A finite pivoting algorithm for minimizing a single criterion over the tricriteria efficient set

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June 30, 1999

ISE-TR-99-161

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* The author was partly supported by Grant-in-Aid for Scientific Research of the Japan Society for the Promotion of Science, Grant No. (C)(2)11650064.

A Finite Pivoting Algorithm for Minimizing a Single Criterion over the Tricriteria Efficient Set

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Abstract. In this paper, we propose a parametric-cost simplex algorithm for minimizing a single criterion over the efficient set of a triobjective linear programming problem. We first characterize potential optimal solutions for this nonconvex program. Then we show that a globally optimal solution can be found among them within a finite number of pivoting operations. Computational results indicates that the algorithm is practical and can solve fairly large scale problems.

Key words: Optimization, multiple objective linear programming, efficient set, global optimum, parametric simplex algorithm.

1. Introduction

The multiple objective linear programming (MOLP) involves the simultaneous maximization of more than one linear objective functions on a polyhedral set. In most cases, the objectives are in conflict with each other and hence cannot all be maximized simultaneously. Instead a set of efficient solutions is often supplied to the decision maker. An efficient solution represents a situation that no one can improve each of the objectives without making at least one of the rest worse. Since the MOLP emerged as a new topic in the early 1970s, the concept of efficiency has played the central role in the analysis and solution [14, 16]. The decision maker is then required to select a compromise from the set of efficient solutions; however, it is a rather troublesome task because the set is usually enormous even if it is only part of the entire efficient set [7, 8].

One reliable way to reduce the decision maker's burden is to optimize a single evaluation function over the efficient set. This approach was first proposed by Philip in 1972; and he developed a cutting plane algorithm to search the efficient set for a minimum point of a linear evaluation function [13]. Since the efficient set is not a convex set, his

*The author was partly supported by Grand-in-Aid for Scientific Research of the Japan Society for the Promotion of Science, Grant No. (C)(2) 11650064.

problem belongs to multiextremal global optimization [9, 10]. In the framework of global optimization, several promising algorithms have been proposed so far [2, 3, 4, 5, 15].

An important special case of Philip's problem is to *minimize* a specified one of the objective functions. Its uses in multiple criteria decision making are discussed in [4, 6, 12, 17]. Solving this problem, one can determine the range of values that the objective function can achieve on the efficient set. With the help of this information, the decision maker can set goals appropriately, and evaluate the utility of the objective function values at individual efficient solutions. If the range is very narrow, he may decide that the objective can be neglected.

In this paper, we will concentrate on the tricriteria case and propose a parametric-cost simplex algorithm for minimizing one of the three objective functions over the efficient set. As shown in [5], the parametric simplex algorithm can serve as a practical method for searching the entire efficient set of a biobjective linear program. In the tricriteria case, however, the efficient set is too large to search entirely. We then exploit the fact that the evaluation function is one of the objectives, and narrow down the set of efficient solutions that can provide an optimal solution. We refer to such an efficient solution as an *marginally efficient solution.* In Section 2, we will characterize the set of marginally efficient solutions of the general MOLP. Section 3 will be devoted to the algorithm for generating a sequence of marginally efficient solutions. We will show that the algorithm terminates within a finite number of simplex pivoting operations and yields a globally optimal solution. In Section 4, we will report computational results of testing the proposed algorithm on randomly generated problems.

2. Structure of the Problem

Let us consider a multiple objective linear program:

$$\text{MP} \quad \left| \begin{array}{ll} \text{`Maximize'} & \mathbf{z} = \mathbf{C}\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}, \end{array} \right.$$

where $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{C} \in \mathbb{R}^{p\times n}$ and $\operatorname{rank} \mathbf{C} = p \leq n$. We assume that the feasible set

$$X = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$$

is nonempty and bounded. We also impose a nondegeneracy assumption on X for simplicity:

Assumption 2.1. The coefficient matrix $\mathbf{A}$ has full row rank. Any subset of columns of $[\mathbf{A}, \mathbf{b}]$ has full rank if the corresponding submatrix of $\mathbf{A}$ has.

The purpose of MP is to find (some or all) feasible points that 'Maximize' the criterion vector $\mathbf{z}$, where the meaning of 'Maximize' is to make $\mathbf{z}$ *nondominated* by other feasible criterion vectors. More precisely, if $\bar{\mathbf{x}}$ is a solution of MP, there does not exist an $\mathbf{x} \in X$ such that

$$\mathbf{C}\mathbf{x} \geq \mathbf{C}\bar{\mathbf{x}} \text{ and } \mathbf{C}\mathbf{x} \neq \mathbf{C}\bar{\mathbf{x}}. \tag{2.1}$$

We refer to such a point $\bar{\mathbf{x}} \in X$ as an *efficient* or *Pareto optimal* solution of MP, and denote by X_E the set of all $\bar{\mathbf{x}}$'s. Our problem is not MP itself but to find an efficient solution $\mathbf{x}^*$ that minimizes a specified criterion $z_i = \mathbf{c}^i\mathbf{x}$ in the usual sense, where $\mathbf{c}^i$ denote the ith row of $\mathbf{C}$. Therefore, the problem can be written as follows:

$$\text{P} \quad \left| \begin{array}{ll} \text{minimize} & z_i = \mathbf{c}^i\mathbf{x} \\ \text{subject to} & \mathbf{x} \in X_E. \end{array} \right.$$

2.1. D.C. REPRESENTATION OF THE EFFICIENT SET

Let

$$Y = X + \{\mathbf{d} \in \mathbb{R}^n \mid \mathbf{C}\mathbf{d} \leq \mathbf{0},\ \mathbf{C}\mathbf{d} \neq \mathbf{0}\}. \tag{2.2}$$

We see from (2.1) that each $\mathbf{y} \in Y$ is dominated by some $\mathbf{x} \in X$. By means of this set Y, we can express the efficient set X_E as

$$X_E = X \setminus Y. \tag{2.3}$$

Since the second term of (2.2), called the *criterion cone*, is convex polyhedral, Y is a convex subset of $\mathbb{R}^n$. Hence, (2.3) implies that the feasible set of P is not a convex set in general but a *d.c. set* (difference of two convex sets). Problems of this kind, called *d.c. programs*, can have multiple locally optimal solutions, many of which fail to be globally optimal [9, 10].

