Generalized (C,A,B)-pair and Robust Disturbance-Rejection Problem with Dynamic Compensator

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June 2, 1998
ISE-TR-98-153

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Abstract

In the so-called geometric approach, the notion of generalized (C, A, B)-pair is introduced for uncertain linear systems and its properties are investigated. Further, the robust disturbance-rejection problem with dynamic compensator is formulated and some sufficient conditions for the problem to be solvable are presented.

Keywords : Generalized (C, A, B)-pair, Uncertain Linear Systems, Robust Disturbance-Rejection, Dynamic Compensator, Geometric Approach

1 Introduction

The notion of (A, B)-invariance was first studied independently by Basil and Marro[1] and Wonham[6], respectively and various control problems have been studied. Further, the notion of (C, A)-invariance was studied by Basil and Marro[1], and then the notion of (C, A, B)-pair was first introduced by Schumacher[5] and this concept has been successfully studied to design dynamic compensators.
After that, simultaneous versions of (A, B)-invariance, (C, A)-invariance and (C, A, B)-pair were investigated by Ghosh[3] and disturbance-rejection problems for uncertain linear systems in the sense that system's matrices are represented as convex combinations of given matrices were investigated. Further, the notions of generalized (A, B)-invariance and generalized (C, A)-invariance were studied by Bhattacharyya[2] and the present author[4], respectively and robust disturbance-rejection problems with state feedback and with output feedback were investigated.

In this paper, the notion of generalized (C, A, B)-pair is introduced and its properties are investigated. Further, a robust disturbance-rejection problem with dynamic compensator for uncertain linear system which was investigated by Bhattacharyya[2] and Otsuka[4] is formulated, and some sufficient conditions for the problem to be solvable are presented. Finally, an illustrative example is given.

2 Generalized Invariant Subspaces

First, we give some notations used throughout this investigation. For a linear map A from a vector space X into a vector space Y and a subspace φ of Y the image, the kernel, the inverse image and the dimension are denoted by $\mathrm{Im}(A)$, $\mathrm{Ker}(A)$, $A^{-1}\varphi := \{x \in X \mid Ax \in \varphi\}$ and $\dim(\varphi)$, respectively. Further, the direct sum, the orthogonal complement and the identity operator on $\boldsymbol{R}^n$ are denoted by $\oplus$, $(\cdot)^\perp$ and I_n, respectively. For vectors $\{v_1, \cdots, v_k\}$ $\mathrm{span}\{v_1, \cdots, v_k\}$ means the linear subspace generated by the vectors $\{v_1, \cdots, v_k\}$.

Next, consider the following linear systems defined in an Euclidean space $X := \boldsymbol{R}^n$:

$$S(\alpha,\beta,\gamma):\begin{cases}\dfrac{d}{dt}x(t)=A(\alpha)x(t)+B(\beta)u(t),\\ y(t)=C(\gamma)x(t),\end{cases}$$

where $x(t)\in X, u(t)\in U:=\boldsymbol{R}^m, y(t)\in Y:=\boldsymbol{R}^\ell$ are the state, the input and the measurement output, respectively. And coefficient matrices $A(\alpha)$, $B(\beta)$ and $C(\gamma)$ have unknown parameters in the sense that

$$\begin{aligned}A(\alpha)&=A_0+\alpha_1A_1+\cdots+\alpha_pA_p:=A_0+\Delta A(\alpha)\in\boldsymbol{R}^{n\times n}\\ B(\beta)&=B_0+\beta_1B_1+\cdots+\beta_qB_q:=B_0+\Delta B(\beta)\in\boldsymbol{R}^{n\times m}\\ C(\gamma)&=C_0+\gamma_1C_1+\cdots+\gamma_rC_r:=C_0+\Delta C(\gamma)\in\boldsymbol{R}^{\ell\times n},\end{aligned}$$

where $\alpha:=(\alpha_1,\cdots,\alpha_p)\in\boldsymbol{R}^p$, $\beta:=(\beta_1,\cdots,\beta_q)\in\boldsymbol{R}^q$, $\gamma:=(\gamma_1,\cdots,\gamma_r)\in\boldsymbol{R}^r$.
In system $S(\alpha,\beta,\gamma)$ (A_0,B_0,C_0) and $(\Delta A(\alpha),\ \Delta B(\beta),\ \Delta C(\gamma))$ represent the nominal system model and a specific uncertain perturbation, respectively.

2.1 Generalized $(A,\ B)$-invariant Subspaces

In this subsection, a generalized $(A,\ B)$-invariant subspace and its properties which were investigated by Bhattacharyya[2] are summarized.

Definition 2.1 Let $V,\Omega\ (\subset X)$ be subspaces.
V is said to be a generalized $(A,\ B)$-invariant if there exists an $F\in\boldsymbol{R}^{m\times n}$ such that

$$(A(\alpha)+B(\beta)F)V\subset V$$

for all $(\alpha,\beta)\in\boldsymbol{R}^p\times\boldsymbol{R}^q$. Further, define

$\boldsymbol{F}(V):=\{F\in\boldsymbol{R}^{m\times n}\mid(A(\alpha)+B(\beta)F)V\subset V$ for all $(\alpha,\beta)\in\boldsymbol{R}^p\times\boldsymbol{R}^q\}$ and
$\boldsymbol{V}(A,B;\Omega):=\{V(\subset\Omega)\mid^\exists F\in\boldsymbol{R}^{m\times n}$ s.t. $(A(\alpha)+B(\beta)F)V\subset V$ for all $(\alpha,\beta)\in\boldsymbol{R}^p\times\boldsymbol{R}^q\}$. ∎

For a subspace V of X define a subspace R_V of $\boldsymbol{R}^m$ and a linear map Q_V on $\boldsymbol{R}^m$ by

$$R_V:=\bigcap_{i=1}^{q}B_i^{-1}V\ \text{ and }\ Q_V:=\boldsymbol{R}^m\ \rightarrow\ \boldsymbol{R}^m,\ \text{a projection map onto } R_V \text{ along } (R_V)^\perp,\ \text{respectively.}$$

Then, the following lemma was proved.

