Belief Revision for Inductive Game Theory

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Abstract. Inductive game theory captures how a player inductively derives his/her personal views from experiences. The player may have multiple views, some of which differ from the objective situation, but may revise them with further experiences. This paper gives a logical formulation of this revision process by focusing on the role of player's beliefs. For this objective, we take the AGM approach of belief revision. The idea behind our logic is that the player's belief state is represented by a belief set of propositional formulas, thereby describing a revision process for the belief states by using a revision operation in AGM theory. In this setting, the player's personal views are described as models for the current belief set. We also present an application of our framework to a class of inductive games, called festival games.

1 Introduction and Overview

Standard game theory assumes players to have sufficient knowledge or common belief of the game they play. However, in a real social or economic situation, such knowledge/belief is not given in advance, but rather emerges from the individual experiences with bounded cognitive abilities and is revised through time. Inductive game theory, originally introduced by Kaneko and Matsui [15] and Kaneko and Kline [12] [13] [14], explores this issue and captures how a player inductively derives his/her knowledge and beliefs of the game from experiences.

Inductive game theory distinguishes an objective situation and players' personal views of the game. Players are assumed to have little initial knowledge about the objective situation, but repetitively face given situation, to accumulate their experiences through their choices of available actions, and to construct their personal views with the experiences. In the players' construction, they have different personal views based on their different experiences even when the players face an identical situation. Moreover, a player may have multiple views, some differing from the objective situation, but may revise them with further experiences.

The difference in players' views is due not only to their experiences, but also to their memories, which are represented by memory functions. By introducing memory functions, players' abilities of memories can be represented in various ways, although extensive games in standard game theory place some restrictions on the representations. Therefore, even when gathering many experiences, a player may only partially recall his experiences.

While inductive game theory provides a framework to derive a personal view consistent with a player's memories of the experiences, it has been less studied how the derived view is revised by additional experiences. This paper provides a framework for revision in inductive game theory based on the idea of AGM (Alchourrón-Gärdenfors-Makinson) theory of belief revision [1]. By introducing the revision process, we reflect a player's inductive inference in decision-making.

Since standard game theory assumes players to have sufficient knowledge or common belief of the structure, it is not good at treating inductive inferences of the structure. Revision of beliefs in (epistemic) logic has also been applied to standard game theory as in Binmore [2], Bonnano [4], van Benthem [17], Board [3], and Feinberg [6] [7][3]. However, such studies usually focus on beliefs about opponent players' actions. While it is important to consider belief formations of opponents' actions in interactive situations, we need to ask whether such beliefs are plausible when the formation heavily depends on (common) knowledge of the structure. Inductive game theory and this paper truly ask the question: How do players cognize the (interactive) situations that they face?

Our logic is based on the standard classical propositional logic. Basic statements in inductive game theory, such as the histories of player's actions and payoffs are associated with propositional atomic formulas, thereby describing causality relations between histories or players' strategies as compound formulas. In terms of this language, the player's belief state is described by a belief set of formulas, thus a revision process for the belief states is described by using a revision operation in AGM theory. In this setting, the player's personal views are described as models for the current belief set (i.e., assignments of truth values which satisfy all the formulas of belief set) in the semantics.

By means of this logic, we also show an application to festival games, which are specific inductive games proposed by Kaneko and Matsui [15]. As explained above, the players in inductive games accumulate their experiences and inductively derive their personal views of the experiences. Festival games capture the mechanisms of prejudices and discriminations resulting from experiences.

This paper is organized as follows. The next section gives the definition of inductive game theory following Kaneko and Kline [14]. Section 3 presents our logic for belief revision. Section 4 presents an application of our belief revision to festival games. Finally, Section 5 gives conclusions and discusses further research.

[3] A theory of belief revision in itself is now a very active area of research, and it connects with dynamic epistemic logic. Comprehensive surveys were published by van Ditmarsch, et al. [5], van der Hoek and Pauly [10].

2 Inductive Game Theory

2.1 Information protocols

We define an *information protocol* Π below. Let W be a nonempty finite set of *information pieces*, A be a nonempty finite set of *actions*, and $\prec$ be a finite subset of $\bigcup_{m=0}^{\infty}((W \times A)^m \times W)$. $A_w \subseteq A$ is the available action at $w \in W$. When $m = 0$, $(W \times A)^0 \times W$ is regarded as W of a unary relation on W.

The relation $\prec$ is called a *causality relation*. Each element $\langle (w_1, a_1), \ldots, (w_m, a_m), w_{m+1} \rangle \in \prec$ is called a *sequence* of length $m + 1$. Expression $\langle \xi, w \rangle$ denotes a generic element of $\bigcup_{m=0}^{\infty}((W \times A)^m \times W)$, and $\langle w \rangle$ is that of $(W \times A)^0 \times W$. Using a causal relation $\prec$, we give a partition on W. That is, $W^D := \{w \in W \mid \langle (w, a), v \rangle$ for some $a \in A$ and $v \in W\}$ is called a set of *decision pieces* and $W^E := W \setminus W^D$ a set of *end pieces*.

Now let $N = \{1, \ldots, n\}$ be the set of players for an information protocol Π. *Player assignment* is a function $\pi : W \to 2^N$ that $\pi(w)$ assigns a single player for any $w \in W^D$ and N for any $w \in W^E$. Player i's payoff assignment is given as a function $h_i : W^E \to \mathbb{R}$ for all $i \in N$. We then complete the definition of *an information protocol* as a quintuple $\Pi = (W, A, \prec, (\pi, N), (h_i)_{i \in N})$.

To describe a player's personal view and an objective situation with information protocols, we require two basic axioms and three non-basic axioms. To stipulate the basic axioms, we define a *subsequence* of a sequence in $\bigcup_{m=0}^{\infty}((W \times A)^m \times W)$. We say that $\langle (w_1, a_1), \ldots, (w_m, a_m), w_{m+1} \rangle$ is a subsequence of $\langle (v_1, b_1), \ldots, (v_k, b_k), v_{k+1} \rangle$ iff $[(w_1, a_1), \ldots, (w_m, a_m), (w_{m+1}, a)]$ is a subsequence of $[(v_1, b_1), \ldots, (v_k, b_k), (v_{k+1}, b)]$ for some a and b. We use the notation $\langle \xi, w_{m+1} \rangle \sqsubseteq \langle \zeta, v_{k+1} \rangle$ to state that $\langle \xi, w_{m+1} \rangle$ is a subsequence of $\langle \zeta, v_{k+1} \rangle$. A *supersequence* is defined likewise. A sequence $\langle \xi, w \rangle$ is *maxial* iff there is no proper supersequence in $\prec$. A *position* $\langle \xi, v \rangle$ is an initial segment of some maximal sequence. The set of positions is denoted Ξ.

