$ and $MoT(S)$ due to moving from S to a particular target state. (Note, however, that the case of $MoT(S) = 1$ needs separate treatment.) Consider a state with the most massive profit among the states technically superior to S. $MoT\Pi(S)$ gives the factor by which the profit increases due to moving from a state S to the state.

We show that MoT and MoST generalize and solidify the inferiority-based foundation of the Farrel-type technical inefficiency measure UTE. Under the seemingly ordinary (proportionality) condition that S has a (weakly) technically-efficient frontier point on its radial expansion, $MoT(S)$ and $MoST(S)$ behave identically with $UTE(S)$ (Theorems 3 and 4).[7] Under the condition, $UTE(S)$ is identical to the maximum degree of technical inferiority of the S to some other.

However, consider the seemingly exceptional cases where the state in question has no (weakly) technically-efficient frontier-point on its radial expansion. Then, $UTE(S)$ has a non-efficient frontier point on the radial expansion of S. The significance of this frontier point is unclear. In contrast, MoST and MoT have a solution that reaches a technically-efficient frontier point. The significance of this point seems evident.

Furthermore, if the state in question has a weakly (but not strongly) technically-efficient frontier-point on its radial expansion, UTE, MoST, and MoT have some difficulty (of slacks). MoTΠ is more complicated but solves this difficulty similarly as additive technical-inefficiency measures do (Färe and Knox Lovell, 1978; Charnes, Cooper, Golany, Seiford and Stutz, 1985).

The rest of our results: We show that $MoST(S)$ distinguishes strict technical inefficiency of S. That is, if $MoST(S) > 1$, S is strictly technically inefficient, and if $MoST(S) = 1$, S is weakly technically efficient (Proposition 2). But $MoST(S)$ is not for distinguishing technical inferiority of S. Contrarily, $MoT(S)$ distinguishes technical inefficiency of S but in a complicated way for $MoT(S) = 1$ (Proposition 3). $MoT\Pi(S)$ distinguishes the technical inefficiency of S (Proposition 4). However, the procedure of calculating MoTΠ is rather complicated (Theorem 1) compared to the above measures. MoΠ is the simplest among them but does always distinguish (strict) technical inefficiency. UTE may have difficulty in distinguishing technical inefficiency for uncommon technology frontiers.

We show a fixed relationship between the values of the measures. That is, we see that for arbitrary state S, $Mo\Pi(S)$ (if exists) $\geq MoT\Pi(S) \geq MoT(S) = MoST(S) \geq UTE(S)$ (Theorem 2). This inequality agrees that from the definitions, profit optimum (if it exists) implies technical-efficient profit optimum that implies technical efficiency and weak technical efficiency, but not vice versa.

Figure 1 shows examples of the technology spaces of two outputs with the fixed inputs. y_i denotes the output i, $i = 1, 2$. S_F denotes the radial projection of S onto the technology frontier achieving $UTE(S)$. S_Π denotes a profit optimum. $S_\prec$ and $S_\leq$ denote, respectively, the aforementioned target-states that achieve the factors $MoST(S)$ and $MoT(S)$. $S_{\Pi|\leq}$ denotes the state profit optimal within the above-mentioned subset of states regarding $Mo\Pi(S)$.[8] We denote by $\overline{AB}$ the length of the line segment from state A to state B. We see that $Mo\Pi(S) = \overline{S'_\Pi 0}/\overline{S\ 0}$ and that $MoT\Pi(S) = \overline{S'_{\Pi|\leq} 0}/\overline{S\ 0}$.[9] In the upper part of the figure, S is technically inferior to S_Π. In the lower, S is not technically inferior to S_Π.

Figure 2 shows examples of the technology spaces of one input $x = z_1^{-1}$ and one output $y = z_2$.

I) We see that in the lower part of Figures 1, $Mo\Pi(S) = \overline{S'_\Pi 0}/\overline{S\ 0} > 1$, although S is not technically inferior to S_Π (profit optimum). Therefore, $Mo\Pi(S)$ may sometimes be inappropriate as the technical-inefficiency measure of S.

II) In the Figures 1 and 2, the technology frontier consists of technically efficient states. The proportionality (radial expansion) condition holds. Then, $MoST(S) = MoT(S) = UTE(S) = \overline{S_\prec 0}/\overline{S\ 0} > 1$, as is shown generally by Theorems 3 and 4. $MoST(S) = MoT(S)$ behave identically with a unified Farrel measure $UTE(S)$.

The paper is organized as follows: Section 2 presents a unified Farrel measure UTE. Section 3 elaborates on the general procedure of inefficiency measures in production theory and, based on the procedure, build the measures: MoST, MoT, and MoTΠ. Section 3.3.2 presents relations among the inefficiency measures. Section 4 shows that MoST and MoT are conceptually more straightforward and behave as UTE in the case where some proportionality (radial expansion) condition holds. Section 5 illustrates the proposed measures by examples of the technology spaces. Section 6 concludes this arti-

A state S is strictly technically *inefficient* if there exists another S' to which S is strictly technically *inferior*. More details are in Section 3.1.

[6] If each input of a state S is not less than of another S', if each output of S is not greater than of S', and if at least one input or one output of S is not equal to that of S', then S is technically inferior to S' ($S \leq S$). A state S is technically *inefficient* if there exists another S' to which S is technically *inferior*. More details are in Section 3.1.

[7] The condition holds if the technology set satisfies Axioms 5 and 6 — strong disposablity of all inputs and outputs (P.15 and P.17, Sickles and Zelenyuk (2019)).

[8] Detailed definitions of $S_\prec$ and $S_\leq$, and $S_{\Pi|\leq}$ are in Section 3.3.1.

[9] We show this as follows: On the upper part of Figure 1, the two parallel thin dotted lines that go through S_Π and S show the sets of points with the same values of $\Pi = p_1 y_1 + p_2 y_2$ as S_Π and S. Thus, we can obtain $Mo\Pi(S)(= MoT\Pi(S))$ by $\overline{S'_\Pi 0}/\overline{S\ 0}$. On the lower, similarly, we have $Mo\Pi(S) = \overline{S'_\Pi 0}/\overline{S\ 0}$ and $MoT\Pi(S) = \overline{S'_{\Pi|\leq} 0}/\overline{S\ 0}$, but $Mo\Pi(S) > MoT\Pi(S)$.

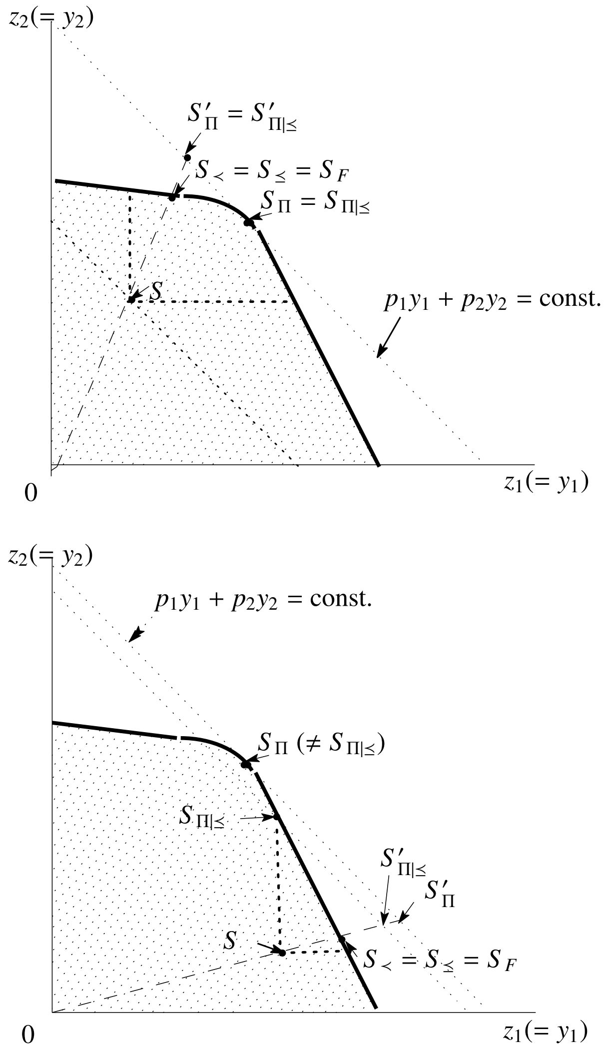

Figure 1: Examples of S_Π (the profit optimum), S_F (achieves $UTE(S)$)), $S_\prec$ (achieves $MoST(S)$), $S_\leq$ (achieves $MoT(S)$), and $S_{\Pi|\leq}$ (achieves $MoT\Pi(S)$. The areas hatched by thin dotted lines present the technology sets. Extra-thick curves are technology frontiers. The thin-dotted lines present the states with the same profit. The subsequent figures have the same conventions.