Lemma 2.2 [2] The following three statements are equivalent.
(i) $V\in\boldsymbol{V}(A,B;\Omega)$.
(ii) There exists an $F\in\boldsymbol{R}^{m\times n}$ such that $(A_0+B_0F)V\subset V$ and $B_iFV\subset V$ $(i=1,\cdots,q)$, and $A_iV\subset V\subset\Omega$ $(i=1,\cdots,p)$.
(iii) $A_0V\subset\mathrm{Im}(B_0Q_V)+V$ and $A_iV\subset V\subset\Omega$ $(i=1,\cdots,p)$. ∎

The following lemma gives the computational algorithm of a unique maximal element of $\boldsymbol{V}(A,B;\Omega)$.

Lemma 2.3 [2] $\boldsymbol{V}(A,B;\Omega)$ has a unique maximal element $V^*(\Omega)$ which may be calculated from the following algorithm.

Step1. $V_0:=\Omega$.

Step2. $R_k := \bigcap_{i=1}^{q} B_i^{-1}V_k \ (\subset \boldsymbol{R}^m)$, where $B_i^{-1}V_k := \{u \in \boldsymbol{R}^m \mid B_i u \in V_k\}$ $(k \geq 0)$.

Step3. $Q_k := \boldsymbol{R}^m \rightarrow \boldsymbol{R}^m$, a projection map onto R_k along $(R_k)^{\perp}$ $(k \geq 0)$.

Step4. $B_{0k} := B_0 Q_k$ $(k \geq 0)$.

Step5. $V_{k+1} := V_k \cap A_0^{-1}(\mathrm{Im}B_{0k} + V_k) \cap A_1^{-1}V_k \cap \cdots \cap A_p^{-1}V_k$ $(k \geq 0)$.

Step6. $V^*(\Omega) := V_n$. ■

2.2 Generalized $(C,\ A)$-invariant Subspaces

In this subsection, a generalized $(C,\ A)$-invariant subspace and its properties which were investigated by the present author[4] are summarized.

Definition 2.4 Let $V, \varepsilon\ (\subset X)$ be subspaces.

V is said to be a generalized $(C,\ A)$-invariant if there exists a $G \in \boldsymbol{R}^{n\times\ell}$ such that

$$(A(\alpha) + GC(\gamma))V \subset V$$

for all $(\alpha, \gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^r$. Further, define

$\boldsymbol{G}(V) := \{G \in \boldsymbol{R}^{n\times\ell} \mid (A(\alpha) + GC(\gamma))V \subset V$ for all $(\alpha,\gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^r\}$ and

$\boldsymbol{V}(C, A; \varepsilon) := \{V(\supset \varepsilon) \mid^{\exists} G \in \boldsymbol{R}^{n\times\ell}$ s.t. $(A(\alpha) + GC(\gamma))V \subset V$ for all $(\alpha,\gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^r\}$. ■

For a subspace V of X let P_V be a linear map on $\boldsymbol{R}^\ell$ satisfying $\mathrm{Ker}P_V = \sum_{i=1}^{r} C_i V$ and $V = \phi \oplus (V \cap \mathrm{Ker}(P_V C_0))$ for some subspace ϕ. Since, $C_0\phi \cap \mathrm{Ker}P_V = \{0\}$, we can define a projection map $P_V\ :\ \boldsymbol{R}^\ell \rightarrow \boldsymbol{R}^\ell$ onto $C_0\phi \oplus \Gamma$ along $\mathrm{Ker}P_V = \sum_{i=1}^{r} C_i V$ for some subspace Γ satisfying $V = \phi \oplus (V \cap \mathrm{Ker}(P_V C_0))$.

Then, the following lemma was proved.

Lemma 2.5 [4] The following three statements are equivalent.

(i) $V \in \boldsymbol{V}(C, A; \varepsilon)$.

(ii) There exists a $G \in \boldsymbol{R}^{n\times\ell}$ such that $(A_0 + GC_0)V \subset V$ and $GC_iV \subset V$ $(i = 1, \cdots, r)$, $A_iV \subset V$ $(i = 1, \cdots, p)$ and $\varepsilon \subset V$.

(iii) $A_0(V \cap \mathrm{Ker}(P_V C_0)) \subset V$, $A_iV \subset V$ $(i = 1, \cdots, p)$ and $\varepsilon \subset V$. ■

The following lemma gives the computational algorithm of a unique minimal element of $\boldsymbol{V}(C, A; \varepsilon)$.

Lemma 2.6 [4] $\boldsymbol{V}(C, A; \varepsilon)$ has a unique minimal element $V_*(\varepsilon)$ which may be calculated from the following algorithm.

Step1. $V_0 := \varepsilon$.

Step2. $P_k := \boldsymbol{R}^\ell \rightarrow \boldsymbol{R}^\ell$, a projection map onto $C_0\phi_k \oplus \Gamma_k$ along $\mathrm{Ker}P_k$ for some Γ_k such that

$$\mathrm{Ker}P_k = \sum_{i=1}^{r} C_i V_k \text{ and } V_k = \phi_k \oplus (V_k \cap \mathrm{Ker}(P_k C_0)) \quad (k \geq 0).$$

Step3. $C_{0k} := P_k C_0$ $(k \geq 0)$.