Let us denote by S the set of optimal solutions of a problem:

$$\left| \begin{array}{ll} \text{minimize} & z_i = \mathbf{c}^i\mathbf{x} \\ \text{subject to} & \mathbf{x} \in X. \end{array} \right.$$

If $S \cap Y = \emptyset$, then any $\mathbf{x} \in S$ is an optimal solution of P; we can obtain it using the simplex or interior-point algorithm. To exclude such a trivial case, we assume hereafter that

$$S \cap Y \neq \emptyset. \tag{2.4}$$

Under this condition, it is known [10] that any d.c. program has at least one globally optimal solution in the intersection of ∂Y with some edge of the polytope X, where $\partial\,\cdot$ denotes the set of relative boundary points. In addition to this, since the efficient set X_E is a connected subset of ∂X [14], problem P must have a globally optimal solution on some efficient edge or extreme point. In fact, we have the following, which was proved by Benson (Theorem 4.5 in [2]) for problems of minimizing a general linear function:

Lemma 2.1. *Among efficient extreme points of X exists a globally optimal solution $\mathbf{x}^*$ of P.*

In our problem P, the objective function is a component of the criterion vector $\mathbf{z} = \mathbf{C}\mathbf{x}$. Exploiting this special structure, we can strengthen Lemma 2.1. To do this, however, we need some preliminaries.

2.2. Marginally and internally efficient extreme points

For any constant vector $\boldsymbol{\lambda}^\circ \in \mathbb{R}^p \setminus \{\mathbf{0}\}$, let us consider a *composite linear program* associated with MP:

$$\mathrm{Q}(\boldsymbol{\lambda}^\circ) \quad \left| \begin{array}{ll} \text{maximize} & z(\boldsymbol{\lambda}^\circ) = \boldsymbol{\lambda}^\circ \mathbf{C}\mathbf{x} \\ \text{subject to} & \mathbf{x} \in X. \end{array} \right.$$

Since X is bounded, the objective function attains its maximum at some extreme point, say $\mathbf{x}^\circ$, of the polytope. By Assumption 2.1, the point $\mathbf{x}^\circ$ corresponds to a unique basis matrix $\mathbf{B} \in \mathbb{R}^{m \times m}$ of $\mathbf{A}$. Let us decompose $\mathbf{A}$, $\mathbf{C}$ and $\mathbf{x}$ accordingly into $[\mathbf{B}, \mathbf{N}]$, $[\mathbf{C}_B, \mathbf{C}_N]$ and $(\mathbf{x}_B, \mathbf{x}_N)$. Then we can write the optimal dictionary of $\mathrm{Q}(\boldsymbol{\lambda}^\circ)$ as follows:

$$\left| \begin{array}{l} \mathbf{x}_B = \bar{\mathbf{b}} - \bar{\mathbf{A}}\mathbf{x}_N \\ z = \boldsymbol{\lambda}^\circ \mathbf{C}_B \bar{\mathbf{b}} + \boldsymbol{\lambda}^\circ \bar{\mathbf{C}} \mathbf{x}_N, \end{array} \right. \tag{2.5}$$

where

$$\bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b}, \quad \bar{\mathbf{A}} = \mathbf{B}^{-1}\mathbf{N}, \quad \bar{\mathbf{C}} = \mathbf{C}_N - \mathbf{C}_B \bar{\mathbf{A}}.$$

Each factor of the relative profit $\boldsymbol{\lambda}^\circ \bar{\mathbf{C}}$ is either zero or negative in (2.5). From the optimality condition in linear programming [1], the point $\mathbf{x}^\circ$ is optimal for $\mathrm{Q}(\boldsymbol{\lambda})$ as well if $\boldsymbol{\lambda}\bar{\mathbf{C}}$ is nonpositive, in other words, if $\boldsymbol{\lambda}$ belongs to a polyhedral cone:

$$\Lambda[\mathbf{B}] = \{\boldsymbol{\lambda} \in \mathbb{R}^p \mid \boldsymbol{\lambda}\bar{\mathbf{C}} \leq \mathbf{0}\}.$$

For any $\boldsymbol{\lambda} \neq \mathbf{0}$, problem $\mathrm{Q}(\boldsymbol{\lambda})$ has an optimal solution. Therefore, the whole space of $\boldsymbol{\lambda}$ except the origin is completely covered by a finite number of polyhedral cones $\Lambda[\mathbf{B}_{(k)}]$'s, where $\mathbf{B}_{(k)}$'s are feasible bases of X. Moreover, we have the following:

Lemma 2.2. *Let r denote the number of feasible bases of X. Then $\{\Lambda[\mathbf{B}_{(k)}] \mid k = 1, \ldots, r\}$ is a partition of $\mathbb{R}^p \setminus \{\mathbf{0}\}$, i.e.,*

$$\bigcup_{j=1}^{r} \Lambda[\mathbf{B}_{(k)}] = \mathbb{R}^p \setminus \{\mathbf{0}\} \tag{2.6}$$

$$\operatorname{int} \Lambda[\mathbf{B}_{(k)}] \cap \operatorname{int} \Lambda[\mathbf{B}_{(\ell)}] = \emptyset \quad \textit{if } k \neq \ell. \tag{2.7}$$

where int· represents the interior.