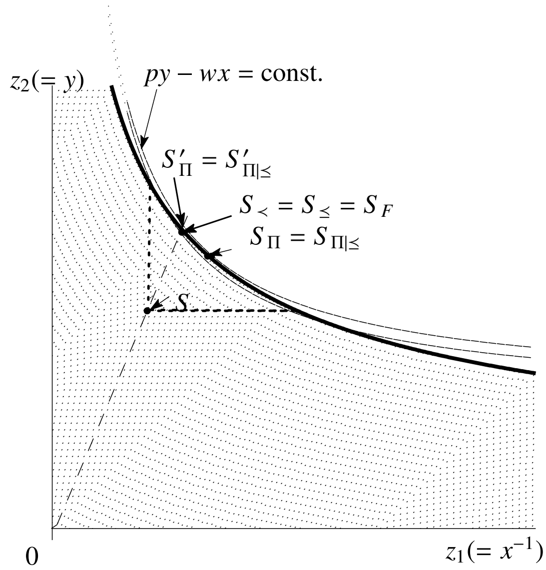

Figure 2: Examples of S_F, S_Π, $S_\prec$, $S_\leq$ and $S_{\Pi|\leq}$. The thin curves present the states with the same profit. The subsequent figures have the same conventions.

cle. Appendix A presents the proofs of theorems. Appendix B presents a glossary of symbols.

2 United Farrel measure of technical inefficiency [UTE]

Recall that $z \equiv (\boldsymbol{x}^{-1}, \boldsymbol{y})$ and that $S(z)$ denotes the state with z. We have the following:

Proposition 1 $\boldsymbol{x}(>0)$ *is a radial expansion of* $\boldsymbol{x}'(>0)$ *iff* $\boldsymbol{x}'^{-1}$ *is a radial expansion of* $\boldsymbol{x}^{-1}$.[10]

[10] We can show this as follows: The former means that $(\theta x_1, \theta x_2, \ldots \theta x_N) = (x'_1, x'_2, \ldots, x'_N)$ for some $\theta \geq 1$. This is equivalent to $(\theta x_1'^{-1}, \theta x_2'^{-1}, \ldots, \theta x_N'^{-1}) = (x_1^{-1}, x_2^{-1}, \ldots, x_N^{-1})$, which means the latter.

This proposition implies that the radial-expansion relation keeps from the space of $\boldsymbol{x}$ to that of $\boldsymbol{x}^{-1}$.

Define the united Farrel measure of technical inefficiency (UTE) as follows:

$$UTE(S(z)) \equiv \max\{\lambda | S(\lambda z) \in \mathcal{T}\}. \tag{1}$$

S_F denotes $S(\lambda z)$ with such λ that maximizes $UTE(S(z))$ of (1). That is, S_F is a technology-front point on a radial expansion of S. If $UTE(S) > 1$, S is technically inefficient. But, if $UTE(S) = 1$, $S(= S_F)$ is not necessarily (weakly) technically efficient in the case of the seemingly uncommon technology front.

3 Magnitude of Inefficiency

Our notion of the *inefficiency* of a state presupposes the *inferiority* of the state to some other. We primarily present the definitions of inferiorities and inefficiencies (Section 3.1).

Our procedure for obtaining the inefficiency measure is as follows:

(1) Firstly, determine the degree (extent) to which a state is *inferior* to some other (Section 3.2).

(2) Then, given the definition of inferiority, determine the *state's inefficiency measure* (section 3.3).

3.1 Definitions of Inferiority and Inefficiency

We think of inferiority relations between two states or combinations and inefficiency of a state. In the following, we consider an arbitrary pair of states (or instances) of the firms, S^a and $S^b \in \mathcal{T}$.

Profit optimality/inefficiency: We denote the prices of the outputs by $\boldsymbol{p} = (p_1, p_2, \ldots, p_M) \in \Re_+^M$ and those of inputs

$\boldsymbol{w} = (w_1, w_2 \ldots, w_N) \in \Re_+^N$. The profit of the firm $(\boldsymbol{x}, \boldsymbol{y})$ is $\Pi(S) \equiv \boldsymbol{p}\boldsymbol{y} - \boldsymbol{w}\boldsymbol{x}$.

If $\Pi(S^a) < \Pi(S^b)$, we say that S^a is profit inferior to S^b (denoted by $S^a \prec_\Pi S^b$). If $\Pi(S^a) > \Pi(S^b)$, we say that S^a is profit superior to S^b (denoted by $S^a \succ_\Pi S^b$).

(i) If there exists some state S' that is profit superior to state S, we call S a *profit inefficient* (profit non-optimal) state. That is, if $S^a \prec_\Pi S$ for some $S \in \mathcal{T}$ (i.e., $\exists S \in \mathcal{T}$ s.t. $S^a \prec_\Pi S$), S^a is profit inefficient (profit non-optimal).

(ii) If there exists no state that is profit superior to S, we call S a *profit optimal* (efficient) state.

Technical inefficiency and efficiency: If $z_i(S^a) \leq z_i(S^b)$ for all $i \in \mathcal{P}$ and $z_j(S^a) < z_j(S^b)$ for some $j \in \mathcal{P}$, then S^a is *technically inferior* to S^b (denoted by $S^a \preceq S^b$). If $z_i(S^a) \geq z_i(S^b)$ for all $i \in \mathcal{P}$ and $z_j(S^a) > z_j(S^b)$ for some $j \in \mathcal{P}$, then S^a is *technically superior* to S^b (denoted by $S^a \succeq S^b$).

(i) If there exists some state S' that is technically superior to state S, we call S a *technically inefficient* state; that is, S is technically inefficient iff $\exists S' \in \mathcal{T}$ s.t. $S \preceq S'$.

(ii) If there exists no state that is technically superior to S, S is called a *technically efficient* state;[11] that is, S is technically efficient iff $\nexists S' \in \mathcal{T}$ s.t. $S \preceq S'$.

Strict technical inefficiency and weak technical efficiency: If $z_i(S^a) < z_i(S^b)$ for all $i \in \mathcal{P}$, then S^a is *strictly technically inferior* to S^b (denoted by $S^a \prec S^b$). If $z_i(S^a) > z_i(S^b)$ for all $i \in \mathcal{P}$, then S^a is *strictly technically superior* to S^b (denoted by $S^a \succ S^b$). $S^a \prec S^b$ implies $S^a \preceq S^b$ but not vice versa.

(i) If there exists some state S' that is strictly technically superior to state S, we call S a *strictly technically inefficient* state; that is, S is strictly technically inefficient iff $\exists S' \in \mathcal{T}$ s.t. $S \prec S'$. (Being strictly technically inefficient implies being technically inefficient but not vice versa.)

(ii) If there exists no state that is strictly technically superior to state S, we call S a *weakly technically efficient* state; that is, S is weakly technically efficient iff $\nexists S' \in \mathcal{T}$ s.t. $S \prec S'$.

Remark 1 We note that technical inferiority lacks uniformity concerning players in contrast to *strict* technical inferiority. This non-uniformity leads to particular difficulty in defining the (*non-strict*) technical-inefficiency measure.

3.2 Degree of the inferiority of one state to another

Here, we present some definitions of the degree of the inferiority of one state to another.

3.2.1 Degree of profit inferiority

As the degree of profit inferiority of a state S^a to another S^b, we have:

$$Q_\Pi(S^a, S^b) \triangleq \frac{\Pi(S^b)}{\Pi(S^a)}. \tag{2}$$

Subsequently, if $Q_\Pi(S^a, S^b) > 1$, then S^a is *profit inferior* to S^b ($\Pi(S^a) < \Pi(S^b)$), and if $Q_\Pi(S^a, S^b) \leq 1$, then S^a is not *profit inferior* to S^b ($\Pi(S^a) \geq \Pi(S^b)$). We note $Q_\Pi(S^a, S^b) = 1$ for $z(S^a) = z(S^b)$. Thus, the measure Q_Π (given in (2)) distinguishes profit inferiority ($\prec_\Pi$).

Then, we have the following proposals.

3.2.2 Degree of (strict) technical inferiority

As a candidate of the measure of the degree of (strict) technical inferiority of state S^a to S^b, we define:

$$Q_\prec(S^a, S^b) \triangleq \min_{k \in \mathcal{M}+\mathcal{N}} \frac{z_k(S^b)}{z_k(S^a)} \tag{3}$$

Namely, as the degree of (strict) technical inferiority of state S^a to S^b, we use the minimum ratio of each player's utility of state S^b to that of state S^a.

Degree of strict technical inferiority We have the following: If $Q_\prec(S^a, S^b) > 1$, then $S^a \prec S^b$ (i.e., S^a is strictly technically inferior to S^b), and if $Q_\prec(S^a, S^b) \leq 1$, then $S^a \nprec S^b$ (i.e., S^a is not strictly technically inferior to S^b). Thus, the measure $Q_\prec$ (given by (3)) distinguishes *strict technical inferiority*. Subsequently, we can use $Q_\prec$ to measure the degree of strict technical inferiority.

