On Factorization of Analytic Functions and its Verification

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Abstract. An interval method for finding a polynomial factor of an analytic function $f(z)$ is proposed. By using a Samelson-like method recursively, we obtain a sequence of polynomials that converges to a factor $p^*(z)$ of $f(z)$ if an initial approximate factor $p(z)$ is sufficiently close to $p^*(z)$. This method includes some well known iterative formulae, and has a close relation to a rational approximation. According to this factoring method, a fixed point relation for $p^*(z)$ is derived. Based on this relation, we obtain a polynomial with complex interval coefficients that includes $p^*(z)$.

Key words: Factoring method, zeros of analytic function, interval method.

1 Introduction

The purpose of this paper is to present a method for finding a set of polynomials which includes a factor of an analytic function $f(z)$ defined for $|z| < R$, where $R > 0$.

For the determination of multiple or close zeros of $f(z)$, iterative methods usually require large number of iterations, or fail by a jump of an approximation. Factoring methods can find such zeros as a polynomial. The computation of coefficients of a polynomial of which zeros are close is more stable than the determination of locations of close zeros.

Bauer and Samelson [2] have proposed a method to find a zero of a polynomial $f(z)$ by considering Newton's method for $f(z)/q(z)$ at an approximation z_0, where $q(z)$ is a polynomial of degree less than $\deg f$. Jenkins and Traub [10] improved the order of convergence by modifying $q(z)$ in each iteration step. These methods can be regarded as a combination of two iterations for approximations $p(z) = z - z_0$ and $q(z)$. Stewart [17] generalized these methods for the case that the degree of $p(z)$ is arbitrary. When $p(z)$ is quadratic, it contains Bairstow's method [1]. In [18], the relation of this method with qd-algorithm and König's theorem is also considered. A factorization of an analytic function by reducing a problem to a solution of infinite block Toeplitz matrix is proposed in [3].

Grau's method [7] improves approximate factors $p_1(z), \ldots, p_N(z)$ for a polynomial simultaneously. When all the approximate factors are linear, this method is just the Durand-Kerner method [6]. When $N = 2$, Grau's method is equivalent to the method in [17]. This method can be extended to a simultaneous factoring method of arbitrary order of convergence by using a rational Hermite interpolation ([4]).

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For the estimation of initial approximations, global methods to find zeros or poles in a given domain ([5, 12, 19]) are used. When there exist dense clusters of zeros in the domain, the problem to find all the zeros in the domain is numerically extremely ill-conditioned. The location and multiplicity of a cluster of zeros in a certain domain is a stable phenomenon. Clustering methods ([9, 11, 15]) that find centers of clusters provide appropriate initial approximations for factoring methods.

In Section 2, we show a method to generate a sequence of polynomials that converges to a factor of an analytic function $f(z)$. In Section 3, we derive a fixed point relation for a factor. Based on this relation, a method for finding a set of polynomials that includes a factor of $f(z)$ is considered. In Section 4, an algorithm with circular arithmetic is proposed. Some examples illustrate numerical features of the presented method in Section 5.

2 A factoring method

First, we consider the case that $f(z)$ is a polynomial of degree $m+n$. Suppose that $f(z) = p^*(z)q^*(z)$, where $p^*(z)$ is a monic polynomial of degree m, and $q^*(z)$ is a polynomial of degree n having no zeros in common with $p^*(z)$. Let $p(z)$ and $q(z)$ be approximations for $p^*(z)$ and $q^*(z)$, respectively. Samelson's method ([17]) defines an improved approximation $p(z)+s(z)$ for $p^*(z)$ by calculating polynomials $s(z)$ and $t(z)$ satisfying

$$sq + tp = r, \quad \deg s < m, \quad \deg t < n, \tag{1}$$

where $r(z) = f(z) - p(z)q(z)$. The polynomials s and t are uniquely determined if p and q are mutually prime. (1) is translated into a linear equation for the coefficients of s and t. These coefficients are also calculated via the extended Euclidean algorithm for p and q ([16, 20]).

Let g be a function defined on zeros of a polynomial p. Let v be a polynomial of degree at most $\deg p - 1$ such that $g - v$ is divisible by p. Then we denote v by $v = \mathrm{mod}(g,p)$. If g is a polynomial then $\mathrm{mod}(g,p)$ is just a polynomial remainder of g divided by p. From (1) we have

$$\left(\frac{r}{q}\right) - s = p\left(\frac{t}{q}\right), \quad \deg s < \deg p.$$

Therefore $s = \mathrm{mod}(r/q, p)$.