Proof: We have already seen (2.6). Let us prove (2.7) by assuming the contrary. Let $\boldsymbol{\lambda}'$ be an interior point of both $\Lambda[\mathbf{B}_{(k)}]$ and $\Lambda[\mathbf{B}_{(\ell)}]$. Associated with $\mathbf{B}_{(k)}$ and $\mathbf{B}_{(\ell)}$ are different dictionaries of $\mathrm{Q}(\boldsymbol{\lambda}')$, where the relative profits are respectively

$$\boldsymbol{\lambda}'\bar{\mathbf{C}}_{(k)} < \mathbf{0} \quad \text{and} \quad \boldsymbol{\lambda}'\bar{\mathbf{C}}_{(\ell)} < \mathbf{0}.$$

These inequalities imply that each basis uniquely determines an optimal solution of $\mathrm{Q}(\boldsymbol{\lambda}')$. However, this can never happen unless the optimal solution is degenerate. ■

Suppose that $\Lambda[\mathbf{B}_{(k)}]$ shares a face defined by $\boldsymbol{\lambda}\bar{\mathbf{C}}^j_{(k)} = 0$ with $\Lambda[\mathbf{B}_{(\ell)}]$ for $\ell \neq k$, where $\bar{\mathbf{C}}^j_{(k)}$ denotes the jth column of $\bar{\mathbf{C}}_{(k)}$. Then we can obtain $\mathbf{B}_{(\ell)}$ from $\mathbf{B}_{(k)}$ in a single pivoting operation; we need only to replace some column of $\mathbf{B}_{(k)}$ by the jth column of $\mathbf{N}_{(k)}$. We say that $\mathbf{B}_{(k)}$ is *adjacent to* $\mathbf{B}_{(\ell)}$ in this case. The extreme points $\mathbf{x}^k$ and $\mathbf{x}^\ell$ associated with adjacent bases $\mathbf{B}_{(k)}$ and $\mathbf{B}_{(\ell)}$ are also adjacent on X with some edge.

It is well known [14] that $\mathbf{x}^\circ \in X$ is an efficient solution of MP if and only if $\mathbf{x}^\circ$ is an optimal solution of $\mathrm{Q}(\boldsymbol{\lambda}^\circ)$ for some $\boldsymbol{\lambda}^\circ > \mathbf{0}$. Therefore, from (2.6), the extreme point $\mathbf{x}^k$ associated with $\mathbf{B}_{(k)}$ is efficient if and only if the cone $\Lambda[\mathbf{B}_{(k)}]$ intersects the positive orthant $\mathbb{R}^p_+ = \{\boldsymbol{\lambda} \in \mathbb{R}^p \mid \boldsymbol{\lambda} > \mathbf{0}\}$. We classify such efficient extreme points $\mathbf{x}^k$'s into two families:

Definition 2.1. Extreme point $\mathbf{x}^k$ of X is *internally efficient (abbr. i.e.e.)* iff

$$\Lambda[\mathbf{B}_{(k)}] \subset \mathbb{R}^p_+.$$

Extreme point $\mathbf{x}^k$ of X is *marginally efficient (abbr. m.e.e.)* iff

$$\Lambda[\mathbf{B}_{(k)}] \not\subset \mathbb{R}^p_+ \quad \text{and} \quad \Lambda[\mathbf{B}_{(k)}] \cap \mathbb{R}^p_+ \neq \emptyset.$$

Note that some of extreme points adjacent to an m.e.e. point might be inefficient while all extreme points adjacent to an i.e.e. point are efficient. We are now ready to strengthen Lemma 2.1.

Theorem 2.3. *Among marginally efficient extreme points of X exists a globally optimal solution $\mathbf{x}^*$ of P.*

Proof: Let $\mathbf{x}^k$ be an arbitrary i.e.e. point. By definition, $\mathbf{x}^k$ is optimal for $\mathrm{Q}(\boldsymbol{\lambda})$ if and only if $\boldsymbol{\lambda} \in \Lambda[\mathbf{B}_{(k)}] = \{\boldsymbol{\lambda} \in \mathbb{R}^p \mid \boldsymbol{\lambda}\bar{\mathbf{C}}_{(k)} \leq \mathbf{0}\} \subset \mathbb{R}^p_+$. Consider a problem $\mathrm{Q}(-\mathbf{e}^i)$ of minimizing $\mathbf{c}^i\mathbf{x} = \mathbf{e}^i\mathbf{C}\mathbf{x}$, where $\mathbf{e}^i \in \mathbb{R}^p$ is the ith unit vector. The relative profit $-\mathbf{e}^i\bar{\mathbf{C}}_{(k)}$ of $\mathrm{Q}(-\mathbf{e}^i)$ with respect to $\mathbf{B}_{(k)}$ has at least one positive factor, since $-\mathbf{e}^i \notin \Lambda[\mathbf{B}_{(k)}]$. This implies that there is an extreme point $\mathbf{x}^\ell$ of X adjacent to $\mathbf{x}^k$ such that $\mathbf{c}^i\mathbf{x}^\ell < \mathbf{c}^i\mathbf{x}^k$. Since $\mathbf{x}^k$ is internally efficient, $\mathbf{x}^\ell$ is efficient and hence belongs to X_E. By the arbitrariness of $\mathbf{x}^k$, no i.e.e. points can be optimal for P. Then we see from Lemma 2.1 that a globally optimal solution of P exists among m.e.e. points. ■

Any $\boldsymbol{\lambda} \in \mathbb{R}^p_+$ can be normalized so that the components add up to one. Therefore, if we choose a $\boldsymbol{\lambda}$ appropriately from $\Delta = \{\boldsymbol{\lambda} \in \mathbb{R}^p_+ \mid \sum_{i=1}^p \lambda_i = 1\}$, any efficient extreme point of X will be given as an optimal solution of $\mathrm{Q}(\boldsymbol{\lambda})$. As Theorem 2.3 suggests, however, we need not search for an optimal $\boldsymbol{\lambda}$ every nook and cranny but only part of Δ along its relative boundary.