We now state the basic axioms.

Axiom B1 If $\langle \xi, w \rangle \in \prec$ and $\langle \zeta, v \rangle \sqsubseteq \langle \xi, w \rangle$, then $\langle \zeta, v \rangle \in \prec$.
Axiom B2 If $\langle \xi, w \rangle \in \prec$ and $w \in W^D$, then there are $a \in A$ and $v \in W$ such that $\langle \xi, (w, a), v \rangle \in \prec$.

Axiom B1 requires that $\prec$ is closed under a subsequence relation, while axiom B2 states that a sequence ending with a decision piece can be extended to a longer sequence in $\prec$. When an information protocol satisfies these basic axioms, we call it a *basic protocol*.

As we shall see in the next subsection, basic protocols are used to describe player's personal views. On the other hand, to describe an objective situation, inductive game theory introduces the concept of *full protocol*, which is a restricted form of basic protocols. For the detailed definition of full protocols, see [14].

2.2 Players' memories and inductively derived views

The central idea behind inductive game theory is the consideration of a player's memories, from which he/she derives a personal view of the objective situation.

Kaneko and Kline [14] formulated a player's memories in terms of a *memory function*, which maps each objective history of his/her play to the recollection in the player's mind.

Definition 1 (Memory function). Let Π be a basic protocol and Ξ^i the set of player i's positions in Π; i.e., $\Xi^i = \{\langle \xi, w\rangle \in \Xi \mid i \in \pi(w)\}$. A memory function m_i for player i is a function mapping each element in a set D_i with $\Xi^i \subseteq D_i \subseteq \Xi$ to a finite sequence $\langle \zeta, v\rangle = \langle (v_1, b_1), \ldots, (v_m, b_m), v\rangle$ satisifying two conditions: (1) $v = w$; and (2) $m \geq 0$ and $v_t \in W$, $b_t \in A_{v_t}$ for all $t = 1, \ldots, m$.

Here, the set D_i, called a *domain of accumulation* (or a *domain*, for short) means the objective description of player i's accumulated experiences. Condition 1 guarantees that the latest information piece is what player i receives at the current position. Condition 2 is a minimal requirement to represent players' memories with information protocols. The above memory functions can represent players' forgetfulness or incorrect recollections. We call each sequence $\langle \zeta, v\rangle$ given by m_i *a memory thread*, and each component (v_t, b_t) or v_{m+1} *a memory knot.* The memory function of player i takes all of player i's perceptions of the objective world; i.e., each player recognizes an objective world only through his memory function.

We now present a basic framework for an objective world.

Definition 2 (Objective situation). An *objective situation* is a pair (Π^o, m^o) such that $\Pi^o = (W^o, A^o, \prec^o, (\pi^0, N), h^o)$ is a full protocol with $h^o = (h^o_1, \ldots, h^o_n)$ and $\mathsf{m}^o = (\mathsf{m}^o_1, \ldots, \mathsf{m}^o_n)$ is an n-tuple of memory functions in Π^o.

On the other hand, a player's personal view derived from his/her memories is formulated in terms of the memory function. Let Ξ^o be a set of positions in an objective situation (Π^o, m^o_i). For a domain D_i with $\Xi^i \subseteq D_i \subseteq \Xi^o$, the *memory kit* T_{D_i}, which describes the accumulated experiences in the mind of player i, is defined as $T_{D_i} := \{\mathsf{m}^o_i\langle \xi, w\rangle \mid \langle \xi, w\rangle \in D_i\}$. *Basic experiences* for player i is the set of all subsequences of every sequence in T_{D_i}, denoted ΔT_{D_i}. In terms of the basic experiences, we define a player's *inductively derived view* (or *i.d.view*, for short) as follows [4].

Definition 3 (I.d.view). Suppose the objective situation (Π^o, m^o) is fixed. A pair (Π^i, m^i) of a protocol and a memory function for player i is called an inductively derived view from a memory kit T_{D_i} iff

ID1 $W^i := \{w \in W^o \mid w$ occurs in some sequence in $T_{D_i}\}$, $W^{iD} \subseteq W^{oD}$ and $W^{iE} \subseteq W^{oE}$;
ID2 $A^i_w \subseteq A^o_w$ for each $w \in W^i$;
ID3 $\Delta T_{D_i} \subseteq \prec^i$;

[4] The original definition in [14] requires one additional condition that m^i is a perfect-information memory function (i.e., $\mathsf{m}^i\langle \xi, w\rangle = \langle \xi, w\rangle$). However, in this paper we omit this condition since our formulation does not explicitly treat the classification of memory functions.

ID4 $\pi^i(w) = \pi^o(w)$ if $w \in W^{iD}$ and $\pi^i(w) = N^i$ if $w \in W^{iE}$, where $N^i := \{j \in N^o \mid j \in \pi^i(w)$ for some $w \in W^{iD}\}$;
ID5 $h^i(w) = h^o_i(w)$ for all $w \in W^{iE}$.

In closing this section, to help readers understand the definition of inductive game theory, we present a simple example called the *absent-minded driver game* [14].