Step4. $V_{k+1} := V_k + A_0(\mathrm{Ker}C_{0k} \cap V_k) + A_1 V_k + \cdots + A_p V_k$ $(k \geq 0)$.

Step5. $V_*(\varepsilon) := V_n$. ■

3 Generalized (C, A, B)-pair

In this section, the notion of generalized (C, A, B)-pair which is a generalization of (C, A, B)-pair investigated by Schumacher[5] is introduced and its properties are investigated.

Consider the system $S(\alpha, \beta, \gamma)$ in Section 2.
Now, introduce a compensator (K, L, M, N) defined in $W := \boldsymbol{R}^w$ of the following form :

$$\begin{cases} \dfrac{d}{dt}w(t) = Nw(t) + My(t), \\ u(t) = Lw(t) + Ky(t), \end{cases} \tag{1}$$

where $N \in \boldsymbol{R}^{w \times w}$, $M \in \boldsymbol{R}^{w \times \ell}$, $L \in \boldsymbol{R}^{m \times w}$ and $K \in \boldsymbol{R}^{m \times \ell}$.

If a compensator of the form (1) is applied to system $S(\alpha, \beta, \gamma)$, the resulting closed-loop system with the extended state space $X^e := X \oplus W$ is easily seen to be

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} A(\alpha) + B(\beta)KC(\gamma) & B(\beta)L \\ MC(\gamma) & N \end{bmatrix}\begin{bmatrix} x(t) \\ w(t) \end{bmatrix}. \tag{2}$$

For the combined system (2), define

$$x^e(t) := \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \text{ and } A^e(\alpha, \beta, \gamma) := \begin{bmatrix} A(\alpha) + B(\beta)KC(\gamma) & B(\beta)L \\ MC(\gamma) & N \end{bmatrix}. \tag{3}$$

Now, the difinition of a generalized (C, A, B)-pair is given.

Definition 3.1 Let V_1 and V_2 be subspaces of X. A pair (V_1, V_2) is said to be a generalized (C, A, B)-pair if the following three conditions are satisfied.

(i) V_1 is a generalized (C, A)-invariant.

(ii) V_2 is a generalized (A, B)-invariant.

(iii) $V_1 \subset V_2$. ∎

For a closed-loop system (2) with (3), we give the following definitions.

Definition 3.2 Let V^e be a subspace of X^e. V^e is said to be an $A^e(\alpha, \beta, \gamma)$-invariant if $A^e(\alpha, \beta, \gamma)V^e \subset V^e$ for all $(\alpha, \beta, \gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^q \times \boldsymbol{R}^r$. ∎

Definition 3.3 Let V^e be a subspace of X^e. The following two subspaces are defined:

$$V_p := \left\{ x \in X \middle| \begin{bmatrix} x \\ w \end{bmatrix} \in [V^e] \text{ for some } w \in W \right\} = P_X([V^e]) \text{ and}$$

$$V_s := \left\{ x \in X \middle| \begin{bmatrix} x \\ 0 \end{bmatrix} \in V^e \right\},$$

where P_X is the projection map from X^e onto X along W. ∎

The following lemma is used to prove Lemma 3.5.

Lemma 3.4 If a pair (V_1, V_2) of subspaces of X is a generalized (C, A, B)-pair such that

$$\sum_{i=1}^{q} \text{Im} B_i \subset V_1 \subset V_2 \subset \bigcap_{i=1}^{r} \text{Ker} C_i,$$

then there exist $G \in \boldsymbol{G}(V_1)$, $G(\beta) \in \boldsymbol{R}^{n\times\ell}$, $F(\gamma) \in \boldsymbol{F}(V_2)$, $F_0 \in \boldsymbol{R}^{m\times n}$ and $K \in \boldsymbol{R}^{m\times\ell}$ such that

$$\begin{cases} G = B(\beta)K + G(\beta), \ \ \mathrm{Im}G(\beta) \subset V_2 \\ F(\gamma) = KC(\gamma) + F_0, \ \ \mathrm{Ker}F_0 \supset V_1 \ \text{ for all } (\beta,\gamma) \in \boldsymbol{R}^q \times \boldsymbol{R}^r. \end{cases}$$

Proof. Suppose that a pair (V_1, V_2) is a generalized (C, A, B)-pair satisfying the stated above conditions. Since $\sum_{i=1}^{q} \mathrm{Im}B_i \subset V_2$, it remarks that $V_2 + \mathrm{Im}B(\beta) = V_2 + \mathrm{Im}B_0$.

Claim 1: $\hat{G}C(\gamma)V_1 \subset V_2 + \mathrm{Im}B_0$ for all $\hat{G} \in \boldsymbol{G}(V_1)$ and $\gamma \in \boldsymbol{R}^r$.
To prove Claim 1, choose an arbitrary element $x \in V_1$. Then,

$$\begin{aligned} \hat{G}C(\gamma)x &= (A(\alpha) + \hat{G}C(\gamma))x - A(\alpha)x \\ &\in V_1 + V_2 + \mathrm{Im}B(\beta) \\ &\in V_2 + \mathrm{Im}B(\beta), \\ &= V_2 + \mathrm{Im}B_0, \end{aligned}$$

which proves Claim 1.