3. Parametric Algorithm for the Tricriteria Case

On the basis of the observation in Section 2, we will develop an algorithm for solving problem P with tricriteria.

In the case of $p = 3$, composite linear programs to be considered are of the form:

$$\mathrm{Q}(\boldsymbol{\lambda}) \quad \left| \begin{array}{ll} \text{maximize} & z(\boldsymbol{\lambda}) = \lambda_1 \mathbf{c}^1\mathbf{x} + \lambda_2 \mathbf{c}^2\mathbf{x} + \lambda_3 \mathbf{c}^3\mathbf{x} \\ \text{subject to} & \mathbf{x} \in X, \end{array} \right.$$

where

$$\boldsymbol{\lambda} \in \Delta = \{\boldsymbol{\lambda} \in \mathbb{R}^3_+ \mid \lambda_1 + \lambda_2 + \lambda_3 = 1\}.$$

As we have seen, $\mathrm{Q}(\boldsymbol{\lambda}^*)$ provides an optimal solution $\mathbf{x}^*$ of P for some $\boldsymbol{\lambda}^* \in \Delta$. Since Δ is a two-dimensional simplex, we can find a $\boldsymbol{\lambda}^*$ by searching Δ only along its three edges. In the first stage of this search, we solve $\mathrm{Q}(\boldsymbol{\lambda})$ parametrically by changing the value of $\boldsymbol{\lambda}$ from $(1-2\epsilon, \epsilon, \epsilon)$ to $(\epsilon, 1-2\epsilon, \epsilon)$ for sufficiently small number $\epsilon > 0$, and evaluate the value of $z_i = \mathbf{c}_i\mathbf{x}$ at the encountered m.e.e. points. Similarly, we carry out the second and third stages, where the value of $\boldsymbol{\lambda}$ is changed from $(\epsilon, 1-2\epsilon, \epsilon)$ to $(\epsilon, \epsilon, 1-2\epsilon)$, and from $(\epsilon, \epsilon, 1-2\epsilon)$ to $(1-2\epsilon, \epsilon, \epsilon)$. In this section, we will show that each of these three stages can be done in almost the same manner as the parametric-cost simplex algorithm for linear programs with a single parameter (see e.g. [1]).

3.1. First stage of the search

In the first stage of the search, we solve the linear program $\mathrm{Q}(1-\epsilon-\mu, \mu, \epsilon)$ as increasing the value of μ from ϵ to $1-\epsilon$. The number ϵ is positive and sufficiently small, but need not be fixed beforehand.

Let $\mathbf{x}^\circ$ be an optimal basic solution of $\mathrm{Q}(1-\epsilon-\mu^\circ, \mu^\circ, \epsilon)$ and $\mathbf{B}$ its associated basis, where $\mu^\circ \in [\epsilon, 1-\epsilon]$. We assume that the cone $\Lambda[\mathbf{B}]$ has a nonempty interior and intersects a boundary edge $(1,0,0)$–$(0,1,0)$ of Δ. This implies that $\mathbf{x}^\circ$ is an m.e.e. point. The dictionary of $\mathrm{Q}(1-\epsilon-\mu^\circ, \mu^\circ, \epsilon)$ with respect to $\mathbf{B}$ is

$$\left| \begin{array}{l} \mathbf{x}_B = \bar{\mathbf{b}} - \bar{\mathbf{A}}\mathbf{x}_N \\ z = [(1-\epsilon-\mu)\mathbf{c}^1_B + \mu\mathbf{c}^2_B + \epsilon\mathbf{c}^3_B]\bar{\mathbf{b}} + [(1-\epsilon-\mu)\bar{\mathbf{c}}^1 + \mu\bar{\mathbf{c}}^2 + \epsilon\bar{\mathbf{c}}^3]\mathbf{x}_N. \end{array} \right. \tag{3.1}$$

Hence, $\Lambda[\mathbf{B}]$ is defined by the system of inequalities:

$$(1-\epsilon-\mu)\bar{c}^1_j + \mu\bar{c}^2_j + \epsilon\bar{c}^3_j \leq 0, \quad j \in J_N, \tag{3.2}$$

where J_N denotes the index set of nonbasic variables. Let

$$\alpha_j = \bar{c}_j^3 - \bar{c}_j^1, \quad \beta_j = \bar{c}_j^2 - \bar{c}_j^1, \quad j \in J_N,$$

and let

$$\overline{J} = \{j \in J_N \mid \beta_j > 0\}, \quad \underline{J} = \{j \in J_N \mid \beta_j < 0\}.$$

If $\overline{J} = \emptyset$, then (3.2) holds for all $\mu \geq \mu^\circ$. In this case, we terminate this stage and proceed to the next, where we can use $\mathbf{x}^\circ$ as the starting m.e.e. point. Similarly, (3.2) holds for all $\mu \leq \mu^\circ$ if $\underline{J} = \emptyset$.