Degree of technical inferiority Note that $Q_\prec(S^a, S^b) = 1$ implies $z(S^a) \leq z(S^b)$. Thus, for $Q_\prec(S^a, S^b) = 1$, if $z(S^a) \neq z(S^b)$, then $S^a \preceq S^b$ (S^a is technically inferior to S^b), otherwise $S^a \npreceq S^b$ (S^a is not technically inferior to S^b). Note also that $Q_\prec(S^a, S^b) = 1$ for $z(S^a) = z(S^b)$. Therefore, If $Q_\prec(S^a, S^b) > 1$, then $S^a \preceq S^b$. If $Q_\prec(S^a, S^b) = 1$, then if $z(S^a) \neq z(S^b)$, then $S^a \preceq S^b$, otherwise $S^a \npreceq S^b$. If $Q_\prec(S^a, S^b) < 1$, then $S^a \npreceq S^b$. Consequently, we can use $Q_\prec(S^a, S^b)$ as a technical-inferiority measure. However, this measure seems complicated.

3.2.3 Degree of technical profit inferiority

We define the following: S^a is *technical profit inferior* to S^b if $S^a \preceq S^b$ (technically inferior) and $S^a \prec_\Pi S^b$ (profit inferior), and this is not the case otherwise. As a candidate of the degree of technical profit inferiority of state S^a to S^b, we define:

$$\begin{aligned} Q_{\Pi|\preceq}(S^a, S^b) &\triangleq \Delta_{Q_\prec(S^a,S^b)\geq 1} \frac{\Pi(S^b)}{\Pi(S^a)} + \Delta_{Q_\prec(S^a,S^b)<1} \\ &= 1 + \Delta_{Q_\prec(S^a,S^b)\geq 1}\Big(\frac{\Pi(S^b)}{\Pi(S^a)} - 1\Big), \end{aligned} \tag{4}$$

where $\Delta_L \triangleq 1$ if L is, and $= 0$ otherwise.

Namely, as the degree of technical profit inferiority of state S^a to S^b, we use the degree of profit inferiority of state S^a to S^b if S^a is technically inferior to S^b ($S^a \preceq S^b$); otherwise, we use the value 1 ($S^a \npreceq S^b$).

Then, if $Q_{\Pi|\preceq}(S^a, S^b) > 1$, S^a is technically profit inferior to S^b, and if $Q_{\Pi|\preceq}(S^a, S^b) = 1$, S^a is not technically profit inferior to S^b. We note that $Q_{\Pi|\preceq}(S^a, S^b) = 1$ for $z(S^a) = z(S^b)$. Thus, the measure $Q_{\Pi|\preceq}$ (given in (4)) distinguishes technical profit inferiority and technical inferiority. $Q_{\Pi|\preceq}(S^a, S^b)$ coincides with $Q_\Pi(S^a, S^b)$ if $S^a \preceq S^b$, and $Q_{\Pi|\preceq}(S^a, S^b) = 1$ if $S^a \npreceq S^b$. Namely, the technical profit inferiority of S^a to S^b

[11] One often calls this technical efficiency the *Pareto-Koopmans efficiency*.

equals the profit inferiority of S^a to S^b, if S^a is technically inferior to S^b. The degree of it is equal to 1 if S^a is not technically inferior to S^b.

3.3 Obtaining Inefficiency Measure by General Procedure

Given the definition of the degree of the inferiority of one state to another, we may have plural candidates for the procedures obtaining the inefficiency measure of a state. We divide the candidates into two categories:

1) One candidate depends on the maximum degree of the inferiority of the state to some other (i.e., the maximum degree to which the inferiority improves by moving from the state to some other.)

2) The other candidate depends on other relations between the state and the set of efficient states.

In this article, we consider that *the inefficiency measures along the line of 1) are preferable*. We justify it is in section 3.3.3.

3.3.1 Inefficiency measure based on the maximum degree of inferiority

As the candidate based on the maximum degree of the inferiority of the state in question to some other, we have the following definition of the inefficiency measure: Given the definition of the degree of (particular) inferiority $Q(S, S')$ of state S to S', we have the (corresponding) inefficiency measure of a state S ($S \in \mathcal{T}$) by

$$M_Q(S) \triangleq \max_{S' \in \mathcal{T}} Q(S, S'). \tag{5}$$

It shows the maximum degree of the inferiority of state S to some other. As the basis of M, we can use various measures of the degree of the inferiority $Q(S, S')$ of state S to S' (given in the previous subsection), as in the following:

I) **The magnitude of profit inefficiency (profit non-optimality)**: It is natural to use the ratio of the optimal profit S_Π to the profit of the state S, $\Pi(S_\Pi)/\Pi(S)$ as the inefficiency measure of a state. In this article, we call it MoΠ. If we use the degree of profit inferiority $Q_\Pi(S, S')$ (2), as $Q(S, S')$ in (5) above, the following $M_{Q_\Pi}(S)$ equals $Mo\Pi(S)$.

$$\begin{aligned} Mo\Pi(S) &= M_{Q_\Pi}(S) = \max_{S' \in \mathcal{T}} Q_\Pi(S, S') \\ &= \max_{S' \in \mathcal{T}} \frac{\sum_k z_k(S')}{\sum_k z_k(S)} = \frac{\max_{S' \in \mathcal{T}} \sum_k z_k(S')}{\sum_k z_k(S)} = \frac{\Pi(S_\Pi)}{\Pi(S)}, \end{aligned} \tag{6}$$

where S_Π denotes a profit optimum. We say that S_Π is the most profit superior to S. Therefore, we have:

If $Mo\Pi(S) > 1$, then S ($S \in \mathcal{T}$) is profit inefficient (non-optimal). If $Mo\Pi(S) = 1$, then S ($S \in \mathcal{T}$) is profit optimal. Therefore, MoΠ distinguishes profit inefficiency (profit non-optimality). *We thus see that* MoΠ *also conforms to the general procedure.*

$Mo\Pi(S)$ may not exist with the prices fixed, if the technology is of nondecreasing returns to scale (NDRS). However, if a firm increases the outputs to obtain more profit, it may decrease the price. The situation seems complicated.

II) **The magnitude of strict technical inefficiency [MoST]**: If we use the degree of strict technical inferiority $Q_<(S, S')$ given in (3), as $Q(S, S')$ in (5), then

$$MoST(S) = M_{Q_<}(S) = \max_{S' \in \mathcal{T}} Q_<(S, S') = \max_{S' \in \mathcal{T}} \min_{k \in \mathcal{P}} \frac{z_k(S')}{z_k(S)}. \tag{7}$$

We call $M_{Q_<}(S)$ the magnitude of strict technical inefficiency (MoST) of S, $MoST(S)$. It shows the maximum degree of strict technical inferiority of state S to some other S'. If there exists $S_<$ s.t. $Q_<(S, S_<) = MoST(S)$, we call $S_<$ the *most strictly technically superior* state to S. $S_<$ must be weakly technically efficient.

Proposition 2 *If $MoST(S) > 1$, then S ($S \in \mathcal{T}$) is strictly technically inefficient, and if $MoST(S) = 1$, then S ($S \in \mathcal{T}$) is weakly technically efficient.*[12]

This proposition implies that *MoST distinguishes strict technical inefficiency*. We note that MoST has a specific emphasis on the strictness of technical inefficiency distinguished from usual technical inefficiency. We also note that $MoST(S)$ does not always distinguish technical inefficiency. As an alternative, we propose the following:

III) **The magnitude of technical inefficiency [MoT]**: If we use the degree of technical inferiority $Q_<(S, S')$ given in (3), as $Q(S, S')$ in (5), then

$$MoT(S) = M_{Q_\leq}(S) = \max_{S' \in \mathcal{T}} Q_<(S, S') = \max_{S' \in \mathcal{T}} \min_{k \in \mathcal{P}} z_k(S')/z_k(S). \tag{8}$$

We use (3) differently here from the case for MoST. We call $M_{Q_\leq}(S)$ the magnitude of technical inefficiency (MoT) of S, $MoT(S)$. It shows the maximum degree of technical inferiority of state S to some other S'.

If there exists $S_\leq$ s.t. $Q_<(S, S_\leq) = MoT(S)$, we call $S_\leq$ the *most technically-superior* state to S.

Proposition 3 *If $MoT(S) > 1$, then S is (strictly) technically inefficient ($S \in \mathcal{T}$). If $MoT(S) = 1$, then if there exists S' s.t. $Q_<(S, S') = 1$ and $z(S) \neq z(S')$, then S is technically inefficient; otherwise, S is technically efficient ($S \in \mathcal{T}$).*[13]

Therefore, while MoT distinguishes technical inefficiency, it does not do so straightforwardly for $MoT(S) = 1$. This difficulty (of slacks) seems parallel to that of the Farrel measure of technical inefficiency. However, If $MoT(S) > 1$, $MoT(S)$ and $MoST(S)$ behave identically.