Next, the following two Claims 2 and 3 hold.
Claim 2: There exists a $G \in \boldsymbol{G}(V_1)$ such that $\mathrm{Im}G \subset V_2 + \mathrm{Im}B_0$.
To prove Claim 2, choose a $\hat{G} \in \boldsymbol{G}(V_1)$ and $x\ (= y + z) \in \boldsymbol{R}^\ell$ such that $y \in \sum_{i=0}^{r} C_i V_1$ and $z \in (\sum_{i=0}^{r} C_i V_1)^\perp$. Define a linear map $G \in \boldsymbol{R}^{n\times\ell}$ by $Gx := \hat{G}y$. Then, for some $x_i \in V_1$

$$\begin{aligned} Gx &= \hat{G}y = \sum_{i=0}^{r} \hat{G}C_i x_i \\ &\in V_2 + \mathrm{Im}B_0, \ \ \text{(by Claim 1)} \end{aligned}$$

which proves Claim 2.

Claim 3: There exists a $K \in \boldsymbol{R}^{m\times\ell}$ and $G(\beta) \in \boldsymbol{R}^{n\times\ell}$ such that $G = B(\beta)K + G(\beta)$, $\mathrm{Im}G(\beta) \subset V_2$.
To prove Claim 3, let $\{y_1, \cdots, y_\ell\}$ be a basis of $\boldsymbol{R}^\ell$. Then, it follows from Claim 2 that there exists an $x_i \in V_2$ and $u_i \in \boldsymbol{R}^m$ such that

$$\begin{aligned} Gy_i &= x_i + B_0 u_i \\ &= x_i - \sum_{i=1}^{q} \beta_i B_i u_i + B_0 u_i + \sum_{i=1}^{q} \beta_i B_i u_i \\ &= (x_i - \sum_{i=1}^{q} \beta_i B_i u_i) + B(\beta) u_i. \end{aligned}$$

Define linear maps $K \in \boldsymbol{R}^{m\times\ell}$ and $G(\beta) \in \boldsymbol{R}^{n\times\ell}$ by

$$Ky_i := u_i \ \text{ and } \ G(\beta)y_i := x_i - \sum_{i=1}^{q} \beta_i B_i u_i, \text{respectively.}$$

Then,

$$Gy_i = G(\beta)y_i + B(\beta)Ky_i \text{ and } G(\beta)y_i \in V_2,$$

which proves Claim 3.

Claim 4: $(A(\alpha)+GC(\gamma))V_2 \subset V_2 + \mathrm{Im}B_0$ for all $(\alpha,\gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^r$.
In fact,

$$\begin{aligned}(A(\alpha)+GC(\gamma))V_2 &= A(\alpha)V_2 + GC(\gamma)V_2 \\ &\subset V_2 + \mathrm{Im}B(\beta) + \mathrm{Im}G \\ &\subset V_2 + \mathrm{Im}B(\beta) + V_2 + \mathrm{Im}B_0 \quad \text{(by Claim 2)} \\ &= V_2 + \mathrm{Im}B_0,\end{aligned}$$

which proves Claim 4.

Now, choose parameters $\alpha_1 = \alpha_2 = \cdots = \alpha_p = \gamma_1 = \gamma_2 = \cdots = \gamma_r = 0$ in Claim 4. Then,

$$(A_0 + GC_0)V_2 \subset V_2 + \mathrm{Im}B_0,$$

which means V_2 is $((A_0+GC_0), B_0)$-invariant. Hence, there exists an $F \in \boldsymbol{R}^{m\times n}$ such that

$$(A_0 + GC_0 + B_0F)V_2 \subset V_2. \tag{4}$$

Now, let $\{v_1, \cdots, v_s\}$ be a basis of V_1 and $\{v_{s+1}, \cdots, v_t\}$ be a basis of $V_2 \cap V_1^{\perp}$. Then, define a linear map $F_0 \in \boldsymbol{R}^{m\times n}$ astisfying

$$\begin{cases} F_0 v_i = 0 \quad (i = 1, \cdots, s), \\ F_0 v_i = F v_i \quad (i = s+1, \cdots, t). \end{cases}$$

Then, the following claim holds.

Claim 5: $(A(\alpha)+GC(\gamma)+B(\beta)F_0)V_2 \subset V_2$ for all $(\alpha,\beta,\gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^q \times \boldsymbol{R}^r$ and $V_1 \subset \mathrm{Ker}F_0$.
Since $V_1 \subset \mathrm{Ker}F_0$ is obvious, we prove the first one. Choose an arbitrary element $x\ (= y+z) \in V_2$ $(y \in V_1, z \in V_2 \cap V_1^{\perp})$. Then,

$$\begin{aligned}&(A(\alpha)+GC(\gamma)+B(\beta)F_0)x \\ &= (A(\alpha)+GC(\gamma))y + (A(\alpha)+GC(\gamma)+B(\beta)F_0)z \\ &= (A(\alpha)+GC(\gamma))y + A_0 z + \sum_{i=1}^{p}\alpha_i A_i z + GC_0 z + G\sum_{i=1}^{r}\gamma_i C_i z + B_0F_0 z + \sum_{i=1}^{q}\beta_i B_i F_0 z \\ &= (A(\alpha)+GC(\gamma))y + (A_0+GC_0+B_0F)z + \sum_{i=1}^{p}\alpha_i A_i z + \sum_{i=1}^{q}\beta_i B_i F z \quad (\text{by } V_2 \subset \bigcap_{i=1}^{r}\mathrm{Ker}C_i) \\ &\in V_2 \quad (\text{by } G \in \boldsymbol{G}(V_1), A_i V_2 \subset V_1\ (i=1,\cdots,p), \sum_{i=1}^{q}\mathrm{Im}B_i \subset V_1 \text{ and } (4)),\end{aligned}$$

which proves Claim 5.