Assuming $\overline{J} \neq \emptyset$, let us define

$$\left.\begin{aligned} \overline{\mu}(\epsilon) &= \min\{-(\alpha_j\epsilon + \bar{c}_j^1)/\beta_j \mid j \in \overline{J}\} \\ \underline{\mu}(\epsilon) &= \max\{-(\alpha_j\epsilon + \bar{c}_j^1)/\beta_j \mid j \in \underline{J}\}, \end{aligned}\right\} \quad (3.3)$$

where $\underline{\mu}(\epsilon)$ is understood to be $-\infty$ if $\underline{J} = \emptyset$. Then (3.2) reduces to $\underline{\mu}(\epsilon) \leq \mu \leq \overline{\mu}(\epsilon)$. Note that the interval $[\underline{\mu}(\epsilon), \overline{\mu}(\epsilon)]$ has μ° and an interior point. Hence, $\mathbf{x}^\circ$ remains an optimal solution of $\mathrm{Q}(1 - \epsilon - \mu, \mu, \epsilon)$ as long as μ lies on $[\underline{\mu}(\epsilon), \overline{\mu}(\epsilon)]$. Once μ exceeds $\overline{\mu}(\epsilon)$, however, the optimality condition (3.2) will be violated at some $j \in \overline{J}$. To obtain an alternative optimal basis, we have to identify a $j \in \overline{J}$ that attains the minimum in (3.3). Let

$$J_c = \arg\max\left\{\bar{c}_j^1/\beta_j \,\middle|\, j \in \overline{J}\right\}, \quad J_\alpha = \arg\max\{\alpha_j/\beta_j \mid j \in J_c\}.$$

Lemma 3.1. *Let $s \in J_\alpha$. There is a number $\delta > 0$ such that*

$$(\alpha_s\epsilon + \bar{c}_s^1)/\beta_s > (\alpha_j\epsilon + \bar{c}_j^1)/\beta_j, \quad \forall\epsilon \in (0, \delta), \quad (3.4)$$

for each $j \in \overline{J} \setminus J_\alpha$.

Proof: Let $\overline{J} \setminus J_\alpha$ partition into $\overline{J} \setminus J_c$ and $J_c \setminus J_\alpha$. If $j \in J_c \setminus J_\alpha$, then $\bar{c}_s^1/\beta_s = \bar{c}_j^1/\beta_j$ and $\alpha_s/\beta_s > \alpha_j/\beta_j$ by definition; hence (3.4) holds for all $\delta > 0$. Suppose that $j \in \overline{J} \setminus J_c$. Then we have

$$\bar{c}_s^1/\beta_s - \bar{c}_j^1/\beta_j > 0. \quad (3.5)$$

If $\alpha_s/\beta_s \geq \alpha_j/\beta_j$, then (3.4) is obvious for all $\delta > 0$. Otherwise, it follows from (3.5) that there is a number $\delta_j > 0$ such that

$$(\alpha_s/\beta_s - \alpha_j/\beta_j)\epsilon + (\bar{c}_s^1/\beta_s - \bar{c}_j^1/\beta_j) > 0, \quad \forall\epsilon \in (0, \delta_j).$$

Therefore, letting $\delta = \min\{\delta_j \mid j \in \overline{J} \setminus J_c \text{ s.t. } \alpha_s/\beta_s < \alpha_j/\beta_j\}$, we have (3.4) for every $j \in \overline{J} \setminus J_c$. ■

Lemma 3.1 implies that $\overline{\mu}(\epsilon)$ is attained at each $s \in J_\alpha$ when ϵ is sufficiently small. If J_α is a singleton, we can obtain an alternative basis $\mathbf{B}'$ adjacent to $\mathbf{B}$ by a single pivoting operation: we exchange a column of $\mathbf{N}$ indexed by $s \in J_\alpha$ for a column of $\mathbf{B}$, which is uniquely determined under Assumption 2.1. The resulting basis $\mathbf{B}'$ generates an m.e.e. point because $\Lambda[\mathbf{B}]$ and $\Lambda[\mathbf{B}']$ share a closed line segment:

$$\{\boldsymbol{\lambda} \in \mathrm{IR}^3 \mid \boldsymbol{\lambda} = (1-\epsilon-\overline{\mu}(\epsilon), \overline{\mu}(\epsilon), \epsilon),\ \epsilon \in [0, \delta]\},$$

which intersects the boundary edge $(1,0,0)$–$(0,1,0)$. Also, we can easily check from (3.4) that $\mathrm{int}\Lambda[\mathbf{B}'] \neq \emptyset$. If J_α contains more than one index, however, problem $\mathrm{Q}(1-\epsilon-\overline{\mu}(\epsilon), \overline{\mu}(\epsilon), \epsilon)$ causes *dual degeneracy.* We need some additional procedures to continue this stage.

3.2. Procedures against dual degeneracy

The relative profit factors of (3.1) at $\mu = \overline{\mu}(\epsilon)$ with $\epsilon \in (0, \delta)$ satisfy

$$\begin{array}{ll} [1-\epsilon-\overline{\mu}(\epsilon)]\bar{c}_j^1 + \overline{\mu}(\epsilon)\bar{c}_j^2 + \epsilon\bar{c}_j^3 = 0 & \text{for each } j \in J_\alpha \\ [1-\epsilon-\overline{\mu}(\epsilon)]\bar{c}_j^1 + \overline{\mu}(\epsilon)\bar{c}_j^2 + \epsilon\bar{c}_j^3 < 0 & \text{for each } j \in J_N \setminus J_\alpha. \end{array}$$

Hence, the solution set of $\mathrm{Q}(1-\epsilon-\overline{\mu}(\epsilon), \overline{\mu}(\epsilon), \epsilon)$ is given by

$$F(J_\alpha) = X \cap \{\mathbf{x} \in \mathrm{IR}^n \mid x_j = 0,\ j \in J_N \setminus J_\alpha\},$$

which is a $|J_\alpha|$-dimensional face of X and can contain more than two m.e.e. points to be tested for optimality if $|J_\alpha| > 1$. In practice, however, we need only to test the solution of a linear program:

$$\mathrm{P}(J_\alpha) \quad \left| \begin{array}{ll} \text{minimize} & z_i = \mathbf{c}^i\mathbf{x} \\ \text{subject to} & \mathbf{x} \in F(J_\alpha). \end{array} \right.$$