Instead, to measure technical inefficiency, we propose the following:

[12] We see this as follows: From the definition, $MoST(S) \geq 1$ as $Q_<(S, S) = 1$. If $MoST(S) > 1$, then $Q_<(S, S') > 1$ for some S' ($S' \in \mathcal{T}$); thus, S is strictly technically inefficient. If $MoST(S) = 1$, then $Q_<(S, S') \leq 1$ for all S' ($S' \in \mathcal{T}$); thus, S is not strictly technically inefficient.

[13] We see this as follows: If $MoT(S) > 1$, then $Q_<(S, S') > 1$ for some S' ($S' \in \mathcal{T}$); Thus, S is (strictly) technically inefficient. If $MoT(S) = 1$, then $Q_<(S, S') \leq 1$. Then, if $Q_<(S, S') = 1$ and $z(S) \neq z(S')$ for some S' ($S' \in \mathcal{T}$), S is technically inferior to S' and, thus, is technically inefficient; otherwise, S is technically efficient.

Remark 2 The formulas of $MoST(S)$ and $MoT(S)$ look the same, but their usages and interpretations, shown in Propositions 2 and 3, are different. Nevertheless, as far as S is strictly technically inefficient, $MoST(S)$ and $MoT(S)$ behave identically.

IV) **The magnitude of technical profit inefficiency [MoTΠ]**: If we use the degree of technical profit inferiority $Q_{\Pi|\leq}(S, S')$ given in (4), as $Q(S, S')$, then,

$$MoT\Pi(S) = M_{Q_{\Pi|\leq}}(S) = \max_{S' \in \mathcal{T}} Q_{\Pi|\leq}(S, S'). \tag{9}$$

We call $M_{Q_{\Pi|\leq}}(S)$ the magnitude of technical profit inefficiency (MoTΠ) of S, $MoT\Pi(S)$. It shows the maximum degree of the technical profit inferiority of state S to some other S'. If there exists $S_{\Pi|\leq}$ s.t. $Q_{\Pi|\leq}(S, S_{\Pi|\leq}) = MoT\Pi(S)$, we call $S_{\Pi|\leq}$ the *most technically profit superior* state to S.

Proposition 4 *If $MoT\Pi(S) > 1$, then S is technically inefficient ($S \in \mathcal{T}$), and if $MoT\Pi(S) = 1$, then S is technically efficient.*[14]

This proposition implies that *MoTΠ distinguishes technical inefficiency*.

In the following, we show a base for calculating MoTΠ of a state S. Denote by $\mathcal{R}(S)$ the subset of $\mathcal{T}$, whose elements $S' \in \mathcal{R}(S)$ are all technically superior or identical to S. Define $\Pi(S_{\Pi|\mathcal{R}(S)}) = \max_{S' \in \mathcal{R}(S)} \Pi(S')$. $S_{\Pi|\mathcal{R}(S)}$ is a profit optimum within $\mathcal{R}(S)$. If $\mathcal{R}(S) \setminus \{S\} \neq \emptyset$ (nonempty), $MoT\Pi(S) = \Pi(S_{\Pi|\mathcal{R}(S)})/\Pi(S)$. If $\mathcal{R}(S) \setminus \{S\} = \emptyset$ (empty), $MoT\Pi(S) = 1$.

Theorem 1 *Assume that we can find a profit optimum $S_{\Pi|\mathcal{R}(S)}$ of $\mathcal{R}(S)$. Then, we have $MoT\Pi(S) = \Pi(S_{\Pi|\mathcal{R}(S)})/\Pi(S) = \sum_k U_k(S_{\Pi|\mathcal{R}(S)})/\sum_k U_k(S)$, and $S_{\Pi|\mathcal{R}(S)} = S_{\Pi|\leq}$. In this case, $MoT\Pi(S) > 1$, iff S is technically inefficient, and $MoT\Pi(S) = 1$, iff S is technically efficient.*

[Proof] See Appendix A. □

This theorem shows *how to obtain MoTΠ of a state*, and *MoTΠ distinguishes technical inefficiency*. See Figure 1. By using the above theorem 1, we have $MoT\Pi(S) = \Pi(S_{\Pi|\mathcal{R}(S)})/\Pi(S)$. If $\mathcal{R}(S) = \mathcal{T}$, a profit optimum $S_{\Pi|\mathcal{R}(S)}$ of $\mathcal{R}(S)$, is a profit optimum S_{Π}.

Corollary 1 *The MoΠ of a state S presents the MoTΠ of S in the case where there exists a profit optimum that is technically superior or identical to S. In that case, $Mo\Pi(S) = MoT\Pi(S) > 1$, if S is technically inefficient, and $Mo\Pi(S) = MoT\Pi(S) = 1$, if S is technically efficient.*

This corollary shows that if a state is technically inferior or equal to a profit optimum, then MoTΠ *is identical to* MoΠ, and that MoTΠ *and* MoΠ *of S distinguish technical inefficiency of S*. Then, in that case, MoΠ *works both as the measures of profit inefficiency and of technical inefficiency*. See Figures 1.

This measure avoids the difficulty (of slacks) of the measure *MoT* by introducing the optimization of a scalar function within the subset of the states technical superior to or equal to the state S. The introduction of optimizing some scalar function looks parallel to the additive measures of technical inefficiencies, such as those of Russel, Charnes et al., etc. (Färe and Knox Lovell, 1978; Charnes et al., 1985).

3.3.2 Relation between MoST, MoT, MoTΠ , MoΠ, and UTE

In the previous section, we have the following:

- If $Mo\Pi(S) > 1$, then S is profit inefficient (profit non-optimal). If $Mo\Pi(S) = 1$ then S is profit optimal.
- If $MoST(S) > 1$, then S is strictly technically inefficient. If $MoST(S) = 1$, then S is weakly technically efficient.
- If $MoT\Pi(S) > 1$, then S is technically inefficient. If $MoT\Pi(S) = 1$ then S is technically efficient.
- If $MoT(S) > 1$, then S is technically inefficient. If $MoT(S) = 1$ then, if $Q(S, S') = 1$ and $z(S) \neq z(S')$ for some $S' \in \mathcal{S}$ then S is technically inefficient, otherwise S is technically efficient.
- If $UTE(S) > 1$, then S is technically inefficient. For $UTE(S) = 1$, we cannot say whether S is technically inefficient or efficient.

We have the following definition:

[**Proportionality – radial expansion**] Consider the case where $z_i(S^a) = K_{S^aS^b} z_i(S^b)$ ($S^a, S^b \in \mathcal{T}$) for all $i \in \mathcal{P}$ and for some constant $K_{S^a,S^b} > 0$. (We denote this by $z(S^a) = K_{S^aS^b} z(S^b)$.) We say that state S^a is *proportional* to state S^b and that state S^a is a *radial* expansion of state S^b for $K_{S^aS^b} > 1$. K_{S^a,S^b} is the *proportionality constant*.

Condition 3.1 [proportionality (radial expansion)] *There exists such a technically-efficient state, S_π, that is proportional to S.*

Condition 3.2 [profit proportionality] *We can find a profit optimum S_Π that is proportional to a state S.*

Condition 3.3 [technical profit proportionality] *We can find S's technical profit optimum $S_{\Pi|<}$ that is proportional to a state S.*

We note that Condition 3.2 implies Condition 3.3 if S_Π exists. Condition 3.3 implies Condition 3.1. Then we have the following:

[14] We see this as follows: From the definition, $MoT\Pi(S) \geq 1$ as $Q_{\Pi|\leq}(S, S) = 1$. If $MoT\Pi(S) > 1$, then $Q_{\Pi|\leq}(S, S') > 1$ for some S' ($S' \in \mathcal{T}$); thus, S is technically inefficient. If $MoT\Pi(S) = 1$, then $Q_{\Pi|\leq}(S, S') = 1$ for all S' ($S' \in \mathcal{T}$); thus, S is neither profit technically inefficient nor technically inefficient (technically efficient).

Theorem 2 *For an arbitrary state S, $Mo\Pi(S)$ (if it exists) $\geq MoT\Pi(S) \geq MoT(S) = MoST(S) \geq UTE(S)$ where the first equality holds if S is technically inferior or equal to a profit optimum, where the second equality holds if Condition 3.3 is satisfied with $K_{S_{\Pi|\leq}S} = 1$, and where the fourth equality holds if Condition 3.1 is satisfied.*

[Proof] See Appendix A. □

This theorem implies that for every realizable state $S \in \mathcal{S}$, $Mo\Pi(S)$ is not less than $MoT\Pi(S)$, which is not less than $MoT(S)$ and $MoST(S)$, which is not less than $UTE(S)$.