Finally, define $F(\gamma) := KC(\gamma) + F_0$.

Then, the following claim holds.

Claim 6: $F(\gamma) \in \boldsymbol{F}(V_2)$.

In fact,

$$\begin{aligned}(A(\alpha)+B(\beta)F(\gamma))V_2 &= (A(\alpha)+B(\beta)KC(\gamma)+B(\beta)F_0)V_2 \\ &= \{A(\alpha) + (B(\beta)K + G(\beta))C(\gamma) - G(\beta)C(\gamma) + B(\beta)F_0\}V_2\end{aligned}$$

$$
\begin{aligned}
&= \{A(\alpha) + GC(\gamma) + B(\beta)F_0 - G(\beta)C(\gamma)\}V_2 \quad \text{(by Claim 3)} \\
&\subset (A(\alpha) + GC(\gamma) + B(\beta)F_0)V_2 + \mathrm{Im}G(\beta) \\
&\subset V_2 \ \text{ (by Claim 3 and Claim 5)},
\end{aligned}
$$

which proves Claim 6.

This completes the proof of Lemma 3.4. ■

The following lemma is used to prove Theorem 4.2.

Lemma 3.5 If a pair (V_1, V_2) of subspaces of X is a generalized (C, A, B)-pair such that

$$
\sum_{i=1}^{q} \mathrm{Im}B_i \subset V_1 \subset V_2 \subset \bigcap_{i=1}^{r} \mathrm{Ker}C_i \text{ and } A_i V_2 \subset V_1 \ (i = 1, \cdots, p),
$$

then there exist a compensator (K, L, M, N) on $W := \boldsymbol{R}^w$ and a subspace V^e of X^e such that $V_1 = V_s$, $V_2 = V_p$ and V^e is $A^e(\alpha, \beta, \gamma)$-invariant, where $w := \mathrm{dim}V_2 - \mathrm{dim}V_1$.

Proof. Suppose that there exists a pair (V_1, V_2) of subspaces satisfying the stated above conditions. Let $\{v_1, \cdots, v_s\}$, $\{v_1, \cdots, v_s, v_{s+1}, \cdots, v_{s+w}\}$ and $\{y_1, \cdots, y_w\}$ be bases of V_1, V_2 and $W = \boldsymbol{R}^w$, respectively. Define a linear map R from V_2 to $\boldsymbol{R}^w$ by

$$
\begin{cases}
Rv_i = 0 \quad (i = 1, \cdots, s), \\
Rv_i = y_{i-s} \quad (i = s+1, \cdots, s+w).
\end{cases}
$$

Then, $\mathrm{Ker}R = V_1$. Define a subspace V^e of $X \oplus \boldsymbol{R}^w$ by

$$
V^e := \left\{ \begin{bmatrix} x \\ Rx \end{bmatrix} \mid x \in V_2 \right\}.
$$

Then, it is obvious that $V_p = V_2$ and $V_s = V_1$. Since (V_1, V_2) is a generalized (C, A, B)-pair, it follows from Lemma 3.4 that there exist $G \in \boldsymbol{G}(V_1)$, $G(\beta) \in \boldsymbol{R}^{n \times \ell}$, $F(\gamma) \in \boldsymbol{F}(V_2)$, $F_0 \in \boldsymbol{R}^{m \times n}$ and $K \in \boldsymbol{R}^{m \times \ell}$ such that

$$
\begin{cases}
G = B(\beta)K + G(\beta), \ \ \mathrm{Im}G(\beta) \subset V_2 \\
F(\gamma) = KC(\gamma) + F_0, \ \mathrm{Ker}F_0 \supset V_1 \ \ \text{for all } (\beta, \gamma) \in \boldsymbol{R}^q \times \boldsymbol{R}^r.
\end{cases}
$$

Let $F_0 |_{V_2}$ be a restriction of F_0 to V_2. Then, since $\mathrm{Ker}R = V_1 \subset \mathrm{Ker}F_0 |_{V_2}$ there exists a linear map $L \in \boldsymbol{R}^{m \times w}$ such that $LR = F_0 |_{V_2}$. Further, for all $(\alpha, \beta, \gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^q \times \boldsymbol{R}^r$

$$
\begin{aligned}
(A(\alpha) + B(\beta)F_0 + GC(\gamma))V_1 &= (A(\alpha) + GC(\gamma))V_1 \\
&\subset V_1
\end{aligned}
$$

and

$$
\begin{aligned}
(A(\alpha) + B(\beta)F_0 + GC(\gamma))V_2 &= (A(\alpha) + B(\beta)(F_0 + KC(\gamma)) + G(\beta)C(\gamma))V_2 \\
&\subset (A(\alpha) + B(\beta)F(\gamma))V_2 + G(\beta)C(\gamma)V_2 \\
&\subset V_2,
\end{aligned}
$$

which imply $(A_0 + B_0F_0 + GC_0)V_i \subset V_i$ $(i = 1, 2)$.

Then, since $\mathrm{Ker}R\ (= V_1) \subset \mathrm{Ker}\{R(A_0 + B_0F_0 + GC_0)\mid_{V_2}\}$, there exists an $N \in \boldsymbol{R}^{w\times w}$ such that

$$\begin{aligned} NR &= R(A_0 + B_0F_0 + GC_0)\mid_{V_2} \\ &= R(A_0 + B_0F(0) + G(0)C_0)\mid_{V_2}. \end{aligned} \tag{5}$$

Now, noticing that $\mathrm{Im}G(0) \subset V_2$, define a linear map $M \in \boldsymbol{R}^{w\times \ell}$ by $M := -RG(0)$.