Let $\mathbf{x}_\alpha$ denote the vector of x_j, $j \in J_\alpha$, and $\mathbf{N}_\alpha$ the corresponding submatrix of $\mathbf{A}$. Then $\mathrm{P}(J_\alpha)$ is essentially a problem with variables $(\mathbf{x}_B, \mathbf{x}_\alpha)$ constrained by

$$\mathbf{B}\mathbf{x}_B + \mathbf{N}_\alpha\mathbf{x}_\alpha = \mathbf{b}, \quad \mathbf{x}_B \geq \mathbf{0}, \quad \mathbf{x}_\alpha \geq \mathbf{0}.$$

By Assumption 2.1, this system has no degenerate basic solution, since $[\mathbf{B}, \mathbf{N}_\alpha, \mathbf{b}]$ is a subset of columns of $[\mathbf{A}, \mathbf{b}]$. This implies that $\mathrm{P}(J_\alpha)$ can be solved in a finite number of the usual simplex pivoting operations [1].

After solving $\mathrm{P}(J_\alpha)$, we have to escape the dual degenerate face $F(J_\alpha)$. For this purpose, we next solve

$$\mathrm{R}(J_\alpha) \quad \left| \begin{array}{ll} \text{maximize} & w = (\mathbf{c}^2 - \mathbf{c}^1)\mathbf{x} \\ \text{subject to} & \mathbf{x} \in F(J_\alpha). \end{array} \right.$$

Since the feasible set is again $F(J_\alpha)$, we can compute an optimal solution $\mathbf{x}'$ and basis $\mathbf{B}'$ in a finite number of pivoting operations. Let us decompose $\mathbf{A}$, $\mathbf{c}^i$ and $\mathbf{x}$ accordingly into $[\mathbf{B}', \mathbf{N}']$, $(\mathbf{c}^i_{B'}, \mathbf{c}^i_{N'})$ and $(\mathbf{x}_{B'}, \mathbf{x}_{N'})$, respectively. Also, let

$$\tilde{\mathbf{c}}^i = \mathbf{c}_{N'} - \mathbf{c}_{B'}[\mathbf{B}']^{-1}\mathbf{N}', \quad i = 1, 2, 3.$$

Then we have

$$\tilde{c}^2_j - \tilde{c}^1_j \leq 0, \quad j \in J_{N'} \cap (J_B \cup J_\alpha), \tag{3.6}$$

where $J_{N'}$ is the index set of $\mathbf{x}_{N'}$. Moreover, the following holds for $\epsilon \in (0, \delta)$ because $\mathbf{x}'$ remains a dual degenerate optimal solution of $\mathrm{Q}(1 - \epsilon - \overline{\mu}(\epsilon), \overline{\mu}(\epsilon), \epsilon)$:

$$\left.\begin{array}{ll} [1 - \epsilon - \overline{\mu}(\epsilon)]\tilde{c}^1_j + \overline{\mu}(\epsilon)\tilde{c}^2_j + \epsilon\tilde{c}^3_j = 0 & \text{for each } j \in J_{N'} \cap (J_B \cup J_\alpha) \\ [1 - \epsilon - \overline{\mu}(\epsilon)]\tilde{c}^1_j + \overline{\mu}(\epsilon)\tilde{c}^2_j + \epsilon\tilde{c}^3_j < 0 & \text{for each } j \in J_{N'} \setminus (J_B \cup J_\alpha). \end{array}\right\} \tag{3.7}$$

As before, letting

$$\alpha'_j = \tilde{c}^3_j - \tilde{c}^1_j, \quad \beta'_j = \tilde{c}^2_j - \tilde{c}^1_j, \quad j \in J_{N'},$$

we define

$$\left.\begin{array}{l} \overline{\mu}'(\epsilon) = \min\{-(\alpha'_j\epsilon + \tilde{c}^1_j)/\beta'_j \mid j \in \overline{J}'\} \\ \underline{\mu}'(\epsilon) = \max\{-(\alpha'_j\epsilon + \tilde{c}^1_j)/\beta'_j \mid j \in \underline{J}'\} \end{array}\right\} \tag{3.8}$$

where

$$\overline{J}' = \{j \in J_{N'} \mid \beta'_j > 0\}, \quad \underline{J}' = \{j \in J_{N'} \mid \beta'_j < 0\}.$$

Note that $\underline{J}' \neq \emptyset$. Since the profit vector $\mathbf{c}^2 - \mathbf{c}^1$ is linearly independent from that of $\mathrm{Q}(1 - \epsilon - \overline{\mu}(\epsilon), \overline{\mu}(\epsilon), \epsilon)$ if $\epsilon > 0$, at least one inequality in (3.6) holds strictly. If $\overline{J}' = \emptyset$, then we set $\overline{\mu}'(\epsilon)$ to be $+\infty$.