The following lemma shown in [16, 20] is essential for the factoring method described below. The similar result is also given in [18]. The symbol $\|\cdot\|$ for a polynomial denotes the vector 1-norm for a vector of coefficients of the polynomial.

Lemma 2.1 *Let p and q be mutually prime polynomials of degree m and n, respectively. Let r be a polynomial of degree at most $m+n$. If $\|p\| = O(1)$, $\|q\| = O(1)$, and $\|r\| = O(\varepsilon)$ with sufficiently small $\varepsilon > 0$, then $\|s\| = O(\varepsilon)$ and $\|t\| = O(\varepsilon)$.*

By applying (1) recursively, we obtain polynomial sequences $\{s^{(k)}\}$ and $\{t^{(k)}\}$ as follows.

$$s^{(k)}(q + t^{(k-1)}) + t^{(k)}p = r, \quad k = 1, 2, \ldots, \tag{2}$$

where $t^{(0)} \equiv 0$. The polynomials $s^{(k)}$ and $t^{(k)}$ have the following property.

Lemma 2.2 *Let $s^{(k)}$ and $t^{(k)}$ be defined by (2). Under the same assumption with Lemma 2.1, we have*

$$\|s^{(k)} - s^{(k-1)}\| = O(\varepsilon^k) \text{ and } \|t^{(k)} - t^{(k-1)}\| = O(\varepsilon^k), \tag{3}$$

for $k = 1, 2, \ldots$, where $s^{(0)} \equiv 0$ and $t^{(0)} \equiv 0$.

Proof. In case of $k = 1$, (3) is obvious by Lemma 2.1. Assume that (3) is valid up to $k-1$. Then $\|s^{(k-1)} - s^{(k-2)}\| = O(\varepsilon^{k-1})$ and $\|t^{(k-1)} - t^{(k-2)}\| = O(\varepsilon^{k-1})$. It follows form (2) that

$$(s^{(k)} - s^{(k-1)})q + (t^{(k)} - t^{(k-1)})p = s^{(k)}(t^{(k-1)} - t^{(k-2)}). \tag{4}$$

Since $\|r\| = O(\varepsilon)$, by Lemma 2.1 we have $\|s^{(k)}\| = O(\varepsilon)$. Hence $\|s^{(k)}(t^{(k-1)} - t^{(k-2)})\| = O(\varepsilon^k)$. Therefore from (4) we obtain (3).
□

Next theorem implies that the procedure (2) defines a factoring method.

Theorem 2.3 *If $\|r\| = O(\varepsilon)$, then for $s^{(k)}$ defined by (2),*

$$\|p + s^{(k)} - p^*\| = O(\varepsilon^{k+1}). \tag{5}$$

Proof. Form (2) and $f = p^*q^*$, we have

$$(p + s^{(k)} - p^*)(q + t^{(k)}) + (q + t^{(k)} - q^*)p^* = s^{(k)}(t^{(k)} - t^{(k-1)}). \tag{6}$$

By Lemma 2.2 we have

$$\|s^{(k)}(t^{(k)} - t^{(k-1)})\| = O(\varepsilon^{k+1}).$$

Since $\|t^{(k)}\| = O(\varepsilon)$, we can regard that $q + t^{(k)}$ and p^* are mutually prime with sufficiently small ε. Hence by (6) we have (5). □

If q and r are chosen so that $f = qp + r$, $\deg r < \deg p$, and if $\|p - p^*\| = O(\varepsilon)$, then $\|r\| = O(\varepsilon)$. Therefore the polynomial sequence $p + s^{(k)}$ converges to p^*, provided the starting factor p is sufficiently near p^*. When $k = 0$ and $m = 2$ this method is just Bairstow's method. Hereafter we denote $p^{(k)} := p + s^{(k)}$ and $q^{(k)} := q + t^{(k)}$.

Now let us consider the case that f is given by a power series $f(z) = \sum_{k=0}^{\infty} c_k z^k$. Let R be a fixed positive number. Let f be analytic for $|z| < R$ with zeros ζ_i, $i = 1, 2, \ldots$ ordered so that $|\zeta_1| \le \cdots \le |\zeta_m| < |\zeta_{m+1}| \le \cdots$, and let $\|f\| = O(1)$. Define $p^* = \prod_{i=1}^{m}(z - \zeta_i)$, and let q^* be an analytic function such that $f = p^*q^*$.