Example 1. Consider the one-player protocol (Π^o, m^o_1) described by the upper figure in Fig 1 (C). Now suppose that player 1 plays the game Π^o three times and

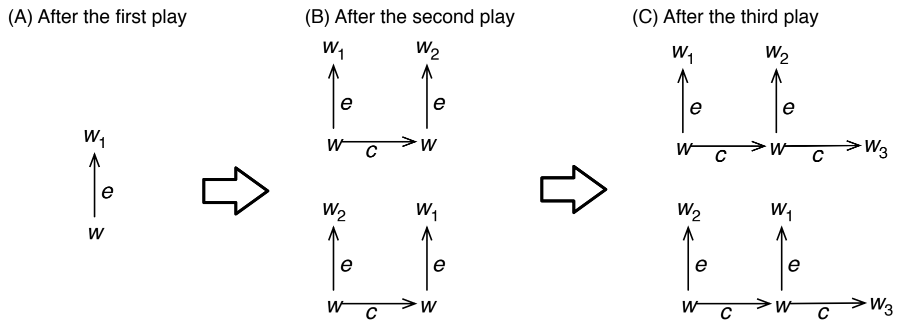

Fig. 1. Player 1's i.d.views in absent-minded driver game

experiences the sequences leading to w_1, w_2, and w_3 in this order. After each of the plays, the objective history of his/her behavior is described as the following sequence: $D^0_1 := \phi$, $D^1_1 := \{\langle (w,e), w_1\rangle\}$, $D^2_1 := D^1_1 \cup \{\langle (w,c),(w,e),w_2\rangle\}$, $D^3_1 := D^2_1 \cup \{\langle (w,c),(w,c),w_3\rangle\}$.

Let us consider the memory functions m^{R1} describing that player 1 can recall only the latest memory knots within his experiences; that is, m^{R1} is defined as $\mathsf{m}^{R1}\langle w\rangle = \langle w\rangle$, $\mathsf{m}^{R1}\langle (w,c),w\rangle = \langle (w,c),w\rangle$, $\mathsf{m}^{R1}\langle (w,e),w_1\rangle = \langle (w,e),w_1\rangle$, $\mathsf{m}^{R1}\langle (w,c),(w,e),w_2\rangle = \langle (w,e),w_2\rangle$, $\mathsf{m}^{R1}\langle (w,c),(w,c),w_3\rangle = \langle (w,c),w_3\rangle$.

Thus the corresponding sequence of basic experiences is $\Delta T^0_{D_1} := \phi$, $\Delta T^1_{D_1} := \{\langle w\rangle, \langle (w,e),w_1\rangle\}$, $\Delta T^2_{D_1} := \Delta T^1_{D_1} \cup \{\langle (w,c),w\rangle, \langle (w,e),w_2\rangle\}$, $\Delta T^3_{D_1} := \Delta T^2_{D_1} \cup \{\langle (w,c),w_3\rangle\}$, and some of the player's i.d.views obtained from the basic experiences are depicted as figures (A)–(C) in Fig 1 .

As the example shows, we may consider (possibly an infinite number of) multiple i.d.views that differ from the objective situation, examples of which are given as the lower figures of (B) and (C) in Fig 1, respectively.

3 Logical Formulation of Inductive Game Theory

In this section, we first briefly overview the AGM theory of belief revision [1] then presents our logical formulation of inductive game theory. We here introduce a minimal setting which formulates a player's beliefs about experienced sequences. An extension, including the concepts of player's strategy and payoff, shall be considered in the next section.

3.1 AGM theory of belief revision

We suppose a propositional language $\mathcal{L}$ over a finite alphabet Σ of propositional atomic variables $s, t, \ldots, s_1, s_2, \ldots$ with the usual sentential connectives ($\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$). Propositional formulas are denoted by $\varphi, \psi, \ldots, \varphi_1, \varphi_2, \ldots$.

As the syntax for $\mathcal{L}$, we suppose the usual inference system of classical propositional logic. Consequence relation $\vdash$ is defined by this logic. The set of all logical consequences of a set $\Gamma \subseteq \mathcal{L}$ (i.e., the set $\{\varphi \mid \Gamma \vdash \varphi\}$) is denoted $Cn(\Gamma)$. Deductively closed sets of propositional formulas, i.e. $K = Cn(K)$, are denoted $K, K', \ldots$ and are called *belief sets*.

As the semantics for $\mathcal{L}$, we suppose the usual truth assignment and models of propositional logic. A *truth assignment* is a function $\sigma : \Sigma \rightarrow \{1, 0\}$. A truth assignment σ is called a *model* of a proposition φ if σ satisfies φ in the classical sense. A model of a set of propositions Γ is a truth assignment σ that satisfies all $\varphi \in \Gamma$.

The AGM theory of belief revision considers three types of operations on belief sets: *expansion* $\dot{+}$, *contraction* $\dot{-}$ and *revision* $\star$. For a belief set K and a formula φ, expansion operation is defined as $K \dot{+} \varphi := Cn(K \cup \{\varphi\})$. Contraction operation is assumed to satisfy the following postulates.

P1 $K \dot{-} \varphi$ is a theory;
P2 $K \dot{-} \varphi \subseteq K$;
P3 If $\varphi \notin K$ then $K \dot{-} \varphi = K$;
P4 If $\nvdash \varphi$ then $\varphi \notin K \dot{-} \varphi$;
P5 If $\varphi \in K$ then $K \subseteq (K \dot{-} \varphi) \dot{+} \varphi$;
P6 If $\vdash \varphi \leftrightarrow \psi$ then $K \dot{-} \varphi = K \dot{-} \psi$;
P7 $(K \dot{-} \varphi) \cap (K \dot{-} \psi) \subseteq K \dot{-} (\varphi \wedge \psi)$;
P8 If $\varphi \notin K \dot{-} (\varphi \wedge \psi)$ then $K \dot{-} (\varphi \wedge \psi) \subseteq K \dot{-} \varphi$.

Revision operation can be defined by *Levy identity*, i.e. $K \star \varphi := (K \dot{-} \neg\varphi) \dot{+} \varphi$ (cf. Chapter 1 in [9]).

3.2 Logical formulation

As explained in the previous section, in inductive game theory, a unique objective situation is described by a pair (Π^o, m^o) of full protocol and memory functions, while player's accumulated memories are described by basic experiences ΔT_{D_i}. Thus, an accumulating process of memories for player i can be represented by a sequence $\Delta T^0_{D_i}, \Delta T^1_{D_i}, \ldots$ which is obtained from a sequence $D^0_i, D^1_i, \ldots$ of his/her objective histories.