Corollary 2 *If state S satisfies the profit-proportionality condition 3.3, although other feasible states may not be proportional to S, $Mo\Pi(S) = MoT\Pi(S) \geq MoT(S) = MoST(S) = UTE(S) = K_{S_\Pi S} > 1$ if S is strictly technically inefficient, and $Mo\Pi(S) = MoT\Pi(S) = MoT(S) = MoST(S) = K_{S_\Pi S} = UTE(S) = 1$ if S is technically efficient.*

This corollary shows that *all measures MoΠ, MoTΠ , MoT, MoST, and UTE of a state S happen to be identical* if S satisfies the profit proportionality condition 3.2 with $K_{S_{\Pi|\leq}S} = 1$. In this proportional case, *MoT, MoTΠ, MoΠ, and UTE of a state distinguish the (strict) technical inefficiency of the state as well as MoST.* Moreover, note that in this proportional case, if a state S is technically inefficient, it is also strictly technically inefficient, and that even if it is weakly technically efficient, it is also technically efficient.

3.3.3 Inefficiency measure based on other relations with the efficient set

See, for example, Figure 3, which has the same convention as Figure 2. The technology frontier consists of technically efficient states. In the figure, for UTE, S_F is the frontier point on the radial expansion of S. S is proportional to the profit optimum (S_Π). MoST, MoT, and MoTΠ give $S_\prec = S_\preceq = S_{\Pi|\leq}$ as the state most strictly technically superior, the most technically superior, and the most technically profit superior to S. $S_\Pi = S_\prec = S_\preceq = S_{\Pi|\leq} = S_F$.

In the figure, X is the point in the technology frontier nearest to S. Thus, as another candidate, one may propose the distance between S to X as the (strict) technical-inefficiency measure. However, $S_\prec = S_\preceq = S_{\Pi|\leq}$ coincide with S_Π under the profit-proportionality condition. We think that it would be natural that the states most superior to S coincide with a profit optimum S_Π (if it exists) under the profit-proportionality condition 3.2. The reason is that MoΠ seems a common measure of profit inefficiency and that $Mo\Pi(S)$ may also serve as a technical-inefficiency measure of S, as far as S satisfies the profit-proportionality condition.

Moreover, we note that $MoST(S)$ and $MoT(S)$ behave identically as the united Farrel measure $UTE(S)$ when $S_\prec = S_\preceq$ are proportional to or radial expansion of S. By Sickles and Zelenyuk (2019) (page 74), the Farrel measure seems more popular in empirical use than all of the alternative measures of technical inefficiency. $MoST(S)$ and $MoT(S)$ are the inefficiency measures based on the maximum inferiority degree but not based on the technically-efficient frontier's distance from other points. Besides, the methods of defining and calculating the least distance in this framework appear uneasy, particularly in the integrated input- and output space united. Thus, we *prefer to the inefficiency measure based on the maximum degree of inferiority to the measure based on the distance from other points of the technically-efficient frontier*.

Furthermore, we note that by Briec (1999), the Hölder distance weighted by S covers the Farrel input technical inefficiency measure of S, where the Hölder norm is Tshebishev norm. Therefore, by the suitable settings in defining the distance measure, the frontier point least distant from S is identical to the state, $S_\prec (= S_\preceq = S_F)$, most technically superior to S.

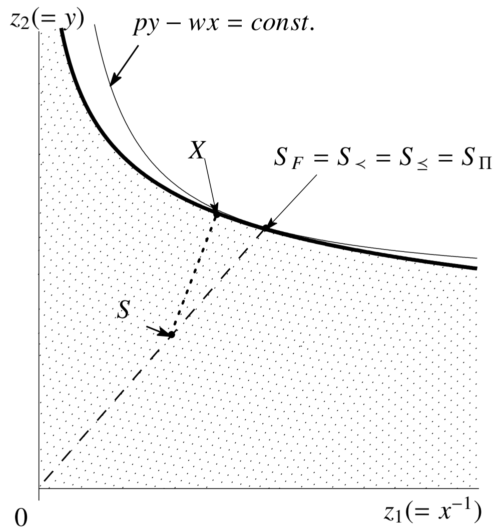

Figure 3: Distance from the technology frontier.

4 Proportional (Radial Expansion) Cases

Consider a state S with $\boldsymbol{z}(S) = (z_1(S), z_2(S), \ldots, z_n(S))$.

Condition 4.1 [weak proportionality] *There exists such a weakly technically efficient state, $S_{\omega\pi}$, that is proportional to S.*

Theorem 3 *Assume that weak-proportionality condition 4.1 is satisfied. That is, $z_i(S_{\omega\pi})/z_i(S) = K_{S_{\omega\pi}S}$, $i \in \mathcal{P}$, for some proportionality constant $K_{S_{\omega\pi}S}$. We have $MoST(S) = K_{S_{\omega\pi}S}$, and $S_{\omega\pi} = S_\prec$. In this case, $MoST(S) = K_{S_{\omega\pi}S} > 1$, and $= 1$ if S is strictly technically inefficient and weakly technically efficient.*

[Proof] See Appendix A. □

This theorem shows that *we can obtain $MoST(S)$ in the way identical to $UTE(S)$, i.e., in an intuitively-simpler way (radial-fashion) than the method following the definition* when the weak-proportionality condition 4.1 is satisfied. Note that among the states S for which Condition 4.1 holds, technically-inefficient states are also strictly technically inefficient. Note also that it is impossible that $MoST(S) = K_{S_{\omega\pi}S} < 1$. We see that $MoST(S)$ behaves in the same way as $UTE(S)$.

A technically efficient state is also weakly technically efficient. Then, we have the following:

Corollary 3 *Assume that proportionality condition 3.1 is satisfied; that is,* $z_i(S_\pi)/z_i(S) = K_{S_\pi S}$, $i \in \mathcal{P}$ *for some proportionality constant* $K_{S_\pi,S}$. *We have* $MoST(S) = K_{S_\pi S}$ *and* $S_\pi = S_<$. *In this case,* $MoST(S) = K_{S_{\omega\pi}S} > 1$ *and* $= 1$, *if* S *is strictly technically inefficient and technically efficient.*

This corollary implies that *we can obtain* $MoST(S)$ *in the way identical to* $UTE(S)$, *i.e., in an intuitively-simpler way (radial-fashion) than the method used following the definition* when the proportionality condition 3.1 is satisfied. Note that if the proportionality condition 3.1 holds for state S, S is technically inefficient iff S is strictly technically inefficient. In the case where Corollary 3 holds, Theorem 3 also holds.

Theorem 4 *Assume that proportionality condition 3.1 is satisfied. That is,* $z_i(S_\pi)/z_i(S) = K_{S_\pi S}$, $i \in \mathcal{P}$ *for some proportionality constant* $K_{S_\pi S}$. *For* $K_{S_\pi S} > 1$, *we have* $MoT(S) = K_{S_\pi S}$, *and* $S_\pi = S_\leq$. *In this case,* $MoT(S) = K_{S_\pi S} > 1$ *and* S *is strictly technically inefficient. For* $K_{S_\pi S} = MoT(S) = 1$, S *is technically efficient.*

[Proof] See Appendix A. □

This theorem implies that *we can obtain* $MoT(S)$ *in the way identical to* $UTE(S)$, *i.e., in an intuitively-simpler way (radial-fashion) than the method following the definition* when the proportionality condition 3.1 is satisfied. Note that when S satisfies the proportionality condition 3.1, $MoST(S)$ and $MoT(S)$ behave identically. See Figures 1 and 2. Note also that it is impossible that $K_{S_\pi S} < 1$.

We present certain sufficient conditions to achieve the proportionality conditions 4.1 and 3.1 as follows: Define $\mathcal{Z} \triangleq \{z(S) \mid S \in \mathcal{T}\}$. We consider the case where $\mathcal{Z}$ is compact (closed and bounded). Then, we have the boundaries of $\mathcal{Z}$ as follows: n boundary hyperplanes $\mathcal{B}_i = \{z(S) \mid z_i(S) = 0, z(S) \in \mathcal{Z}\}, i \in \mathcal{P}$ and a boundary hypersurface η connecting all $\mathcal{B}_i$, $i \in \mathcal{P}$.

Condition 4.2 [weak technically efficient frontier] *The hypersurface* η *is composed of weakly technically efficient points.*

Condition 4.3 [technically efficient frontier] *The hypersurface* η *is composed of (strongly) technically efficient points.*

Proposition 5 *If Conditions 4.2 and 4.3 are satisfied, then, for an arbitrary* $z(S) \in \mathcal{Z}$, *any line* $z_i = tz_i(S), i \in \mathcal{P}, t \geq 0$, *crosses weakly and strongly technically efficient points that are both denoted by* $z(S_{\omega\pi}) = K_{S_{\omega\pi}S}\, z(S)$. *Then, Theorem 3 and both of Theorems 4 and Corollary 3 hold.*

Consider systems that satisfy the technically-efficient frontier conditions 4.2 and 4.3. This proposition implies that *for the systems, we can obtain MoST (and also MoT) of* S as $MoST(S) = MoT(S) = \overline{S_<0}/\overline{S0}$ in the graph of $\mathcal{Z}$. See Figures 1 and 2.