Suppose that $\zeta_1, \ldots, \zeta_m$ form a cluster covered by a small disk with the radius $\delta < R$ around the origin. Then $p = z^m$ can be regarded as a good initial approximation for p^*. In this case, we can calculate $s^{(k)}$ and $t^{(k)}$ that satisfy (2) easily by setting

$$r(z) = \sum_{k=0}^{m-1} c_k z^k \quad \text{and} \quad q(z) = \sum_{k=m}^{n+m} c_k z^{k-m}. \tag{7}$$

Moreover let

$$h(z) = \sum_{k=m+n+1}^{\infty} c_k z^{k-m-n-1} \tag{8}$$

then

$$f = p^*q^* = r + pq + z^{m+n+1}h. \tag{9}$$

Define

$$s^{(k)}(z) = \sigma_0^{(k)} + \sigma_1^{(k)} z + \cdots + \sigma_{m-1}^{(k)} z^{m-1},$$

and

$$t^{(k)}(z) = \tau_0^{(k)} + \tau_1^{(k)} z + \cdots + \tau_{n-1}^{(k)} z^{n-1}.$$

By comparing the coefficients in (2) we have the following relations.

$$\begin{pmatrix} c_m^{(k-1)} & & & \\ c_{m+1}^{(k-1)} & c_m^{(k-1)} & & \\ \vdots & & \ddots & \\ c_{2m-1}^{(k-1)} & \cdots & \cdots & c_m^{(k-1)} \end{pmatrix} \begin{pmatrix} \sigma_0^{(k)} \\ \sigma_1^{(k)} \\ \vdots \\ \sigma_{m-1}^{(k)} \end{pmatrix} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{m-1} \end{pmatrix} \tag{10}$$

and

$$\begin{pmatrix} \tau_0^{(k)} \\ \tau_1^{(k)} \\ \vdots \\ \tau_{n-1}^{(k)} \end{pmatrix} = - \begin{pmatrix} c_{2m}^{(k-1)} & c_{2m-1}^{(k-1)} & \cdots & c_{m+1}^{(k-1)} \\ c_{2m+1}^{(k-1)} & c_{2m}^{(k-1)} & \cdots & c_{m+2}^{(k-1)} \\ \vdots & & & \vdots \\ c_{2m+n-1}^{(k-1)} & c_{2m+n-2}^{(k-1)} & \cdots & c_{m+n}^{(k-1)} \end{pmatrix} \begin{pmatrix} \sigma_0^{(k)} \\ \sigma_1^{(k)} \\ \vdots \\ \sigma_{m-1}^{(k)} \end{pmatrix}, \tag{11}$$

where $c_{m+j}^{(k-1)} = c_{m+j} + \tau_j^{(k-1)}$, $j \geq 0$.

Let $1/f = \sum_{k=0}^{\infty} d_k z^k$. In case of $m = 1$, we can verify that $\sigma_0^{(k)} = -d_{k-1}/d_k$ from (10) and (11). Therefore $p^{(k)} = z + \sigma_0^{(k)}$ is just the numerator of the $[1/k-1]$-Pade approximant for f at $z = 0$.

When f is not a polynomial, we should take account of influence of $z^{m+n+1}h$ to discuss the convergence order of the method.

Theorem 2.4 *Let $p = z^m$, and let r, q and h be defined by (7) and (8). Let $n = mK - 1$. If $\|\Delta p\| := \|p - p^*\| = O(\varepsilon)$, then*

$$\|p^{(k)} - p^*\| = O(\varepsilon^{\hat{k}+1})$$

where $\hat{k} = \min(k, K)$.

Proof. Since

$$f = p^* q^* = (p - \Delta p)q^* = z^m q^* - \Delta p q^*$$

and $\|q^*\| = O(1)$, it follows that

$$\|r\| = O(\|\Delta p\|) = O(\varepsilon).$$

From (2) and (9) we have

$$(p^{(k)} - p^*)q^{(k)} + (q^{(k)} - q^*)p^* = s^{(k)}(t^{(k)} - t^{(k-1)}) - z^{m+n+1}h. \tag{12}$$