Our logical formulation of inductive game theory is given along with the following steps. We first fix the propositional language $\mathcal{L}$ whose atomic formulas are used to denote statements of the form "a sequence $\langle \xi, w\rangle$ may occur". Thus, relations over sequences, such as negation and causality, can be represented by (compound) formulas. We next define a player's belief state as a belief set, which consists of beliefs about experienced sequences and some ex-ante beliefs about causality relations over sequences. For a given sequence $\Delta T^0_{D_i}, \Delta T^1_{D_i}, \ldots$ of basic experiences, the corresponding sequence $K^0_i, K^1_i, \ldots$ of belief sets is defined by the revision operator in AGM theory. On the other hand, as explained in the previous subsection, we can obtain (possibly multiple) i.d.views from a given basic experiences $\Delta T^j_{D_i}$. In our framework, an i.d.view (Π^i, m^i) is defined by an assignment function σ such that $\sigma(\langle \xi, w\rangle) = 1$ if and only if $\langle \xi, w\rangle \in \prec^i$. Finally, we show that any assignment σ obtained from ΔT_{D_i} is a model for belief set K_i obtained from ΔT_{D_i}. This means that a player's i.d.view can be regarded as a personal view constructed from the current belief state.

We here fix the objective situation $(\Pi^o, \mathsf{m}^o) = (W^o, A^o, \prec^o, (\pi^o, N^o), h^o, \mathsf{m}^o)$, sequence $D^0_i, D^1_i, \ldots$ of domains, and the corresponding sequences $\Delta T^0_{D_i}, \Delta T^1_{D_i}, \ldots$ of basic experiences. For convention, we consider the initial domain $D^0_i := \phi$.

The language $\mathcal{L}$ is defined as follows.

Definition 4 (Language). The propositional language $\mathcal{L}$ is defined by considering an alphabet Σ that provides a sufficient number of atomic propositions to denote any sequence in $\prec^o$. That is, we suppose that there is a bijective mapping $* : \Sigma \to \prec^o$.

To simplify our discussion, throughout we omit sequences whose length is 1; i.e., sequences of the form $\langle w\rangle$.

Next, for $\Delta T^0_{D_i}, \Delta T^1_{D_2}, \ldots$, we define the corresponding sequence of the player's belief sets.

Definition 5 (Belief sets). Suppose that $\Delta T^{j+1}_{D_i}$ is obtained from $\Delta T^j_{D_i}$ by adding a sequence $\langle \zeta, v\rangle \in \prec^o$ for each $j = 0, 1, \ldots$. That is, $\Delta T^{j+1}_{D_i} := \Delta T^j_{D_i} \cup \{\langle \zeta, v\rangle\}$. (If $\Delta T^{j+1}_{D_i} \setminus \Delta T^j_{D_i}$ includes multiple sequences, the following rules are applied one by one to each of the sequences.) The corresponding sequence of the player i's belief sets $K^0_i, K^1_i, \ldots$ is inductively defined as follows.

- For each $j = 0, 1, \ldots$, K^j_i is defined to be the deductive closure of the union of two kinds of sets $K^j_{i,EX}$ (called *experienced beliefs*) and $K^j_{i,CA}$ (called *beliefs about causality*), and K^0_i is fixed as
  R1-1 $K^0_{i,EX} := \{\neg s \mid s \in \Sigma$ and the length of s^* is $2\}$,
  R1-2 $K^0_{i,CA} := \{s \to t \mid s, t \in \Sigma$ and $t^* \sqsubseteq s^*\}$.
- For given $K^j_i = Cn(K^j_{i,EX} \cup K^j_{i,CA})$, the belief set K^{j+1}_i is defined by
  R2-1 $K^{j+1}_{i,EX} := K^j_{i,EX} \setminus \{\neg t \mid t \in \Sigma,$ and $s^* \sqsubseteq \langle \zeta, v\rangle\} \cup \{s \mid s^* = \langle \zeta, v\rangle\}$
  R2-2 $K^{j+1}_{i,CA} := K^j_{i,CA}$.

Intuitively, R1-1 represents the player's initial belief in the impossibility of any elementary sequence. R1-2 and R2-2 represent the persistent belief about causality; i.e., if a sequence may occur then its subsequence also may occur. R2-1 represents the belief that any experienced sequence may occur.

For this formulation, the following theorem holds.

Theorem 1. *For a given objective situation* (Π^o, m^o) *and sequence of basic experiences* $\Delta T^0_{D_i}, \Delta T^1_{D_i}, \ldots$ *obtained from a sequence of domains* $D^0_i, D^1_i, \ldots$*, the corresponding sequence* $K^0_i, K^1_i, \ldots$ *can be characterized by revision operation* $\star$ *in AGM theory.*

Proof. By Levy identity, it is enough to show that R2-1 in Definition 5 can be characterized by contraction and expansion operations. That is, for any $s \in \Sigma$ with $s^* = \langle \xi, w \rangle$, if we define the operation of R2-1 as $K^{j+1}_i := (K^j_i \dot{-} \neg s) \dot{+} s$, then $\dot{-}$ and $\dot{+}$ satisfy postulates P1–P8 and the condition that $(K^j_i \dot{+} s) = Cn(K^j_i \cup \{s\})$, respectively. Clearly, operation $\dot{+}$ is an expansion operator in AGM theory. We here only consider the case for P3 of $\dot{-}$, since the other cases are trivial or shown by similar argument. If $\neg s \notin K^j_i$, then $\neg t \notin K^j_i$ for any t with $t^* \sqsubseteq s^*$ because $s \rightarrow t \in K^j_i$. Therefore, by definition R2-1, $K^j_i \dot{-} \neg s = K^j_i$. □

As a corollary of this theorem, any belief set appearing in a sequence $K^0_i, K^1_i, \ldots$ is consistent. This indicates that our logic formulates a revision process where the player always constructs consistent belief state from experiences.

Finally, for given basic experiences $\Delta T^j_{D_i}$ $(j = 0, 1, \ldots)$, we define the corresponding assignment function as follows.

Definition 6 (Assignment function). For each $\Delta T^j_{D_i}$ $(j = 0, 1, \ldots)$, the corresponding assignment σ^j_i is an assignment satisfying the following conditions.