Remark 3 If we consider the nature of technology and production, it would be natural to think that the technology frontier consists of technically efficient states. However, we need to show how our proposals handle exceptional cases where the technology frontier includes technically inefficient states. We offer it in the next section. Note, in passing, that in the context of games one may consider the efficient frontier point that is on the radial expansion of an NE a 'Nash-proportionate fair' allocation (Kameda, Altman, Touati and Legrand, 2012).

5 Exceptional Cases

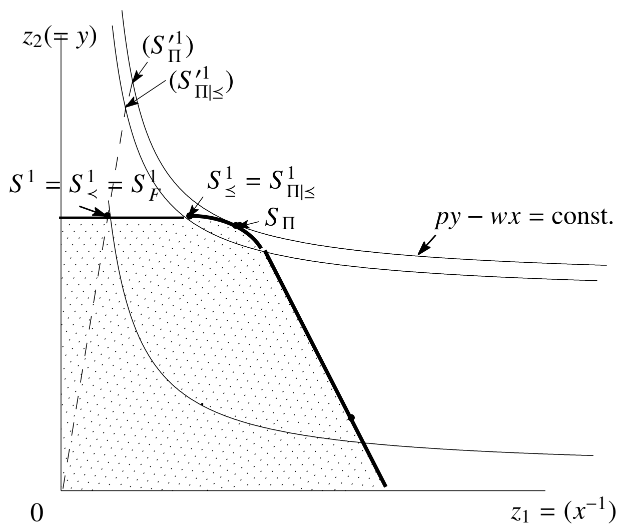

Figure 4: The technology frontier is all (weak) technically efficient. Even by consuming more than a certain amount of input x, the technology allows no increase in producing y. (Besides, to produce output y, the technology requires no less than a certain amount of input x.)

In Figures 4 and 5, we illustrate some cases of UTE, MoST, MoT, MoTΠ, and MoΠ with the seemingly exceptional technology frontier in the two dimensional technology spaces. S^1 is *technically inefficient but not strictly technically inefficient*. S^2 is technically inefficient and also strictly technically inefficient. Again, we denote by $\overline{S^a S^b}$ the length of the line segment from S^a to S^b.

(1) Figure 4 has the same convention as Figure 2. The frontier states of the extra-thick and thick curves, respectively, are technically efficient and weakly technically efficient. S^1, $S^1_<$ and S^1_F are on the *weakly efficient* technology frontier. $S^1_\leq$ is technically efficient, and $Q_<(S^1, S^1_\leq) = 1$ with $MoT(S^1) = UTE(S^1) = 1$ although S^1 is technically inferior to $S^1_\leq$. This situation shows the difficulty (of slacks) because of the non-uniformity of the definition of technical inferiority concerning players. In contrast, $MoT\Pi(S^1) = \Pi(S'^1_{\Pi|\leq})/\Pi(S^1) > 1$

determines the technical inefficiency of S^1 although S^1 is not technically inferior to S_Π.

$Mo\Pi(S^1) \equiv \Pi(S'^1_\Pi)/\Pi(S^1) > MoT\Pi(S^1) \equiv \Pi(S'^1_{\Pi|\leq})/\Pi(S^1) >^{15} \overline{S'^1_{\Pi|\leq}0}/\overline{S^10} > \overline{S^1_<0}/\overline{S^10} \equiv MoST(S^1) = MoT(S^1) = UTE(S^1) = 1$.

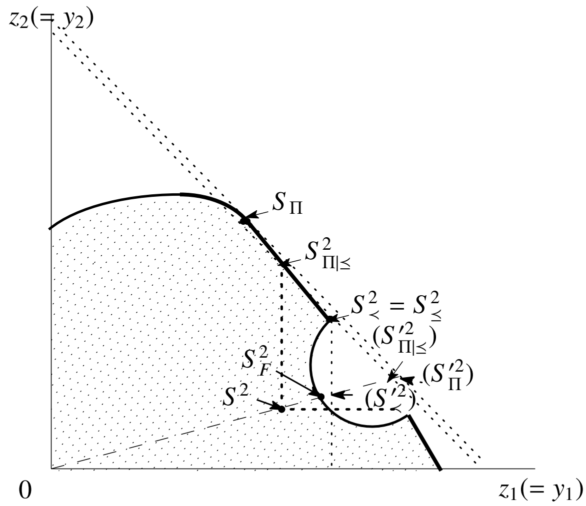

Figure 5: The technology frontier is not all (weak) technically efficient. The technology does not allow to produce certain combinations of outputs (y_1, y_2). Those combinations look possible in standard cases. (Besides, decreasing the production of y_1 less than a certain amount causes the technology to reduce the production capacity of y_2.)

(2) Figure 5 has the same convention as Figure 1 (two outputs with the inputs given). Only some frontier points (extra-thick curve) are technically efficient. Others (thick curve) are not technically efficient which may be uncommon. But for that case, $UTE(S^2)$ gives S^2_F as the radial expansion of S^2 in the technologically-inefficient technology frontier. Thus, $UTE(S^2) = \overline{S^2_F0}/\overline{S^10}$. It is difficult to find the significance of S^2_F. In contrast, MoST and MoT give $S^2_<(= S^2_\leq)$ on the technically-efficient frontier. The significance of $S^2_<$ and $S^2_\leq$) seems clear.

Proportionality conditions 4.1 and 3.1 do not hold for S^2. Then, in these cases, we rely on the definitions (3) and (7), to obtain MoST and MoT. We have $MoST(S^2) = MoT(S^2) = \overline{S'^2_<\,0}/\overline{S^2\,0} > 1$.[16] $MoST(S^2) = MoT(S^2) = \overline{S'^2_<\,0}/\overline{S^2\,0} > \overline{S^2_F0}/\overline{S^10} = UTE(S^2)$. Also $S^2_<$ is technically efficient whereas S^2_F is not. Therefore, we think that in this extraordinary case, MoT and MoST behave more reasonably than UTE. Similarly as S^1 we see that S^2 is not technically inferior to S_Π. Then,

$Mo\Pi(S^2) \equiv \Pi(S'^2_{\Pi|\leq})/\Pi(S^2) > MoT\Pi(S^2) \equiv \Pi(S'^2_{\Pi|\leq})/\Pi(S^2) >^{17} \overline{S'^2_{\Pi|\leq}0}/\overline{S^20} > \overline{S'^2_\leq0}/\overline{S^20} \equiv MoST(S^2) = MoT(S^2) > 1$.

We thus have:

$Mo\Pi(S^1) > MoT\Pi(S^1) > MoT(S^1) = MoST(S^1) = UTE(S^1) = 1$.

$MoST(S^2) > MoT\Pi(S^2) > MoT(S^2) = MoST(S^2) > UTE(S^2) > 1$.

We see that S^1 and S^2 are not technically inferior to S_Π. Nevertheless, $Mo\Pi > 1$ for both of S^1 and S^2. Then, MoΠ is again an unsuitable technical-inefficiency measure sometimes. MoST distinguishes strict technical inefficiency. Contrarily, in some instances ($MoT(S) = 1$), MoT has some difficulty (of slacks) distinguishing technical inefficiency of S, which MoTΠ can distinguish.

6 Concluding Remarks

Preliminarily, we present a united Farrel-type (radial) technical-inefficiency measure (UTE) in integrated input-output spaces. UTE integrates both the input-oriented and output-oriented Farrel inefficiency measures to one radial-type inefficiency measure. This integration seems similar to the 'hyperbolic' inefficiency measure by Färe et al. (Färe, Margaritis, Rouse and Roshdi, 2016), but ours remains radial.

We have had a general procedure for various inefficiency types to obtain each inefficiency measure based on the corresponding inferiority. According to the general procedure, we have obtained inefficiency measures of production:

- MoΠ for the profit-inefficiency measure,
- MoST for the strict-technical-inefficiency measure,
- MoT for the technical-inefficiency measure,
- MoTΠ for the technical-profit-inefficiency measure.

We have confirmed that each measure distinguishes its inefficiency but does not always distinguish the inefficiency of others.

MoST and MoT generalize and solidify the inferiority-based foundation of Farrel type (radial) measures of technical inefficiency. Consider the seemingly ordinary case where a state S has a technically-efficient technology-frontier point on its radial expansion. This case holds for the seemingly familiar technology-front of production. In that case, MoST and MoT behave identically in a radial way as UTE and look conceptually straightforward.