Since h is analytic for $|z| < R$, and all the zeros of p^* lie in the disk with the radius $\delta < R$, $w = \mathrm{mod}(z^{m+n+1}h, p^*)$ is well defined. Let $u = (z^{m+n+1}h - w)/p^*$, then

$$z^{m+n+1}h = w + p^* u.$$

Substituting it for (12) derives

$$(p^{(k)} - p^*)q^{(k)} + (q^{(k)} - q^* + u)p^* = s^{(k)}(t^{(k)} - t^{(k-1)}) - w. \tag{13}$$

Since $\operatorname{mod}(z^{m+n+1}, p^*) = \operatorname{mod}((\Delta p)^{K+1}, p^*)$,

$$\|w\| = \|\operatorname{mod}(z^{m+n+1}h, p^*)\| = O(\varepsilon^{K+1}).$$

Moreover, by Lemma 2.2 we have

$$\|s^{(k)}(t^{(k)} - t^{(k-1)})\| = O(\varepsilon^{k+1}).$$

These relations conclude the theorem. □

Therefore the polynomial $p^{(k)}$ approaches p^*, provided the starting factor p is sufficiently near p^*, and the degree of q is sufficiently large.

3 Validation for an approximate factor

In this section we show a method to give a validation for coefficients of a factor obtained by the method given in the previous section.

From (13) we have a fixed point relation for p^*.

Theorem 3.1 *Let $w = \operatorname{mod}(z^{m+n+1}h, p^*)$. If $q^{(k)}$ has no zeros in common with p^*, then*

$$p^* = p^{(k)} - \operatorname{mod}\left(\frac{s^{(k)}(t^{(k)} - t^{(k-1)}) - w}{q^{(k)}}, p^*\right). \tag{14}$$

Let $\boldsymbol{p}$ be a set of polynomials so that $p^* \in \boldsymbol{p}$, and let $\boldsymbol{w}$ be a set of polynomials so that $w \in \boldsymbol{w}$.

Theorem 3.2 *Let $\|r\| = O(\varepsilon)$ with sufficiently small $\varepsilon > 0$. If q has no zeros in common with any polynomial $\tilde{p} \in \boldsymbol{p}$, then*

$$p^* \in p^{(k)} - \operatorname{mod}\left(\frac{s^{(k)}(t^{(k)} - t^{(k-1)}) - \boldsymbol{w}}{q^{(k)}}, \boldsymbol{p}\right). \tag{15}$$

Proof. Since $\|t^{(k)}\| = O(\varepsilon)$, and q has no zeros in common with any polynomial $\tilde{p} \in \boldsymbol{p}$, we can assume that $q^{(k)} = q + t^{(k)}$ has no zeros in common with $\tilde{p}$ for sufficiently small ε. Therefore

$$\operatorname{mod}\left(\frac{s^{(k)}(t^{(k)} - t^{(k-1)}) - w}{q^{(k)}}, \tilde{p}\right)$$

is well defined. Substituting $\boldsymbol{p}$ for p^*, and $\boldsymbol{w}$ for w in (14) derives (15) by the inclusion property. □

Therefore if we can calculate $\boldsymbol{w}$, and can also calculate $\boldsymbol{s}^{(k)}$ and $\boldsymbol{t}^{(k)}$ that satisfy

$$\boldsymbol{s}^{(k)} q^{(k)} + \boldsymbol{t}^{(k)} \boldsymbol{p} = s^{(k)}(t^{(k)} - t^{(k-1)}) - \boldsymbol{w}$$

then

$$p^* \in \boldsymbol{p}^{(k)} := \boldsymbol{p} + \boldsymbol{s}^{(k)}.$$

Now let us consider a method to calculate $\boldsymbol{w}$. Let

$$C_p = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0/a_m \\ 1 & 0 & \cdots & 0 & -a_1/a_m \\ 0 & 1 & \cdots & 0 & -a_2/a_m \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{m-1}/a_m \end{pmatrix}$$

denotes the companion matrix of a polynomial

$$p(z) = a_0 + a_1 z + \cdots + a_m z^m \quad (a_m \neq 0).$$

The notation $\hat{p}$ implies the vector $(a_0, a_1, \ldots, a_m)^T$ of the coefficients of $p(z)$.

The following interpolation theorem was established in [18].

Theorem 3.3 *Let p be a polynomial of degree m, and let $z_1, \ldots, z_m$ be distinct zeros of p. Let g be a rational function defined at the z_i. Let the coefficients of the polynomial v be defined by*

$$\hat{v} = g(C_p)e_1,$$

where $e_1 = (1, 0, \ldots, 0)^T$. Then

$$v(z_i) = g(z_i), \quad i = 1, 2, \ldots, m.$$

Even when some of the z_i coincide, the polynomial v of the above theorem is well defined ([18]). In this case, it represents the appropriate Hermite interpolant of g over the zeros of p. This implies that $g - v$ is divisible by p, and hence $v = \mathrm{mod}(g, p)$.