- **A1** For any $s, t \in \Sigma$ with $t^* \sqsubseteq s^*$, if $\sigma^j_i(s) = 1$ then $\sigma^j_i(t) = 1$.
- **A2** For any $s \in \Sigma$, if $\sigma^j_i(s) = 1$ and $s^* \in W^D$ then there exists $t \in \Sigma$ such that $s^* \sqsubseteq t^*$ and $\sigma^j_i(t) = 1$.
- **A3** For any $s \in \Sigma$ such that the length of s^* is 2, if $s \not\sqsubseteq \langle \xi, w \rangle$ for all $\langle \xi, w \rangle \in \Delta T^j_{D_i}$ then $\sigma^j_i(s) = 0$.
- **A4** For any $s \in \Sigma$, if $s^* \in \Delta T^j_{D_i}$ then $\sigma^j_i(s) = 1$.
- **A5** For any s^* with $\sigma(s) = 1$ satisfies conditions ID1, ID2, and ID4 in the definition of i.d.view (i.e., Definition 3).

Intuitively, A1 and A2 respectively correspond to the basic axioms B1 and B2. A3 means that a sequence consisting of any unexperienced sequence does not appear in the i.d.view. A4 represents condition ID3 in the definition of i.d.view. A5 stipulates that the set of sequences determined by σ^j_i satisfies all the conditions except for ID3 in the definition of i.d.view. By A5, we restrict our attention to the assignments which can be regarded as an i.d.view.

By condition A4 in Definition 6, the following theorem holds.

Theorem 2. *Suppose that an objective situation* (Π^o, m^o) *and a sequence of domains* $D_i^0, D_i^1, \ldots$ *in* (Π^o, m^o) *are given. For* $j = 0, 1, \ldots$*, let* $\Delta T^j_{D_i}$ *be basic experiences and* K_i^j *be the belief set obtained from* $\Delta T^j_{D_i}$*. If* σ_i^j *is an assignment function obtained from* $\Delta T^j_{D_i}$*, then* σ_i^j *is a model for* K_i^j*.*

In closing this section, we demonstrate a revision process for player's beliefs in terms of our logic in the case of the absent-minded driver game presented in Example 1.

Example 2. Consider the protocol (Π^o, m_1^o) described as the left figure in Fig 1. Propositional atoms and the bijective mapping $*$ from Σ to $\prec^o$ are fixed as

$$\begin{array}{ll} s_1^* = \langle (w,e), w_1 \rangle, & s_5^* = \langle (w,e), w_2 \rangle, \\ s_2^* = \langle (w,c), (w,e), w_2 \rangle, & s_6^* = \langle (w,c), w_3 \rangle, \\ s_3^* = \langle (w,c), (w,c), w_3 \rangle, & s_7^* = \langle (w,c), (w,e), w_1 \rangle. \\ s_4^* = \langle (w,c), w \rangle, & \end{array}$$

In fact, infinitely many sequences other than those in the list above can be considered, but for simplicity, we here focus attention on sequences whose length is less than 3. Let us consider the situation that player 1 repeatedly plays the game and accumulates experiences. Suppose that the process of accumulation is $D^0, \ldots, D^3$, such that $D^0 := \phi$, and $D^j := D^{j-1} \cup \{s_j^*\}$ for each j =1, 2, and 3. If the player's memory function is assumed to be m^{R1}, the corresponding sequence of basic experiences $\Delta T^0_D, \ldots, \Delta T^3_D$ is ϕ, $\{s_1^*\}$, $\{s_1^*, s_4^*, s_5^*\}$, $\{s_1^*, s_4^*, s_5^*, s_6^*\}$. For this sequence, the revision process of the belief sets is

$$\begin{array}{l} K^0_{Ex} := \{\neg s_1, \neg s_4, \neg s_5, \neg s_6\}, \\ K^0_{CA} := \{s_2 \rightarrow s_4 \wedge s_5, s_3 \rightarrow s_4 \wedge s_6, s_7 \rightarrow s_1 \wedge s_4\}, \\ K^1_{Ex} := \{s_1, \neg s_4, \neg s_5, \neg s_6\}, \\ K^2_{Ex} := \{s_1, s_4, s_5, \neg s_6\}, \\ K^3_{Ex} := \{s_1, s_4, s_5, s_6\}, \end{array}$$

where $K^j_{CA} = K^0_{CA}$ for $j = 1, 2, 3$. Note that for ΔT^1_D, both $\neg s_2$ and $\neg s_3$ are derivable, while for ΔT^2_D and for ΔT^3_D, none of s_2, s_3, s_7 or their negations can be derived. On the other hand, the corresponding sequence of assignment functions (denoted σ^j) is defined as

$$\begin{array}{l} \sigma^0(s_i) = 0 \text{ for } i = 1, \ldots, 7, \\ \sigma^1(s_1) = 1,\ \sigma(s_i) = 0 \text{ for } i = 2, \ldots, 7, \\ \sigma^2(s_1) = \sigma^2(s_4) = \sigma^2(s_5) = 1,\ \sigma^2(s_3) = \sigma^2(s_6) = 0, \\ \sigma^3(s_1) = \sigma(s_3) = \sigma^3(s_4) = \sigma^3(s_5) = \sigma^3(s_6) = 1. \end{array}$$

For both σ^2 and σ^3, any value is possible for s_2 and s_7. This results in multiple i.d.views. For example, in the case that $\sigma^2(s_2) = 1$ and $\sigma^2(s_7) = 0$, the corresponding i.d.view is the upper figure in Fig 1 (B), while in the case that $\sigma^2(s_2) = 0$ and $\sigma^2(s_7) = 0$, it is the lower figure in Fig 1 (C).

4 Application to Festival Games

In this section, we apply our logic to a specific inductive game, the *festival game*, which was developed by Kaneko and Matsui [15] and Kaneko and Mitra [16]. The main objective in this section is to formulate the festival games in terms of our logic, thereby showing the revision process of players' beliefs. Especially, we focus attention on the process where players' prejudice is exposed as discriminatory behaviors caused by their experiences. For this objective, we first introduce the definition of festival games following [15] and then present the formulation of festival games in terms of our logic.