Then, we have the following Claim.

Claim 1 $R(A(\alpha) + B(\beta)F(\gamma))x = R(A_0 + B_0F(0))x$ for all $x \in V_2$ and $(\alpha, \beta, \gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^q \times \boldsymbol{R}^r$.

To prove Claim 1, let x be an arbitrary element of V_2. Then,

$$\begin{aligned} &R(A(\alpha) + B(\beta)F(\gamma))x - R(A_0 + B_0F(0))x \\ &= R(A(\alpha) - A_0 + B(\beta)F(\gamma) - B_0F(0))x \\ &= R(\alpha_1A_1 + \cdots + \alpha_pA_p)x + R\{B(\beta)(KC(\gamma) + F_0) - B_0(KC_0 + F_0)\}x \\ &= R\{B(\beta)KC(\gamma) - B_0KC_0\}x + R(B(\beta) - B_0)F_0x \quad (\text{by } A_iV_2 \subset V_1 = \mathrm{Ker}R\ (i = 1, \cdots, p)) \\ &= 0 \quad (\text{by } \sum_{i=1}^{q}\mathrm{Im}B_i \subset V_1 = \mathrm{Ker}R \text{ and } V_2 \subset \bigcap_{i=1}^{r}\mathrm{Ker}C_i), \end{aligned}$$

which proves Claim 1.

Next, we have the following Claim.

Claim 2 $(MC(\gamma) + NR)x = R(A(\alpha) + B(\beta)F(\gamma))x$ for all $x \in V_2$ and $(\alpha, \beta, \gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^q \times \boldsymbol{R}^r$.

To prove Claim 2, let x be an arbitrary element of V_2. Then,

$$\begin{aligned} (MC(\gamma) + NR)x &= (MC_0 + NR)x \quad (\text{by } V_2 \subset \bigcap_{i=1}^{r}\mathrm{Ker}C_i) \\ &= (-RG(0)C_0 + R(A_0 + B_0F(0) + G(0)C_0))x \quad (\text{by (5)}) \\ &= R(A_0 + B_0F(0))x \\ &= R(A(\alpha) + B(\beta)F(\gamma))x \quad (\text{by Claim 1}), \end{aligned}$$

which proves Claim 2.

Hence, for an arbitrary element $\begin{bmatrix} x \\ Rx \end{bmatrix}$ of V^e

$$\begin{aligned} \begin{bmatrix} A(\alpha) + B(\beta)KC(\gamma) & B(\beta)L \\ MC(\gamma) & N \end{bmatrix} \begin{bmatrix} x \\ Rx \end{bmatrix} &= \begin{bmatrix} (A(\alpha) + B(\beta)KC(\gamma))x + B(\beta)LRx \\ (MC(\gamma) + NR)x \end{bmatrix} \\ &= \begin{bmatrix} (A(\alpha) + B(\beta)(KC(\gamma) + F_0))x \\ R(A(\alpha) + B(\beta)F(\gamma))x \end{bmatrix} \quad (\text{by Claim 2}) \\ &= \begin{bmatrix} (A(\alpha) + B(\beta)F(\gamma))x \\ R(A(\alpha) + B(\beta)F(\gamma))x \end{bmatrix} \\ &\in V^e \text{ for all } (\alpha, \beta, \gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^q \times \boldsymbol{R}^r, \end{aligned}$$

which proves that V^e is $A^e(\alpha, \beta, \gamma)$-invariant.

This completes the proof of this lemma. ■

4 Robust Disturbance-Rejection

In this section, the robust disturbance-rejection problem with dynamic compensator is studied. Consider the system $S(\alpha,\beta,\gamma)$ in Section 2 with the controlled output $z(t)$ and disturbance $\xi(t)$ as follows.

$$S(\alpha,\beta,\gamma,\delta,\sigma):\begin{cases}\dfrac{d}{dt}x(t)=A(\alpha)x(t)+B(\beta)u(t)+E(\sigma)\xi(t),\\ y(t)=C(\gamma)x(t),\\ z(t)=D(\delta)x(t),\end{cases}$$

where $x(t)\in X:=\boldsymbol{R}^n, u(t)\in U:=\boldsymbol{R}^m, y(t)\in Y:=\boldsymbol{R}^\ell$ and $z(t)\in Z:=\boldsymbol{R}^\mu$ are the state, the input, the measurement output and the controlled output, respectively, $\xi(t)\in\boldsymbol{R}^\eta$ is the disturbance which can not be measured by controller. It is assumed that coefficient matrices have the following unknown parameters.

$$\begin{aligned}A(\alpha)&=A_0+\alpha_1A_1+\cdots+\alpha_pA_p:=A_0+\Delta A(\alpha)\in\boldsymbol{R}^{n\times n},\\ B(\beta)&=B_0+\beta_1B_1+\cdots+\beta_qB_q:=B_0+\Delta B(\beta)\in\boldsymbol{R}^{n\times m},\\ C(\gamma)&=C_0+\gamma_1C_1+\cdots+\gamma_rC_r:=C_0+\Delta C(\gamma)\in\boldsymbol{R}^{\ell\times n},\\ D(\delta)&=D_0+\delta_1D_1+\cdots+\delta_sD_s:=D_0+\Delta D(\delta)\in\boldsymbol{R}^{\mu\times n},\\ E(\sigma)&=E_0+\sigma_1E_1+\cdots+\sigma_tE_t:=E_0+\Delta E(\sigma)\in\boldsymbol{R}^{n\times\eta},\end{aligned}$$

where $\alpha:=(\alpha_1,\cdots,\alpha_p)\in\boldsymbol{R}^p$, $\beta:=(\beta_1,\cdots,\beta_q)\in\boldsymbol{R}^q$, $\gamma:=(\gamma_1,\cdots,\gamma_r)\in\boldsymbol{R}^r$, $\delta:=(\delta_1,\cdots,\delta_s)\in\boldsymbol{R}^s$, $\sigma:=(\sigma_1,\cdots,\sigma_t)\in\boldsymbol{R}^t$.