Here we shall use the following notations. The matrix $|A|$ has elements $|\alpha_{ij}|$, that is, absolute value of the elements of $A = (\alpha_{ij})$. The notation $A \leq B$ implies $\alpha_{ij} \leq \beta_{ij}$ for every i and j, where $B = (\beta_{ij})$. We also define $|\boldsymbol{\alpha}| := \max_{\alpha \in \boldsymbol{\alpha}} |\alpha|$ for a closed set $\boldsymbol{\alpha}$ of complex numbers .

Theorem 3.4 *Let $h = \sum_{k=0}^{\infty} \gamma_k z^k$ with $|\gamma_k| < \eta^k$ where $0 < \eta < 1$. Let $v = \mathrm{mod}(h, p^*)$, and let $\hat{v}$ be the vector of the coefficients of v. If the spectral radius of $|C_{\boldsymbol{p}}|$, denoted by $\rho(|C_{\boldsymbol{p}}|)$, is smaller than η^{-1}, then*

$$|\hat{v}| \leq (I - \eta |C_{\boldsymbol{p}}|)^{-1} e_1.$$

Proof. Let $v^{(k)}(z) = \mathrm{mod}(z^k, p^*)$. By Theorem 3.3 we have the vector $\hat{v}^{(k)}$ of the coefficients of $v^{(k)}$ by

$$\hat{v}^{(k)} = (C_{p^*})^k e_1.$$

Therefore

$$\hat{v} = \sum_{k=0}^{\infty} \gamma_k \hat{v}^{(k)} = \sum_{k=0}^{\infty} \gamma_k (C_{p^*})^k e_1.$$

Since $|\gamma_k| < \eta^k$ for every k, and $|C_{p^*}| \leq |C_{\boldsymbol{p}}|$ for $p^* \in \boldsymbol{p}$,

$$|\hat{v}| \leq \sum_{k=0}^{\infty} \eta^k |C_{p^*}|^k e_1 \leq \sum_{k=0}^{\infty} \eta^k |C_{\boldsymbol{p}}|^k e_1.$$

By the hypothesis $\rho(|C\boldsymbol{p}|) < \eta^{-1}$, $\sum_{k=0}^{\infty} \eta^k |C\boldsymbol{p}|^k$ is well defined (for example see [8]), hence we obtain the result of the theorem. □

If we define the polynomial $\boldsymbol{v}$ so that

$$|\hat{\boldsymbol{v}}| = (I - \eta|C\boldsymbol{p}|)^{-1} e_1$$

then above theorem implies that $v \in \boldsymbol{v}$. Hence we can obtain $\boldsymbol{w}$ by $\boldsymbol{w} = \mathrm{mod}(z^{m+n+1}\boldsymbol{v}, \boldsymbol{p})$. There are some estimations for the upper bound of the radius of a circle that includes all the zeros of the corresponding polynomial of $|C\boldsymbol{p}|$ by using the coefficients of $\boldsymbol{p}$ (for example see [13]). Therefore if all the zeros of any polynomial that belongs to $\boldsymbol{p}$ lie in a small disk then we can expect that $\rho(|C\boldsymbol{p}|)$ is also small.

4 The algorithm

We show the algorithm to calculate a factor by using circular arithmetic. We denote a circular closed region $\boldsymbol{z} := \{z \mid |z-c| \le d\}$ by $\boldsymbol{z} := \{c, d\}$ with center $c = \mathrm{mid}(\boldsymbol{z})$ and radius $d = \mathrm{rad}(\boldsymbol{z})$. For a polynomial $\boldsymbol{p} = \sum_{k=0}^{m} \boldsymbol{a}_k z^k$, the notation $\mathrm{mid}(\boldsymbol{p})$ gives the polynomial $\sum_{k=0}^{m} \mathrm{mid}(\boldsymbol{a}_k) z^k$.

Suppose that the coefficients $\boldsymbol{c}_k$, $0 \le k \le m+n$ are given. For $k > m+n$, we assume that only the parameters M and η that satisfy $|\boldsymbol{c}_k| < M\eta^{k-m-n-1}$ are given. Suppose that the radius δ of a disk around the origin which includes m zeros of f is also given. Then the following algorithm finds a polynomial with circular coefficients that includes a polynomial factor of f.