4.1 Festival games

The festival game considers that each of the players belongs to an ethnic group, and the player's festival location is chosen. Each player then decides his attitude, *friendly* or *unfriendly*, after observing ethnic groups at his location. Using this framework, Kaneko and Matsui [15] studied how prejudices, as a fallacious image of ethnic groups, arise from players' experiences and how discrimination arises as an unfriendly attitude.

Let us define the objective situation $(\Pi^o, \mathsf{m}^o) = (W^o, A^o, \prec^o, (\pi^o, N), h^o, \mathsf{m}^o)$ for festival games. We consider the number of ethnic groups ϵ. The set $N = \{1, \ldots, n\}$ of players is then partitioned into ethnic groups, $e_1, \ldots, e_\epsilon$ with $|N_j| \geq 2$ for $j = 1, \ldots, \epsilon$. The set A^o of actions consists of the choices of festival locations, $\{\mathsf{f}_1, \ldots, \mathsf{f}_m\}$, and the attitude, *friendly* or *unfriendly*, denoted by frd and unfrd, respectively. That is, $A^o = \{\mathsf{f}_1, \ldots, \mathsf{f}_m\} \cup \{\mathsf{frd}, \mathsf{unfrd}\}$.

The festival game consists of two stages: the first stage of *choosing festival locations* and the second stage of *acting in festivals*. Since the first stage is indeed a simultaneous decision stage, we assume that player i chooses his/her festival location at the position

$$\langle \xi^1, w_i^1 \rangle = \langle (w_1^1, l_1), \ldots, (w_{i-1}^1, l_n), w_i^2 \rangle$$

with $i \leq n$, and $l_1, \ldots, l_{i-1} \in \{\mathsf{f}_1, \ldots, \mathsf{f}_m\}$. At that position, the player obtains a memory thread $\mathsf{m}_i\langle \xi^1, w_i^1 \rangle = \langle w_i^1 \rangle$. This means that the players decide their festival locations in order of their indices, but each cannot observe the choices of the other players before making the decision.

In the second stage, player i's position is given as

$$\langle \xi_i^2, w_i^2 \rangle = \langle (w_1^1, l_1), \ldots, (w_n^1, l_n), (w_1^2, a_1), \ldots, (w_{i-1}^2, a_{i-1}), w_i^2 \rangle$$

with $i \leq n$, and $a_1, \ldots, a_{i-1} \in \{\mathsf{frd}, \mathsf{unfrd}\}$. For the position $\langle \xi_i^2, w_i^2 \rangle$, player i obtains a memory thread as $\mathsf{m}_i\langle \xi_i^2, w_i^2 \rangle = \langle (w_i^1, l_i), w_i^2 \rangle$. Unlike the setting of Kaneko and Matsui [15], information piece w_i^2 conveys to player i information about the number of participants in festival f_j with $l_i = \mathsf{f}_j$ for each ethnicity group. Formally, this information (denoted E_j) is defined by $E_j = (x_1, x_2, \ldots, x_\epsilon)_j$, where x_k indicates the number of participants in ethnic group

$k = 1, \dots, \epsilon$. We use notation $w_i^2 = E_j$ to denote that player i receives ethnic configuration E_j.

Finally, player i's position after the second stage is given as

$$\langle \xi^3, w_j^3 \rangle = \langle (w_1^1, l_1), \dots, (w_n^1, l_n), (w_1^2, a_1), \dots, (w_n^2, a_n), w_j^3 \rangle$$

with $w_j^3 \in W^E$. Here, $j = 1, \dots, m^{|N|} \cdot 2^{|N|}$ since the resulting position is determined by the players' choices of locations (among m alternatives) and attitude (between frd and unfrd). For the position $\langle \xi^3, w_j^3 \rangle$, player i obtains a memory thread as $\mathsf{m}_i\langle \xi^3, w_j^3 \rangle = \langle (w_i^1, l_i), (w_i^2, a_i), w_j^3 \rangle$. The information piece w_j^3 provides the numerical payoffs.

According to the above setting, $\prec^o$ consists of $m^{|N|} \cdot 2^{|N|}$ maximal sequences of the form $\langle \xi^3, w_j^3 \rangle$ and their subsequences. The set W^o is determined by $\prec^o$, and π is defined such that $\pi(w_i^j) = \{i\}$ for any $w_i^j \in W^{oD}$ and $\pi(w_j^3) = N$ for any $w_j^3 \in W^E$.

For determination of players' payoffs $h^o = (h_1^o, \dots, h_n^o)$, we first introduce players' *strategies*. A strategy of player i (denoted st_i) is a pair (l_i, r_i) of choices for the first and second stages. Here, r_i is a function mapping $\{\mathsf{f}_1, \dots, \mathsf{f}_m\} \times \mathsf{E}$ to $\{\mathsf{frd}, \mathsf{unfrd}\}$, where E is the collection of all possible ethnicity configurations. We here note that every player may change strategy in the recurrent plays of the game $(\mathit{\Pi}^o, \mathsf{m}^o)$. To indicate strategy st_i' that is deviated from st_i by replacing the choice of location l_i with l_i', we use expression $st_i[l_i/l_i']$. The replacement of a player's attitude is indicated analogously.

Let S_i be the set of strategies for player i. For a strategy profile $st \in S_1 \times \cdots \times S_n$, the player's payoff is determined by his/her attitude and the *mood* of the location he chose. The mood of festival f_k with $\mathsf{f}_k = l_i$ for player i (denoted μ_i) is given by the number of friendly people at l_i other than player i; that is, $\mu_i(l, r) = \sum_{l_j = l_i, j \neq i} r_j(l_j, E_j)$, where $l = (l_1, \dots, l_n)$, $r = (r_1, \dots, r_n)$, and frd and unfrd are interpreted as 1 and 0, respectively. We then define the payoff function of player i as $h_i(l_i, r_i) = r_i(l_i, E_i) \cdot \mu(l, r)$.