In another case of the weakly efficient technology frontier, UTE, MoST, and MoT may have the problem of slacks. Our proposed $MoT\Pi(S)$ is complicated but solves the problem like the additive measures. We have shown a fixed relation between the values of the measures. Namely, in general, $Mo\Pi(S)$ (if it exists) $\geq MoT\Pi(S) \geq MoT(S) = MoST(S) \geq UTE(S)$. This relation agrees that from the definitions, strict

[15] We see this inequality by the following: Assume $(z_2 =)\ y' = ky$ and $(z_1 =)\ 1/x' = k/x$ for $k \geq 1$ (proportionality). Then $(py' - wx')/(py - wx) = k + (k - 1/k)wx/(py - wx) \geq k$ where the equality holds for $k = 1$.

[16] We obtain them as follows: If a certain state S is in the area above the dashed line from 0 through S^2, $z_1(S)/z_1(S^2) < z_2(S)/z_2(S^2)$. Then, $Q_<(S^2, S) = z_1(S)/z_1(S^2)$ (see (3)), which is the largest when $S = S^2_<$. If S is in the area below the dashed line from 0 through S^2, then $Q_<(S^2, S) = z_2(S)/z_2(S^2)$, which is less than $z_2(S'^2_<)/z_2(S^2) = z_1(S'^2_<)/z_1(S^2) = z_1(S^2_<)/z_1(S^2)$. Therefore, $MoST(S^2)(= MoT(S^2))$ is given by $z_1(S^2_<)/z_1(S^2) = \overline{S'^2_<\,0}/\overline{S^2\,0}$ (see (7)).

[17] We see this inequality by the following: Assume $(z_2 =) y'_2 = ky_2$, $(z_1 =) y'_1 = ky_1$ (proportionality) and $x' = x$ (fixed). Then $(p_1y'_1 + p_2y'_2 - wx)/(p_1y_1 + p_2y_2 - wx) = k + (k - 1)wx/(p_1y_1 + p_2y_2 - wx) \geq k$. where the equality holds for $k = 1$.

technical inefficiency implies technical inefficiency that implies technical profit inefficiency that implies profit inefficiency, not vice versa.

We have had an overall characterization as follows: MoΠ (if it exists) is the most straightforward measure, but it cannot always distinguish technical inefficiency. UTE and MoST look secondly simple. But UTE does not seem proper in distinguishing (strict) technical inefficiency in general. MoST is the second-most simple measure and can distinguish strict technical inefficiency, but it does not always distinguish technical inefficiency. MoT is similar to MoST in simplicity, but $MoT = 1$ requires an extra procedure in distinguishing technical inefficiency. In contrast, in the case of $MoT > 1$, MoT has no problem, MoST and MoT are identical, and MoST also distinguishes technical inefficiency. MoTΠ is more complicated but distinguishes technical inefficiency.

We plan to pursue the suitability of MoST, MoT, and MoTΠ by examining several other concrete examples and seek the possibility of the inefficiency measures of the types not discussed here. We hope that our study will serve as a stepping-stone to future developments.

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Appendix A. Proofs of Theorems

A Proof of Theorem 1 Note that $\Pi(S_{\Pi|\mathcal{R}(S)}) = \max_{S'\in\mathcal{R}(S)} \Pi(S')$. Namely, we can find a profit optimum of $\mathcal{R}(S)$, $S_{\Pi|\mathcal{R}(S)}$, that is technically superior or identical to the state S.

Case 1: $\mathcal{R}(S)\setminus\{S\} \neq \emptyset$ (nonempty). Then, $Q_{<}(S, S_{\Pi|\mathcal{R}(S)}) \geq 1$.

Consider an arbitrary state $S' \in \mathcal{R}(S)$. $Q_{<}(S,S') \geq 1$. $\Pi(S') \leq \Pi(S_{\Pi|\mathcal{R}(S)})$ since $S_{\Pi|\mathcal{R}(S)}$ is a profit optimum within $\mathcal{R}(S)$. That is, $\Pi(S')/\Pi(S) \leq \Pi(S_{\Pi|\mathcal{R}(S)})/\Pi(S)$. Then, $Q_{\Pi|\leq}(S,S') = \Delta_{Q_{<}(S,S')\geq 1}[\Pi(S')/\Pi(S)] + \Delta_{Q_{<}(S,S')<1} = \Pi(S')/\Pi(S)$. Then, $MoT\Pi(S) = \max[\{\max_{S'\in\mathcal{R}(S),S'\neq S_{\Pi|\mathcal{R}(S)}} Q_{\Pi|\leq}(S,S')\}, Q_{\Pi|\leq}(S, S_{\Pi|\mathcal{R}(S)})]$ $= \Pi(S_{\Pi|\mathcal{R}(S)})/\Pi(S)$

(by noting that $Q_{\Pi|\leq}(S, S_{\Pi|\mathcal{R}(S)}) = \Pi(S_{\Pi|\mathcal{R}(S)})/\Pi(S)$ since $Q_{<}(S, S_{\Pi|\mathcal{R}(S)}) \geq 1$ by assumption).

Thus, $MoT\Pi(S)$ is given by $\Pi(S_{\Pi|\mathcal{R}(S)})/\Pi(S)$.

If $S_{\Pi|\mathcal{R}(S)}$ is technically superior to S, S is technically inefficient and $MoT\Pi(S) = \Pi(S_{\Pi|\mathcal{R}(S)})/\Pi(S) > 1$.

If $S_{\Pi|\mathcal{R}(S)}$ is identical to S, S is technically efficient and $MoT\Pi(S) = \Pi(S_{\Pi|\mathcal{R}(S)})/\Pi(S) = 1$.

Case 2: $\mathcal{R}(S)\setminus\{S\} = \emptyset$ (empty). Then, S is technically efficient, $S_{\Pi|\mathcal{R}(S)} = S$, and

$MoT\Pi(S) = \Pi(S_{\Pi|\mathcal{R}(S)})/\Pi(S) = 1$.

From Proposition 4, $MoT\Pi(S) > 1$ means that S is technically inefficient. Furthermore, from Proposition 4, $MoT\Pi(S)$ $= 1$ means that S is technically efficient. □

A Proof of Theorem 2

1) $Mo\Pi(S) \geq MoT\Pi(S)$: It is clear if we note the following. Recall that $Mo\Pi(S)$ is the ratio of the optimal profit to the profit of state $S \in \mathcal{T}$ and that $MoT\Pi(S)$ is the ratio to the profit of state S, of the optimal profit within the subset $\mathcal{T}_{\mathcal{R}(S)}$ of states that are technically superior or equal to S. Since $\mathcal{T}_{\mathcal{R}(S)} \subset \mathcal{T}$, then $\max_{S'\in\mathcal{T}_{\mathcal{R}(S)}} \Pi(S') \leq \max_{S'\in\mathcal{T}} \Pi(S')$. We, therefore, see that $Mo\Pi(S) \geq MoT\Pi(S)$. The equality holds if S is technically inferior or equal to the profit optimum S_{Π}.

2) $MoT\Pi(S) \geq MoT(S)$: Denote $K \equiv Q_{<}(S,S')$ $= \min\{\min_{k\in\mathcal{M}} y_k(S')/y_k(S), \min_{k\in\mathcal{N}} x_k(S)/x_k(S')\}$. We note that for $K \geq 1$

$$\frac{\Pi(S')}{\Pi(S)} = \frac{\sum_{k\in\mathcal{M}} p_k y_k(S') - \sum_{k\in\mathcal{N}} w_k x_k(S')}{\sum_{k\in\mathcal{M}} p_k y_k(S) - \sum_{k\in\mathcal{N}} w_k x_k(S)}$$
$$\geq \frac{\sum_{k\in\mathcal{M}} p_k K y_k(S) - \sum_{k\in\mathcal{N}} w_k x_k(S)/K}{\sum_{k\in\mathcal{M}} p_k y_k(S) - \sum_{k\in\mathcal{N}} w_k x_k(S)}$$
$$= \frac{K\{\sum_{k\in\mathcal{M}} p_k y_k(S) - \sum_{k\in\mathcal{N}} w_k x_k(S)\} + \dfrac{K-1}{K}\sum_{k\in\mathcal{N}} w_k x_k(S)}{\sum_{k\in\mathcal{M}} p_k y_k(S) - \sum_{k\in\mathcal{N}} w_k x_k(S)}$$
$$\geq K,$$

where the first equality holds if S' is proportional to S and where the second equality holds if $K = 1$. Then, from (3) and (4), $Q_{\Pi|\leq}(S,S') \geq Q_{<}(S,S')$. Therefore, from (8) and (9), we have $MoT\Pi(S) \geq MoT(S)$, where the equality holds if Condition 3.3 is satisfied with $K_{S_{\Pi|<}S} = 1$.

3) It is evident that $MoT(S) = MoST(S)$.