Algorithm

Input: $\{\boldsymbol{c}_k\}_{k=0}^{m+n}$, M, η, δ, m, n, ϵ, k_{max}
Output: $\boldsymbol{p}^{(k)}$
$p \leftarrow z^m$
$r \leftarrow \sum_{k=0}^{m-1} \boldsymbol{c}_k z^k$
$q \leftarrow \sum_{k=m}^{m+n} \boldsymbol{c}_k z^{k-m}$
$\boldsymbol{p} \leftarrow (z - \{0, \delta\})^m$
$s^{(0)} \leftarrow 0$
$t^{(0)} \leftarrow 0$
for $k = 1, 2, \dots, k_{max}$
  compute $s^{(k)}$ and $t^{(k)}$ such that
    $s^{(k)}(q + \mathrm{mid}(t^{(k-1)})) + t^{(k)} p = r$
  If $\|s^{(k)} - s^{(k-1)}\| \le \epsilon$ **then** exit for loop
end for
$\boldsymbol{v} \leftarrow (1, z, \dots, z^{m-1})(I - \eta|C\boldsymbol{p}|)^{-1} e_1$
$\boldsymbol{w} \leftarrow \mathrm{mod}(M z^{m+n+1} \boldsymbol{v}, \boldsymbol{p})$
$\boldsymbol{s}^{(k)} \leftarrow \mathrm{mod}\left(\frac{s^{(k)}(t^{(k)} - t^{(k-1)}) - \boldsymbol{w}}{q^{(k)}}, \boldsymbol{p}\right)$
$\boldsymbol{p}^{(k)} \leftarrow (p + s^{(k)} - \boldsymbol{s}^{(k)}) \cap \boldsymbol{p}$

5 Numerical examples

We implemented our algorithm in MATLAB with INTLAB package [14] which provides circular arithmetic facilities for MATLAB.

Example 1 Let

$$\begin{aligned} p^*(z) &= (z-10^{-3})(z+10^{-3}/2)(z-10^{-3}/4) \\ &= z^3 - 7.50\times10^{-4}z^2 - 3.75\times10^{-7}z + 1.25\times10^{-10}, \end{aligned}$$

and let

$$q^*(z) = e^x \prod_{k=1}^{5}(z-k)\prod_{k=1}^{3}(2z+k).$$

Coefficients $\boldsymbol{c}_k$ were calculated by multiplying the polynomials and the truncated polynomial of Maclaurin expansion of e^x.

Parameters were $m=3$, $n=12$, $\delta=10^{-2}$, $\eta=1/2$ and $M=1$. Underlines show the significant figures of coefficients.

$$\begin{aligned} \boldsymbol{p}^{(1)} &= z^3 - \{\underline{7.4999}80503774\times10^{-4}, 8.5\times10^{-8}\}z^2 \\ &\quad -\{\underline{3.74999}0236502\times10^{-7}, 8.4\times10^{-10}\}z \\ &\quad +\{\underline{1.24999}6747551\times10^{-10}, 2.8\times10^{-12}\}, \end{aligned}$$

$$\begin{aligned} \boldsymbol{p}^{(2)} &= z^3 - \{\underline{7.50000000}27622\times10^{-4}, 1.2\times10^{-10}\}z^2 \\ &\quad -\{\underline{3.75000000}13836\times10^{-7}, 1.2\times10^{-12}\}z \\ &\quad +\{\underline{1.250000000}4609\times10^{-10}, 4.0\times10^{-15}\}, \end{aligned}$$

$$\begin{aligned} \boldsymbol{p}^{(3)} &= z^3 - \{\underline{7.4999999999}56\times10^{-4}, 1.9\times10^{-13}\}z^2 \\ &\quad -\{\underline{3.7499999999}78\times10^{-7}, 1.9\times10^{-15}\}z \\ &\quad +\{\underline{1.24999999999}3\times10^{-10}, 6.3\times10^{-18}\}. \end{aligned}$$

These polynomials include p^*, and give sharp bounds for coefficients of p^*.