4.2 Logical formulation of festival games

To give a logical formulation of the above example, we here extend our logic introduced in the previous section to capture the concepts of strategies and payoffs. We first introduce the language as follows:

Definition 7 (Language for festival games). The language $\mathcal{L}$ is defined by fixing the alphabet Σ to denote:

- occurrence of any sequence in $\prec^o$,
- statements of the form "player i chooses f_j as his festival location," denoted $l_i = \mathsf{f}_j$,
- statements of the form "the ethnicity configuration at festival f_j is E_j" (denoted E_j),
- statements of the form "player i chooses frd (unfrd) as his attitude in the festival he chose," denoted $a_i = \mathsf{frd}$ ($a_i = \mathsf{unfrd}$, respectively),

– statements of the form "player i's payoff is x", denoted $\mathsf{payoff}_i = x$.

For readability, we introduce equational expressions instead of single characters to denote these propositional atoms.

In terms of this language, we next give the definition of the sequence of player i's belief sets, $K_i^0, K_i^1, \ldots$. The idea behind our definition is as follows. In addition to Definition 5, we also consider the player's belief about his/her current strategy, experienced ethnicity configurations. Strategy $st_i = (l_i, r_i)$ with $l_i = \mathsf{f}_k$, $r_i(l_i, E_j) = \mathsf{frd}/\mathsf{unfrd}$ is described as the set of formulas $l_i = \mathsf{f}_k \wedge (l_i = \mathsf{f}_k \wedge E_k) \rightarrow \{\mathsf{frd}, \mathsf{unfrd}\}$ for all $E_k \in \mathsf{E}$, where the expression $\{\mathsf{frd}, \mathsf{unfrd}\}$ denotes one of frd and unfrd. The latest decisions of location and attitude, as well as the resulting payoff, are uniquely determined in the current belief set.

Definition 8 (Belief sets in festival games). Suppose that $\Delta T_{D_i}^{j+1}$ is obtained by adding a sequence $\langle \zeta, v \rangle \in \prec^o$ for each $j = 0, 1, \ldots$. That is, $\Delta T_{D_i}^{j+1} := \Delta T_{D_i}^{j} \cup \{\langle \zeta, v \rangle\}$. Let $st_i^j = (\mathsf{f}_j, r_i)$ be a strategy of player i at the j-th play of the game. (For convention, we consider $st_i^0 := st_i^1$.) The corresponding sequence of his/her belief sets $K_i^1, K_i^2, \ldots$ is defined as follows.

– For each $j = 0, 1, \ldots$, K_i^j is defined to be the deductive closure of the union of four kinds of sets $K_{i,EX}^j$, $K_{i,CA}^j$, $K_{i,ST}^j$, $K_{i,AUX}^j$ (where the third and fourth are respectively called *beliefs about strategy* and *auxiliary beliefs*), and K_i^0 is fixed as:

R1-1 $K_{i,EX}^0 := \{\neg s \mid s \in \Sigma \text{ and the length of } s^* \text{ is } 2\}$,

R1-2 $K_{i,CA}^0 := \{s \rightarrow t \mid s, t \in \Sigma \text{ and } t^* \sqsubseteq s^*\}$
$\cup\{l_i = \mathsf{f}_j \leftrightarrow \neg l_i = \mathsf{f}_k \mid k \neq j,\ j \geq l,\ k \geq m\}$
$\cup\{E_j \leftrightarrow \neg E_k \mid E_j \neq E_k,\ E_j, E_k \in \mathsf{E}\}$
$\cup\{a_i = \{\mathsf{frd}, \mathsf{unfrd}\} \leftrightarrow \neg a_i = \{\mathsf{unfrd}, \mathsf{frd}\}\}$
$\cup\{l_i = \mathsf{f}_k \wedge E_h \rightarrow \{\mathsf{frd}, \mathsf{unfrd}\} \leftrightarrow \neg(l_i = \mathsf{f}_k \wedge E_h \rightarrow \{\mathsf{frd}, \mathsf{unfrd}\})\}$
$\cup\{\mathsf{payoff}_i = x \rightarrow \neg \mathsf{payoff}_i = x' \mid\ x \neq x'\}$,

R1-3 $K_{i,ST}^0 := \{\mathsf{f}_j\} \cup \{\mathsf{f}_j \wedge E_j \rightarrow a_i = \{\mathsf{frd}, \mathsf{unfrd}\} \mid E_j \in \mathsf{E},\ \{\mathsf{frd}, \mathsf{unfrd}\}$ is determined by r_i^0 in $st_i^0\}$,

R1-4 $K_{i,AUX}^0 := \{(s \rightarrow \mathsf{payoff}_i = x) \wedge (s' \rightarrow \mathsf{payoff}_i = x') \rightarrow \neg s \mid\ x < x'\}$.

– For given $K_i^j = Cn(K_{i,EX}^j \cup K_{i,CA}^j \cup K_{i,ST}^j \cup K_{i,AUX}^j)$ the belief set K_i^{j+1} is defined by

R2-1 (1) $K_i^{j+1} := \Theta \setminus \{\neg s \mid s \in \Sigma \text{ and } s^* \sqsubseteq \langle \zeta, v \rangle\} \cup \{s \mid s^* = \langle \zeta, v \rangle\}$, where Θ is obtained by the following rules R2-1 (2)–(4):

R2-1 (2) if $\langle \zeta, v \rangle = \langle (w_i^1, \mathsf{f}_k), w_i^2 \rangle$, then $\Theta := K_{i,EX}^j \setminus \{\neg l_i = \mathsf{f}_k\} \cup \{l_i = \mathsf{f}_k\}$,

R2-1 (3) if $\langle \zeta, v \rangle = \langle (w_i^1, \mathsf{f}_k), w_i^2 \rangle$ where $w_i^2 = E_k$, then $\Theta := K_{i,EX}^j \setminus \{\neg E_k\} \cup \{E_k\}$,

R2-1 (4) if $\langle \zeta, v \rangle = \langle (w_i^1, \mathsf{f}_k), (w_i^2, a_i), w^3 \rangle$ where $w_i^2 = E_k$ and $h_i(w^3) = x$, then $\Theta := K_{i,EX}^j \setminus \{\neg\varphi\} \cup \{\varphi\}$ where $\varphi \equiv l_i = \mathsf{f}_k \wedge (l_i = \mathsf{f}_k \wedge E_k \rightarrow a_i = \{\mathsf{frd}, \mathsf{unfrd}\}) \rightarrow \mathsf{payoff}_i = x\}$,