In system $S(\alpha,\beta,\gamma,\delta,\sigma)$ (A_0,B_0,C_0,D_0,E_0) and $(\Delta A(\alpha), \Delta B(\beta), \Delta C(\gamma), \Delta D(\delta), \Delta E(\sigma))$ represent the nominal system model and a specific uncertain perturbation, respectively.

Now, we apply to system $S(\alpha,\beta,\gamma,\delta,\sigma)$ a compensator (K,L,M,N) in Section 3. Then, the closed-loop system $S_{cl}(\alpha,\beta,\gamma,\delta,\sigma)$ is given by

$$\begin{cases}\dfrac{d}{dt}\begin{bmatrix}x(t)\\ w(t)\end{bmatrix}=\begin{bmatrix}A(\alpha)+B(\beta)KC(\gamma) & B(\beta)L\\ MC(\gamma) & N\end{bmatrix}\begin{bmatrix}x(t)\\ w(t)\end{bmatrix}+\begin{bmatrix}E(\sigma)\\ 0\end{bmatrix}\xi(t),\\ z(t)=\begin{bmatrix}D(\delta) & 0\end{bmatrix}\begin{bmatrix}x(t)\\ w(t)\end{bmatrix}.\end{cases}$$

The robust disturbance-rejection problem with dynamic compensator (RDRPDC) for system $S(\alpha, \beta, \gamma, \delta, \sigma)$ is stated as follows : Given matrices A_i, B_i, C_i, D_i, E_i for system $S(\alpha,\beta,\gamma,\delta,\sigma)$, find if possible a dynamic compensator (K,L,M,N) such that the closed-loop system $S_{cl}(\alpha,\beta,\gamma,\delta,\sigma)$ rejects the disturbances ξ from the controlled output z for all parameters $(\alpha,\beta,\gamma,\delta,\sigma)\in\boldsymbol{R}^p\times\boldsymbol{R}^q\times\boldsymbol{R}^r\times\boldsymbol{R}^s\times\boldsymbol{R}^t$.

For convenience, define

$$E^e:=\begin{bmatrix}E(\sigma)\\ 0\end{bmatrix}\quad\text{and}\quad D^e:=\begin{bmatrix}D(\delta) & 0\end{bmatrix}.$$

Then, the problem can be rephrased as follows.

Problem 4.1 (RDRPDC) Given matrices A_i, B_i, C_i, D_i, E_i for system $S(\alpha,\ \beta,\ \gamma,\ \delta,\ \sigma)$, find if possible a compensator (K, L, M, N) such that

$$\begin{aligned} < A^e(\alpha,\beta,\gamma)|\mathrm{Im}E^e > &:= \mathrm{Im}E^e + A^e(\alpha,\beta,\gamma)\mathrm{Im}E^e + \cdots + (A^e(\alpha,\beta,\gamma))^{n+w-1}\mathrm{Im}E^e \\ &\subset \mathrm{Ker}D^e \end{aligned}$$

for all parameters $(\alpha,\beta,\gamma,\delta,\sigma) \in \boldsymbol{R}^p \times \boldsymbol{R}^q \times \boldsymbol{R}^r \times \boldsymbol{R}^s \times \boldsymbol{R}^t$. ■

Now, some sufficient conditions for the RDRPDC to be solvable are given.

Theorem 4.2 If there exists a generalized (C, A, B)-pair (V_1, V_2) such that

$$(\sum_{i=1}^{q}\mathrm{Im}B_i + \sum_{i=0}^{t}\mathrm{Im}E_i) \subset V_1 \subset V_2 \subset (\bigcap_{i=1}^{r}\mathrm{Ker}C_i \cap \bigcap_{i=0}^{s}\mathrm{Ker}D_i) \text{ and } A_iV_2 \subset V_1 \ (i = 1,\cdots,p),$$

then the RDRPDC is solvable.

Proof. Suppose that there exists a generalized (C, A, B)-pair (V_1, V_2) satisfying the stated above conditions. First, the following inclusions can be easily obtained.

$$\mathrm{Im}E(\sigma) \subset V_1 \subset V_2 \subset \mathrm{Ker}D(\delta) \ \text{ for all } \ (\delta,\sigma) \in \boldsymbol{R}^s \times \boldsymbol{R}^t.$$

Further, it follows from Lemma 3.5 that there exist a compensator (K, L, M, N) on $W := \boldsymbol{R}^w$ and a subspace V^e of X^e such that $V_1 = V_s$, $V_2 = V_p$ and

$$\begin{bmatrix} A(\alpha) + B(\beta)KC(\gamma) & B(\beta)L \\ MC(\gamma) & N \end{bmatrix} V^e \subset V^e, \text{ where } V^e := \left\{ \begin{bmatrix} x \\ Rx \end{bmatrix} \mid x \in V_2 \right\}.$$

Since $\mathrm{Im}E(\sigma) \subset V_1 = \mathrm{Ker}R$ and $V_2 \subset \bigcap_{i=0}^{s}\mathrm{Ker}D_i, \mathrm{Im}\begin{bmatrix} E(\sigma) \\ 0 \end{bmatrix} \subset V^e \subset \mathrm{Ker}\begin{bmatrix} D(\delta) & 0 \end{bmatrix}$.