Example 2 Let

$$p^*(z) = (z-10^{-3})\left(z+\frac{10^{-3}}{2}\right)\left(z-\frac{10^{-3}}{4}\right)\left(z+\frac{10^{-3}}{6}\right)\left(z-\frac{10^{-3}}{8}\right).$$

q^* is same as that of Example 1. Parameters were $m=5$, $n=15$, $\delta=10^{-2}$, $\eta=1/2$ and $M=1$.

$$\begin{aligned} \boldsymbol{p}^{(1)} &= z^5 - \{\underline{7.0833}16917107\times10^{-4}, 1.4\times10^{-7}\}z^4 \\ &\quad -\{\underline{4.2708}23418524\times10^{-7}, 2.7\times10^{-9}\}z^3 \\ &\quad +\{\underline{1.24999}7100176\times10^{-10}, 2.6\times10^{-11}\}z^2 \\ &\quad +\{\underline{1.30208}0955610\times10^{-14}, 1.3\times10^{-13}\}z \\ &\quad -\{\underline{2.610}812137458\times10^{-18}, 2.6\times10^{-16}\}, \end{aligned}$$

$$
\begin{aligned}
\boldsymbol{p}^{(2)} &= z^5 - \{\underline{7.0833333}55294 \times 10^{-4}, 1.9 \times 10^{-10}\}z^4 \\
&\quad -\{\underline{4.2708333}46599 \times 10^{-7}, 3.6 \times 10^{-12}\}z^3 \\
&\quad +\{\underline{1.250000000}3879 \times 10^{-10}, 3.6 \times 10^{-14}\}z^2 \\
&\quad +\{\underline{1.302083333}7377 \times 10^{-14}, 1.8 \times 10^{-16}\}z \\
&\quad -\{\underline{2.6041666}74751 \times 10^{-18}, 3.5 \times 10^{-19}\},
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{p}^{(3)} &= z^5 - \{\underline{7.0833333333}01 \times 10^{-4}, 2.7 \times 10^{-13}\}z^4 \\
&\quad -\{\underline{4.2708333333}14 \times 10^{-7}, 5.4 \times 10^{-15}\}z^3 \\
&\quad +\{\underline{1.24999999999}4 \times 10^{-10}, 5.3 \times 10^{-17}\}z^2 \\
&\quad +\{\underline{1.3020833333}27 \times 10^{-14}, 2.6 \times 10^{-19}\}z \\
&\quad -\{\underline{2.6041666666}55 \times 10^{-18}, 5.3 \times 10^{-22}\}.
\end{aligned}
$$

Example 3 Let

$$
\begin{aligned}
f &= (\sinh(2z^2) + \sinh(10z) - 1) \times (\sinh(2z^2) + \sinh(10z) - 1.01) \times \\
&\quad (\sinh(2z^2) + \sinh(10z) - 1.02).
\end{aligned}
$$

This function has 21 simple zeros inside the unit circle. They form 7 clusters, where each cluster consists of 3 zeros. This function was studied in [11, 15] as an example for finding the center of each clusters. Their results show that one of the clusters is located at $z = 8.777826159 \times 10^{-2}$, it contains 3 zeros, and its size is $O(10^{-3})$. The distance to the center of the nearest cluster is about 0.32.

From these results, we estimated the coefficients $\boldsymbol{c}_k$ by using the FFT with size 64 at the equi-distributed points on the circle with radius 0.1. We estimated p^* by using multiple precision arithmetic in Mathematica to verify the numerical results.

$$
\begin{aligned}
p^* &= z^3 + 7.3711680121192 \times 10^{-4} z^2 \\
&\quad -4.7678119480547 \times 10^{-5} z \\
&\quad -1.1197980731788 \times 10^{-8}.
\end{aligned}
$$

Parameters were $m = 3$, $n = 12$, $\delta = 10^{-1}$, $\eta = 0.5$ and $M = 1$.

$$
\begin{aligned}
\boldsymbol{p}^{(1)} &= z^3 + \{\underline{7.3711}670951 \times 10^{-4}, 1.6 \times 10^{-7}\}z^2 \\
&\quad -\{\underline{4.76781}13222 \times 10^{-5}, 1.4 \times 10^{-8}\}z \\
&\quad -\{\underline{1.11979}79540 \times 10^{-8}, 4.4 \times 10^{-10}\},
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{p}^{(2)} &= z^3 + \{\underline{7.3711680}211 \times 10^{-4}, 5.4 \times 10^{-11}\}z^2 \\
&\quad -\{\underline{4.7678119}478 \times 10^{-5}, 4.8 \times 10^{-12}\}z \\
&\quad -\{\underline{1.119798}1010 \times 10^{-8}, 1.6 \times 10^{-13}\},
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{p}^{(3)} &= z^3 + \{\underline{7.3711680}206 \times 10^{-4}, 3.9 \times 10^{-12}\}z^2 \\
&\quad -\{\underline{4.7678119}474 \times 10^{-5}, 3.5 \times 10^{-13}\}z \\
&\quad -\{\underline{1.119798}1009 \times 10^{-8}, 2.0 \times 10^{-14}\}.
\end{aligned}
$$