4) $MoT(S) \geq UTE(S)$: If S_F is technically efficient, naturally, $S_F = S_{<} = S_{\leq}$. Then $MoT(S) = MoST(S) = UTE(S)$. If If S_F is technically inefficient, there must exist S' s.t. $S \leq S'$. Then,

$$UTE(S) \equiv K_{SS_F} = \frac{z_k(S_F)}{z_k(S)} \leq \min_{k\in\mathcal{N}\cup\mathcal{M}} \frac{z_k(S')}{z_k(S)}$$
$$\leq \max_{S''\in\mathcal{T}} \min_{k\in\mathcal{N}\cup\mathcal{M}} \frac{z_k(S'')}{z_k(S)} \equiv MoT(S) = MoST(S).$$

Thus, we have $MoT(S) \geq UTE(S)$. The equality holds if the proportionality condition 3.2 is satisfied. □

A proof of Theorem 3 Note that we can find such a weak technically efficient state $S_{\omega\pi}$ that satisfies $z_i(S_{\omega\pi})/z_i(S) = K_{S_{\omega\pi}S}$, $i \in \mathcal{P}$. Consider another state $S' \in \mathcal{T}$. Since $S_{\omega\pi}$ is weakly technically efficient. Then there must exist some i ($i \in \mathcal{P}$) such that $z_i(S') \leq z_i(S_{\omega\pi})$ and, thus, such that $z_i(S')/z_i(S) \leq z_i(S_{\omega\pi})/z_i(S) = K_{S_{\omega\pi}S}$. Then, $Q_{<}(S,S') = \min_k z_k(S')/z_k(S) \leq K_{S_{\omega\pi}S}$. Then

$MoST(S) = \max[\{\max_{S'\in\mathcal{T},S'\neq S_{\omega\pi}} Q_{<}(S,S')\}, Q_{<}(S,S_{\omega\pi})]$ $= K_{S_{\omega\pi}S}$ (by noting that $Q_{<}(S,S_{\omega\pi}) = K_{S_{\omega\pi}S}$), and $S_{\omega\pi} = S_{<}$. Thus, $MoST(S)$ is given by $K_{S_{\omega\pi}S}$.

Naturally, $MoST(S) = K_{S_{\omega\pi}S} > 1$ means that S is strictly technically inefficient, and $MoST(S) = K_{S_{\omega\pi}S} = 1$ means that $z(S_{\omega\pi}) = z(S)$. Thus, S is also weakly technically efficient. □

A Proof of Theorem 4 Note that for $K_{S_\pi S} > 1$, we can find such a technically efficient state $z(S_\pi)$ that satisfies $z_i(S_\pi)/z_i(S) = K_{S_\pi S} > 1$, $i \in \mathcal{P}$. Consider another state $S' \in \mathcal{T}$. Since S_π is a technically efficient state, there must exist some i ($i \in \mathcal{P}$) such that $z_i(S') \leq z_i(S_\pi)$ and, thus, such that $z_i(S')/z_i(S) \leq z_i(S_\pi)/z_i(S) = K_{S_\pi S}$. Then, $Q_<(S, S') = \min_k z_k(S')/z_k(S) \leq K_{S_\pi S}$. Then,

$MoT(S) = \max[\{\max_{S' \in \mathcal{T}, z(S')) \neq z(S_\pi)} Q_<(S, S')\}, Q_<(S, S_\pi)] = K_{S_\pi S}$ (by noting that $Q_<(S, S_\pi) = K_{S_\pi S}$). Thus, $MoT(S)$ is given by $K_{S_\pi S}$.

Naturally, $MoT(S) = K_{S_\pi S} > 1$ means that S is strictly technically inefficient. For $K_{S_\pi S} = MoT(S) = 1$, $z(S_\pi) = z(S)$ and, thus, S is also technically efficient. (We note that if there were S' s.t. $z(S_\pi) = z(S) \neq z(S')$ and $Q_<(S_\pi, S') = Q_<(S, S') = 1$, then $z_i(S_\pi) < z_i(S')$ for some i and $z_j(S_\pi) \not> z_j(S')$ for all j, thus, S_π should be technically inferior to S. Therefore, S_π should not be technically efficient, which is a contradiction.) □

Appendix B: Glossary of Symbols

We have the following notation:

- $\mathcal{M}$ — the set of output indexes $\{1, 2, \cdots, m\}$.
- $\mathcal{N}$ — the set of input indexes $\{1, 2, \cdots, n\}$.
- $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ — inputs
- $\boldsymbol{y} = (x_1, x_2, \cdots, x_m)$ — outputs
- $\boldsymbol{z} = (\boldsymbol{x}^{-1}, \boldsymbol{y})$ — inputs and outputs
- $z(S)$ — $(z_1(S), z_2(S), \ldots, z_{m+n}(S))$.
- $\mathcal{T}$ — the set of feasible states (instances of firms). $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{T}$ iff $\boldsymbol{y}$ is producible from $\boldsymbol{x}$.
- $\Pi(S)$ — $\boldsymbol{py}(S) - \boldsymbol{wx}(S) \equiv \sum_k p_k y_k(S) - \sum_k w_k x_k(S)$.
- $S^a <_\Pi S^b$ — $\Pi(S^a) < \Pi(S^b)$ (S^a is profit inferior to S^b).
- S_Π — a profit optimum: $\Pi(S_\Pi) = \max_{S' \in \mathcal{T}} \Pi(S')$.
- $S^a \leq S^b$ — S^a is technically inferior to S^b.
- $S^b \geq S^a$ — S^b is technically superior to S^a.
- $S^a < S^b$ — S^a is strictly technically inferior to S^b.
- $S^b > S^a$ — S^b is strictly technically superior to S^a.
- $K_{S^a S^b} > 0$ ($S^a, S^b \in \mathcal{T}$) — the proportionality constant: $z_i(S^a) = K_{S^a S^b} z_i(S^b)$ for all $i \in \mathcal{P}$.
- $Q_\Pi(S^a, S^b)$ — the degree of profit inferiority of S^a to S^b: $\Pi(S^b)/\Pi(S^a) = \sum_k z_k(S^b)/\sum_k z_k(S^a)$.
- $Q_<(S^a, S^b)$ — the degree of technical inferiority of S^a to S^b: $\min_{k \in \mathcal{P}} z_k(S^b)/z_k(S^a)$.
- $Q_{\Pi|\leq}(S^a, S^b)$ — the degree of technical profit inferiority of state S^a to S^b: $\Delta_{Q_<(S^a,S^b) \geq 1} \Pi(S^b)/\Pi(S^a) + \Delta_{Q_<(S^a,S^b) < 1}$ $= 1 + \Delta_{Q_<(S^a,S^b) \geq 1}(\Pi(S^b)/\Pi(S^a) - 1)$.
- $M_Q(S)$ — $\max_{S' \in \mathcal{T}} Q(S, S')$: the inefficiency measure of S ($S \in \mathcal{T}$) with the degree of the inferiority of S to S' being $Q(S, S')$.
- $Mo\Pi(S)$ — the magnitude of profit inefficiency of S: $\Pi(S_\Pi)/\Pi(S)$.
- $MoST(S)$ — the magnitude of strict technical inefficiency of S: $M_{Q_<}(S) = \max_{S' \in \mathcal{T}} Q_<(S, S')$.
- $S_<$ — the most strictly technically superior state to S: $Q_<(S, S_<) = MoST(S)$.
- $MoT(S)$ — the magnitude of technical inefficiency of S: $M_{Q_\leq}(S) = \max_{S' \in \mathcal{T}} Q_<(S, S')$.
- $S_\leq$ — the most technically superior state to S: $Q_<(S, S_\leq) = MoT(S)$.
- $MoTS(S)$ — the magnitude of technical profit inefficiency of S: $M_{Q_{\Pi|\leq}}(S) = \max_{S' \in \mathcal{T}} Q_{\Pi|\leq}(S, S')$.
- $S_{\Pi|\leq}$ — the most profit technically superior state to S: $Q_{\Pi|\leq}(S, S_{\Pi|\leq}) = MoT\Pi(S)$.
- $\overline{AB}$ — the length of the line segment from A to B.
- $\mathcal{Z}$ — the set of z of feasible combinations: $\{z(S) \mid S \in \mathcal{T}\}$.
- $\mathcal{B}_i$, $i \in \mathcal{P}$ — a boundary hyperplane of $\mathcal{Z}$: $\{z(S) \mid z_i(S) = 0, z(S) \in \mathcal{Z}\}, i \in \mathcal{P}$.
- η — a boundary hypersurface of $\mathcal{Z}$ that connects all $\mathcal{B}_i$.
- $\mathcal{R}(S)$ — the subset of $\mathcal{T}$ whose elements are all technically superior or identical to S: the set $\{S' \in \mathcal{T} | S' \geq S \text{ or } S' = S\}$.
- $S_{\Pi|\mathcal{R}(S)}$ — a profit optimum within $\mathcal{R}(S)$: $\Pi(S_{\Pi|\mathcal{R}(S)}) = \max_{S' \in \mathcal{R}(S)} \Pi(S')$.

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