R2-2 $K_{i,CA}^{j+1} := K_{i,CA}^j$,

R2-3 (1) if $st_i^{j+1} = st_i^j[\mathsf{f}_k/\mathsf{f}_h]$, then $K_{i,ST}^{j+1} := K_{i,ST}^j \setminus \{\neg l_i = \mathsf{f}_h\} \cup \{l_i = \mathsf{f}_h\}$,

R2-3 (2) if $st_i^{j+1} = st_i^j[r_i/r_i']$, then $K_{i,ST}^{j+1} := K_{i,ST}^j \setminus \{\neg\rho_i\} \cup \{\rho_i\}$ where ρ_i is formula of the form $\bigwedge_{k=1,\ldots,m,\ E\in\mathsf{E}}(l_i = \mathsf{f}_k \wedge E \to a_i = \{\mathsf{frd}, \mathsf{unfrd}\})$ determined by r_i,

R2-4 $K_{i,AUX}^{j+1} := K_{i,AUX}^j$.

Intuitively, R1-1 and R2-1 revise the beliefs about experienced sequences, while R1-2 and R2-2 mean the beliefs about causality. These are essentially the same as Definition 5, but R1-2 is extended to maintain the uniqueness of choices of locations, attitude, the resulting payoff, and ethncity configurations. R1-3 and R2-3 revise the current strategy. Finally, R1-4 and R2-4 revise the player's strategy if there is another strategy which improves the latest payoff.

We here note that for all the operations in the above definition except for the belief revision for experienced sequences, every removed formula is identical to the negation of corresponding added formula. Thus, if we consider the subtraction ($\setminus$) and the addition ($\cup$) to be $\dot{-}$ and $\dot{+}$, respectively, these operations clearly satisfy postulates P1–P8 and the condition that $K_i^j \dot{+} \varphi = Cn(K_i^j \cup \{\varphi\})$. Therefore, by this fact and Theorem 1, we can prove the following theorem.

Theorem 3. *For a given objective situation* (Π^o, m^o) *in a festival game, sequence of basic experiences* $\Delta T_{D_i}^0, \Delta T_{D_i}^1, \ldots$*, and sequence of strategies* $st_i^0, st_i^1, \ldots$*, the corresponding sequence* $K_i^0, K_i^1, \ldots$ *defined by Definition 8 can be realized by revision operation* $\star$ *in AGM theory.*

Moreover, by our construction of the initial belief set, K_i^0, and by this theorem, every belief set appearing in the sequence is guaranteed to be consistent.

We finally define assignment functions that are obtained from given basic experiences and a player's strategy.

Definition 9 (Assignment functions). For each $\Delta T_{D_i}^j$ and the strategy st_i^j of the j-th play of the game for player i, the corresponding assignment σ_i^j is an assignment function satisfying conditions A1–A5 in Definition 6 and the following.

A6 $\sigma(l_i = \mathsf{f}_k) = 1$ iff (w_i^1, f_k) appears in the sequence $\langle \xi^{j,3}, w^{j,3} \rangle$.

A7 $\sigma(E_k) = 1$ iff $\mathsf{f}_1, \ldots, \mathsf{f}_m$ appears in the sequence $\langle \xi^{j,3}, w^{j,3} \rangle$ and E_k is the ethnicity configuration derived from $\mathsf{f}_1, \ldots, \mathsf{f}_m$.

A8 $\sigma(a_i = \mathsf{frd}) = 1$ iff $(w_i^{j,2}, \mathsf{frd})$ appears in the sequence $\langle \xi^{j,3}, w^{j,3} \rangle$.

Intuitively, A6 means that player i chooses f_j as his/her location. A7 represents the location configuration. A8 means that player i chooses frd (or unfrd) as the attitude. For this semantics, the following theorem still holds.

Theorem 4. *Suppose that an objective situation* (Π^o, m_i^o) *and a sequence of domains* $D_i^0, D_i^1, \ldots$ *in* (Π^o, m_i^o) *are given. For* $j = 0, 1, \ldots,$ *let* $\Delta T_{D_i}^j$ *be basic experiences,* st_i^j *be the strategy of player i, and* K_i^j *be the belief set obtained from* $\Delta T_{D_i}^j$*. If* σ_i^j *is an assignment function obtained from* $\Delta T_{D_i}^j$*, then* σ_i^j *is a model for* K_i^j*.*

5 Concluding Remarks

This paper provided a dynamic framework to revise players' personal views of their experiences following inductive game theory. In addition, we applied our framework to festival games to explain how prejudices and discrimination emerge. In a subsequent paper, we will investigate festival games within our framework.

We finally comment on our findings. First, our inductive derivation differs from learning theory approaches in the literature such as those of Fudenberg and Levine [8]. Standard learning theories do not focus on the learning of structure, but on the learning of beliefs of opponents' actions. Second, while making use of the framework of inductive game theory developed by Kaneko and Kline and Kaneko and Matsui, our theory focuses on a permanent revision process based on the player's experiences. While the developers of inductive game theory focused on how to construct a player's view consistent with his experiences, we focused on how to change a playerfs personal view when he has a new experience.

Finally, the treatment of experiences in our theory slightly differs from the standard belief revision theory pioneered by Alchourrón, Gärdenfors and Makinson [1] (so-called AGM theory). In contrast to AGM theory, our theory distinguishes between what a player originally believes and what he logically derives from the original belief. This is based on our motivation that the players are not simply a database, but have some logical abilities. This approach is similar to the belief base theory, which distinguishes between the belief base and the consequence.[5] However, the theory of the belief base focuses neither on the new observations as the experiences in our theory nor on their accumulation. Our theory combines the use of accumulated observations and beliefs logically derived from them for decision-making.

In further research, we will investigate (i) how players with different views make decisions in our theory and (ii) the direction of various players' views after repetitive revision. In standard game theory, players face an identical situation and know that even when considering incomplete information games. Our theory is a first step to inquiring whether it is possible to achieve and to analyze misunderstandings pointed out by Kaneko [11]. In a society in which people do not necessarily recognize identical environments, we wonder how people harmonize with each other. This question will drive future research.

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[5] See Chapter 1 in Gärdenfors [9].

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