6 Conclusions

We discussed a method to find a factor of an analytic function $f(z)$. A fixed point relation for a polynomial factor p^* is derived. Based on this relation, an algorithm to find a factor of $f(z)$ with circular arithmetic is proposed. The presented method finds good bounds for coefficients of a factor in some numerical examples.

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References

[1] Bairstow, L.: The solution of algebraic equations with numerical coefficients in the case where several pairs of complex roots exist, *Advisory Committee for Aeronautics*, pp. 239–252, 1914-1995.

[2] Bauer, F. L., Samelson, K.: Polynomkerne und iterationsverfahren, *Math. Z.* **Vol. 67**, pp. 93–98 (1957).

[3] Bini, D. A., Gemignani, L., Meini, B.: Factorization of analytic functions by means of Koenig's theorem and Toeplitz computations (preprint), (1998).

[4] Carstensen, C., Sakurai, T.: Simultaneous factorization of a polynomial by rational approximation, *J. Comput. Appl. Math.* **Vol. 61**, pp. 165–178 (1995).

[5] Delves, L. M., Lyness, J. N.: A numerical method for locating the zeros of an analytic function, *Math. Comp.* **Vol. 21**, pp. 543–560 (1967).

[6] Durand, E.: *Solutions Numeriques des Equations Algebriques*, Masson, Paris, 1960.

[7] Grau, A. A.: The simultaneous Newton improvement of a complete set of approximate factors of a polynomial, *SIAM J. Numer. Anal.* **Vol. 8**, pp. 425–438 (1971).

[8] Householder, A. S.: *The Theory of Matrices in Numerical Analysis*, Blaisdell, New York, 1964.

[9] Hribernig, V., Stetter, H. J.: Detection and validation of clusters of polynomial zeros, *J. Symbolic Computation* **Vol. 24**, pp. 667–681 (1997).

[10] Jenkins, M. A., Traub, J. F.: A three-stage variable-shift iteration for polynomial zeros and its relation to generalized Rayleigh iteration, *Numer. Math.* **Vol. 14**, pp. 252–263 (1970).

[11] Kravanja, P., Sakurai, T., Van Barel, M.: A method for finding clusters of zeros of analytic function, *K.U.Leuven, Dept. Computer Science Report*, TW 280 (1998).

[12] Li, T. Y.: On locating all zeros of an analytic function within a bounded domain by a revised Delves/Lyness method, *SIAM J. Numer. Anal.* **Vol. 20**, pp. 865–871 (1983).

[13] Petković, M., Herceg, D., Ilić, S.: *Point Estimation Theory and its Applications*, Institute of Mathematics, Movi Sad, 1997.

[14] Rump, S. M.: INTLAB - INTerval LABoratory, http://www.ti3.tu-harburg.de/~rump/intlab/index.html.

[15] Sakurai, T., Torii, T., Ohsako, N., Sugiura, H.: A method for finding clusters of zeros of analytic function, *Proc. ICIAM'95*, Hamburg, pp. 515–516 (1996).

[16] Sonoda, S., Sakurai, T., Sugiura, H., Torii, T.: Numerical factorization of polynomial by the divide and conquer method (in Japanese), *Trans. Japan SIAM* **Vol. 1**, pp. 277–290 (1991).

[17] Stewart, G.W.: On Samelson's iteration for factoring polynomials, *Numer. Math.* **Vol. 15**, pp. 306–314 (1970).

[18] Stewart, G. W.: On a companion operator for analytic functions, *Numer. Math.* **Vol. 18**, pp. 26–43 (1971).

[19] Torii, T., Sakurai, T.: Global method for the poles of analytic function by rational interpolant on the unit circle, *World Sci. Ser. Appl. Anal.* **Vol. 2**, pp. 389–398 (1993).

[20] Torii, T., Sakurai, T., Sugiura, H.: An application of Sunzi's theorem for solving algebraic equations, *Proc. 1st China-Japan Seminar on Numerical Mathematics*, pp. 155–167 (1993).