Thus, we have

$$\begin{aligned} \left\langle \begin{bmatrix} A(\alpha) + B(\beta)KC(\gamma) & B(\beta)L \\ MC(\gamma) & N \end{bmatrix} \mid \mathrm{Im}\begin{bmatrix} E(\sigma) \\ 0 \end{bmatrix} \right\rangle &\subset \left\langle \begin{bmatrix} A(\alpha) + B(\beta)KC(\gamma) & B(\beta)L \\ MC(\gamma) & N \end{bmatrix} \mid V^e \right\rangle \\ &\subset V^e \\ &\subset \mathrm{Ker}\begin{bmatrix} D(\delta) & 0 \end{bmatrix}, \end{aligned}$$

for all $(\alpha,\beta,\gamma,\delta,\sigma) \in \boldsymbol{R}^p \times \boldsymbol{R}^q \times \boldsymbol{R}^r \times \boldsymbol{R}^s \times \boldsymbol{R}^t$.

Thus, the RDRPDC is solvable. ■

Corollary 4.3 Suppose that V_2^* is a maximal element of $\boldsymbol{V}(A, B; \bigcap_{i=0}^{s}\mathrm{Ker}D_i)$ and V_{1*} is a minimal element of $\boldsymbol{V}(C, A; \sum_{i=0}^{t}\mathrm{Im}E_i)$. If

$$\sum_{i=1}^{q}\mathrm{Im}B_i \subset V_{1*} \subset V_2^* \subset \bigcap_{i=1}^{r}\mathrm{Ker}C_i \text{ and } A_iV_2^* \subset V_{1*} \ (i = 1,\cdots,p),$$

then the RDRPDC is solvable.

Proof. The proof follows from Theorem 4.2. ■

If we assume that $(\alpha,\beta,\gamma,\delta,\sigma) = (0,0,0,0,0)$, Theorem 4.2 and Corollary 4.3 reduce to the following cororally which was studied by Schumacher[5].

Corollary 4.4 [5] Assume that $(\alpha,\beta,\gamma,\delta,\sigma) = (\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0})$. Suppose that V_2^* is a maximal element of $\boldsymbol{V}(A_0,B_0;\mathrm{Ker}D_0)$ and V_{1*} is a minimal element of $\boldsymbol{V}(C_0,A_0;\mathrm{Im}E_0)$. Then, the following three statements are equivalent.
(i) The Disturbance-Rejection Problem with Dynamic Compensator is solvable.
(ii) There exists a (C_0,A_0,B_0)-pair (V_1,V_2) such that $\mathrm{Im}E_0 \subset V_1 \subset V_2 \subset \mathrm{Ker}D_0$.
(iii) $V_{1*} \subset V_2^*$. ■

5 An Illustrative Example

Consider the following systems given by

$$A(\alpha) = \begin{bmatrix} 1+\alpha_1 & 0 & 0 & -\alpha_1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix},\ B(\beta) = \begin{bmatrix} 0 & \beta_1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix},\ C(\gamma) = \begin{bmatrix} 0 & 0 & 0 & \gamma_1 \\ 1 & 1 & 0 & 0 \end{bmatrix},$$

$$D(\delta) = \begin{bmatrix} 0 & 0 & 1+\delta_1 & \delta_2 \end{bmatrix},\ E(\sigma) = \begin{bmatrix} 1+\sigma_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

First, it remarks that the controlled output of the original system is influenced by disturbances. Define the following matrices as follows.

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix},\ A_1 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ B_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix},\ C_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ D_0 = D_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix},\ D_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix},$$

$$E_0 = E_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since $\bigcap_{i=0}^{2} \mathrm{Ker}D_i = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ and $\sum_{i=0}^{1} \mathrm{Im}E_i = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$, it follows from Lemmas 2.3 and 2.6 that

$$V_2^* = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} \text{ and } V_{1*} = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\},$$

which satisfy

$$\left(\mathrm{Im}B_1 + \sum_{i=0}^{1}\mathrm{Im}E_i\right) \subset V_{1*} \subset V_2^* \subset \left((\mathrm{Ker}C_1 \cap \bigcap_{i=0}^{2}\mathrm{Ker}D_i\right) \text{ and } A_1 V_2^* \subset V_{1*}.$$

Thus, it follows from Corollary 4.3 and the proofs of Lemmas 3.4 and 3.5 that the RDRPDC is solvable with dynamic compensator

$$(K, L, M, N) = (\begin{bmatrix} k & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} \ell \\ 1-\ell \end{bmatrix}, \begin{bmatrix} -1 & 0 \end{bmatrix}, 0),$$

where $k, \ell \in \boldsymbol{R}$ are arbitrary elements.

6 Concluding Remarks

In this paper, the notion of generalized (C, A, B)-pair which is a generalization of (C, A, B)-pair investigated by Schumacher[5] was introduced and its properties were investigated. Further, the robust disturbance-rejection problem with dynamic compensator for uncertain linear systems was formulated, and then some sufficient conditions for the problem to be solvable were presented. The results are generalizations of the results of Schumacher[5] to uncertain linear systems.

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