Lemma 3.2. *Let δ denote the number defined in Lemma 3.1. Then*

$$\overline{\mu}(\epsilon) = \underline{\mu}'(\epsilon) < \overline{\mu}'(\epsilon), \quad \forall \epsilon \in (0, \delta). \tag{3.9}$$

Proof: It follows from (3.7) that

$$\begin{array}{ll} \overline{\mu}(\epsilon) = -(\alpha'_j\epsilon + \tilde{c}^1_j)/\beta'_j & \text{for each } j \in \underline{J}' \cap (J_B \cup J_\alpha) \\ \overline{\mu}(\epsilon) > -(\alpha'_j\epsilon + \tilde{c}^1_j)/\beta'_j & \text{for each } j \in \underline{J}' \setminus (J_B \cup J_\alpha), \end{array}$$

which proves the equality in (3.9). We obtain the inequality from

$$\overline{\mu}(\epsilon) < -(\alpha'_j\epsilon + \tilde{c}^1_j)/\beta'_j \quad \text{for each } j \in \overline{J}' = \overline{J}' \setminus (J_B \cup J_\alpha),$$

by noting $\overline{J}' \cap (J_B \cup J_\alpha) = \emptyset$. ■

We see from (3.9) that the cone $\Lambda[\mathbf{B}']$ has an interior point and intersects the boundary edge $(1,0,0)$–$(0,1,0)$ of Δ. Therefore, the optimal basis $\mathbf{B}'$ of $\mathrm{R}(J_\alpha)$ defines an m.e.e. point.

3.3. Algorithm description

The second and third stages of the search are similar to the first one. The rest to be discussed is how to find an initial m.e.e. point for the first stage. This can also be done by solving a linear program parametrically.

Consider a composite linear program $Q(1-2\epsilon, \epsilon, \epsilon)$, i.e.,

$$\left| \begin{array}{ll} \text{maximize} & \mathbf{c}^1\mathbf{x} + \epsilon(\mathbf{c}^2 + \mathbf{c}^3 - 2\mathbf{c}^1)\mathbf{x} \\ \text{subject to} & \mathbf{x} \in X. \end{array} \right. \tag{3.10}$$

We solve this problem parametrically as increasing the value of ϵ from zero. The feasible basic solution first encountered for $\epsilon > 0$ is an m.e.e. point and can be used as the starting point in the first stage.

Let us summarize the algorithm:

algorithm MIN_CRITERION.
begin
  solve (3.10) parametrically until an m.e.e. point $\mathbf{x}^\circ$ is found;
  $(\mathbf{x}^*, z_i^*) := (\mathbf{x}^\circ, \mathbf{c}^i\mathbf{x}^\circ)$;
  let $\mathbf{B}$ and $\mathbf{N}$ denote the basis and nonbasis of $\mathbf{A}$ associated with $\mathbf{x}^\circ$;
  compute the relative profit $\bar{\mathbf{c}}^i$ with respect to $\mathbf{B}$ for each $i = 1, 2, 3$;
  first $:= \mathbf{c}^1$;
  repeat
    set $\alpha_j := \bar{c}_j^3 - \bar{c}_j^1$ and $\beta_j := \bar{c}_j^2 - \bar{c}_j^1$ for each $j \in J_N$;
    $\overline{J} := \{j \in J_N \mid \beta_j > 0\}$;
    SEARCH_EDGE;
    temp $:= \mathbf{c}^1$ $\mathbf{c}^1 := \mathbf{c}^2$; $\mathbf{c}^2 := \mathbf{c}^3$; $\mathbf{c}^3 :=$ *temp*
  until $\mathbf{c}^1 =$ *first*
end;

procedure SEARCH_EDGE.
begin
  while $\overline{J} \neq \emptyset$ **do begin**
    $J_c := \arg\max\{\bar{c}_j^1/\beta_j \mid j \in \overline{J}\}$; $J_\alpha := \arg\max\{\alpha_j/\beta_j \mid j \in J_c\}$;
    if $|J_\alpha| = 1$ **then begin**
      perform a pivoting operation which brings a column of $\mathbf{N}$ indexed by $s \in J_\alpha$ into $\mathbf{B}$;
      update $\mathbf{x}^\circ$ according to the new partition $[\mathbf{B}, \mathbf{N}]$ of $\mathbf{A}$;
    end
    else DEGENERACY;
    if $\mathbf{c}^i\mathbf{x}^\circ < z_i^*$ **then** $(\mathbf{x}^*, z_i^*) := (\mathbf{x}^\circ, \mathbf{c}^i\mathbf{x}^\circ)$;
    compute the relative profit $\bar{\mathbf{c}}^i$ with respect to $\mathbf{B}$ for each $i = 1, 2, 3$;

set $\alpha_j := \bar{c}_j^3 - \bar{c}_j^1$ and $\beta_j := \bar{c}_j^2 - \bar{c}_j^1$ for each $j \in J_N$;
$\overline{J} := \{j \in J_N \mid \beta_j > 0\}$;
end;
end;

procedure DEGENERACY.
begin
compute an optimal basic solution $\mathbf{x}^\circ$ of $\mathrm{P}(J_\alpha)$;
if $\mathbf{c}^i\mathbf{x}^\circ < z_i^*$ **then** $(\mathbf{x}^*, z_i^*) := (\mathbf{x}^\circ, \mathbf{c}^i\mathbf{x}^\circ)$;
compute an optimal basic solution $\mathbf{x}^\circ$ of $\mathrm{R}(J_\alpha)$;
update the partition $[\mathbf{B}, \mathbf{N}]$ of $\mathbf{A}$ according to $\mathbf{x}^\circ$;
end;

Although the algorithm requires one to process a number of linear programs, we need not solve all of them from scratch. If $\mathbf{B}$ is an optimal basis of the preceding linear program, it is also a feasible basis of the current one; hence $\mathbf{B}$ would recover optimality within a fewer pivoting operations. It is also worth noting that the minimum values of all criteria $z_i = \mathbf{c}^i\mathbf{x}$, $i = 1, 2, 3$, can be computed simultaneously if we update three incumbents z_i^*, $i = 1, 2, 3$, appropriately in the algorithm.

Theorem 3.3. *The algorithm MIN_CRITERION terminates within a finite number of pivoting operations and yields a globally optimal solution* $\mathbf{x}^*$ *of P.*

Proof: In each of the three stages, the procedure SEARCH_EDGE generates a sequence of adjacent bases $\mathbf{B}_{(1)}, \mathbf{B}_{(2)}, \ldots$, such that the associated cones $\Lambda[\mathbf{B}_{(k)}]$'s cover a boundary edge of Δ. Some of $\mathbf{B}_{(k)}$'s might be dual degenerate and define cones with empty interior. However, the procedure DEGENERACY tests them for optimality and provides a nondegenerate basis in a finite number of pivoting operations. Since each edge of Δ has a finite length, we see from (2.7) in Lemma 2.2 that the number of nondegenerate $\mathbf{B}_{(k)}$'s is finite. Thus, the algorithm inspects all bases that define m.e.e. points in a finite number of pivoting operations. The output $\mathbf{x}^*$ of minimum value is a globally optimal solution of P by Lemma 2.3. ■

4. Computational Results

We will report computational results of testing the algorithm MIN_CRITERION on randomly generated problems in this section.

The feasible set X_E of each test problem was the efficient set of a triobjective linear program:

$$\left|\begin{array}{ll} \text{`maximize'} & \mathbf{z} = (\hat{\mathbf{c}}^1\mathbf{x}, \hat{\mathbf{c}}^2\mathbf{x}, \hat{\mathbf{c}}^3\mathbf{x}) \\ \text{subject to} & \hat{\mathbf{A}}\mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0}, \end{array}\right. \tag{4.1}$$

Table 4.1. Average number of pivoting operations.

$m \times d$	preprocess	stage I	stage II	stage III	total
80×100	126.9	139.4	126.0	124.3	516.6
120×100	194.0	151.8	151.2	135.2	632.2
130×150	256.5	242.1	200.8	215.7	915.1
170×150	274.5	242.0	255.3	174.8	946.6
180×200	382.9	298.6	323.8	255.1	1260
220×200	435.8	347.7	343.0	316.2	1443

Table 4.2. Average CPU time in seconds.

$m \times d$	preprocess	stage I	stage II	stage III	total
80×100	3.152	5.720	5.201	5.152	19.23
120×100	12.31	14.94	14.66	13.33	55.24
130×150	21.38	32.07	26.60	28.61	108.7
170×150	42.68	56.34	58.79	40.72	198.5
180×200	71.00	84.37	91.65	72.63	319.7
220×200	131.7	150.7	148.1	136.7	567.2

where $\hat{\mathbf{A}} \in \mathbb{R}^{m \times d}$, $\mathbf{b} \in \mathbb{R}^m$ and $\hat{\mathbf{c}}^i \in \mathbb{R}^d$, $i = 1, 2, 3$. All data of $\hat{\mathbf{c}}^i$'s and $\hat{\mathbf{A}}$ were drawn randomly from integers uniformly distributed on the interval $[1, 100]$; and those of $\mathbf{b}$ were fixed at 100. The size of (m, d) ranged from $(80, 100)$ to $(220, 200)$. If we introduce slack variables, (4.1) reduces to MP of size $(m, n, p) = (m, d + m, 3)$; hence (m, n) was between $(80, 180)$ and $(220, 420)$. For each (m, d), ten bounded instances were selected and composed a subclass of P:

$$\left| \begin{array}{ll} \text{minimize} & z_i = \hat{\mathbf{c}}^i \mathbf{x} \\ \text{subject to} & \mathbf{x} \in X_E. \end{array} \right. \tag{4.2}$$

The algorithm MIN_CRITERION was coded in double precision C language and solved (4.2) on a Unix workstation (HyperSPARC, 150MHz).

Table 4.1 shows the average numbers of pivoting operations taken in three stages (*I*, *II* and *III*) for each size (m, d). It also contains the number spent for solving (3.10) (*preprocess*) and the total (*total*). Table 4.2 shows the average CPU times in seconds, in the same manner. We can roughly estimate that *preprocess* indicates the computational time needed for solving a linear program of size (m, d). According to this estimation, the tables tell that the algorithm MIN_CRITERION solves a randomly generated class

Table 4.3. Relative frequency of finding $\mathbf{x}^*$.

$m \times d$	$\min \hat{\mathbf{c}}^1\mathbf{x}$ stage I	II	III	$\min \hat{\mathbf{c}}^2\mathbf{x}$ stage I	II	III	$\min \hat{\mathbf{c}}^3\mathbf{x}$ stage I	II	III
80×100	0.5	0.3	0.2	0.5	0.2	0.3	0.7	0.2	0.1
120×100	0.3	0.7	0.0	0.6	0.1	0.3	1.0	0.0	0.0
130×150	0.4	0.5	0.1	0.4	0.1	0.5	0.9	0.1	0.0
170×150	0.2	0.8	0.0	0.6	0.2	0.2	0.8	0.2	0.0
180×200	0.5	0.5	0.0	0.7	0.2	0.1	0.6	0.3	0.1
220×200	0.2	0.8	0.0	0.4	0.2	0.4	0.8	0.2	0.0

(4.2) in almost the same computational time as needed for solving four linear programs.

Table 4.3 shows the relative frequency of finding the output $\mathbf{x}^*$ in each stage. We see from it that each of the three stages can provide optimal solutions whichever criterion is minimized in (4.2). Therefore, we can never omit any stages to obtain a globally optimal solution of P.

Since all the test problems were nondegenerate, the performance of the procedure DEGENERACY is still open. Hence, we cannot make a final conclusion about the computational properties of the algorithm MIN_CRITERION. However, it would serve as a practical approach to global optimization at least for degeneracy-free problems.

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