Generalized Invariant Subspaces for Infinite-Dimensional Systems

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Abstract

In this paper, some generalized invariant subspaces for infinite-dimensional systems are investigated, and then some sufficient conditions for robust disturbance-rejection problems to be solvable are studied.

Key words : Generalized Invariant Subspaces, Uncertain Systems, Robust Disturbance-Rejection, Geometric Approach

1 Introduction

The notion of invariant subspaces have been extended to study disturbance-rejection problems for uncertain finite-dimensional systems[1], [2], [4], [5]. Further, for infinite-dimensional uncertain systems in the sense that systems operators are represented as convex combinations, robust disturbance-rejection problems have also been studied[6],[7]. In order to study the problems the notin of simultaneous invariant subspaces were introduced and were used to give their solvability conditions.

The main objective of this paper is to investigate the infinite-dimensional version of generalized invariant subspaces which were partly investigated by Bhattacharyya[1]. Further, as an application of this study robust disturbance-rejection problems are formulated and their solvability conditions are presented. Finally, some concluding remarks are given.

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2 Generalized Invariant Subspaces

First, some notations used throughout this investigation are given. Let $\boldsymbol{B}(X;Y)$ denote the set of all bounded linear operators from a Hilbert space X into another Hilbert space Y and for notational simplicity, $\boldsymbol{B}(X;X)$ is written as $\boldsymbol{B}(X)$. For a linear operator A the domain, the image, the kernel and C_0-semigroup generated by A are denoted by D(A), ImA, Ker$A$ and $\{S_A(t); t \geq 0\}$, respectively. The notation $\boldsymbol{R}^n$ denotes the n-th dimensional Euclidean space.

Next, consider the following linear system defined in a Hilbert space X :

$$S(\alpha,\beta,\gamma): \begin{cases} \dfrac{d}{dt}x(t) = A(\alpha)x(t) + B(\beta)u(t), \\ y(t) = C(\gamma)x(t), \end{cases}$$

where $x(t) \in X, u(t) \in U := \boldsymbol{R}^m$ and $y(t) \in Y := \boldsymbol{R}^\ell$ are the state, the input and the measurement output, respectively. And operators $A(\alpha), B(\beta)$ and $C(\gamma)$ are unknown in the sense that they are represented as the forms:

$$\begin{aligned} A(\alpha) &= A_0 + \alpha_1 A_1 + \cdots + \alpha_p A_p, \\ B(\beta) &= B_0 + \beta_1 B_1 + \cdots + \beta_q B_q, \\ C(\gamma) &= C_0 + \gamma_1 C_1 + \cdots + \gamma_r C_r, \end{aligned}$$

where $\alpha := (\alpha_1, \cdots, \alpha_p) \in \boldsymbol{R}^p$, $\beta := (\beta_1, \cdots, \beta_q) \in \boldsymbol{R}^q$, $\gamma := (\gamma_1, \cdots, \gamma_r) \in \boldsymbol{R}^r$, A_0 is the infinitesimal generator of a C_0-semigroup $\{S_{A_0}(t); t \geq 0\}$ on X, $A_i \in \boldsymbol{B}(X)$ $(i = 1, \cdots, p), B_i \in \boldsymbol{B}(\boldsymbol{R}^m; X)$ $(i = 1, \cdots, q)$ and $C_i \in \boldsymbol{B}(X; \boldsymbol{R}^\ell)$ $(i = 1, \cdots, q)$.

Now, since A_i $(i = 1, \cdots, p)$ are bounded, it remarks that operator $A(\alpha)$ generates the C_0-semigroup and has the domain D($A(\alpha)$) =D(A_0) for all $\alpha \in \boldsymbol{R}^p$.

Definition 2.1 Let $V (\subset X)$ be a closed subspace.

(i) V is said to be a generalized controlled $S(A, B)$-invariant if there exists an $F \in \boldsymbol{B}(X; \boldsymbol{R}^m)$ such that

$$S_{A(\alpha)+B(\beta)F}(t)V \subset V \quad (t \geq 0)$$

for all $(\alpha, \beta) \in \boldsymbol{R}^p \times \boldsymbol{R}^q$.

(ii) V is said to be a generalized conditioned $S(C, A)$-invariant if there exists a $G \in \boldsymbol{B}(\boldsymbol{R}^\ell; X)$ such that

$$S_{A(\alpha)+GC(\gamma)}(t)V \subset V \ (t \geq 0)$$

for all $(\alpha,\gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^r$.

(iii) V is said to be a generalized $S(A,B,C)$-invariant if there exists an $H \in \boldsymbol{B}(\boldsymbol{R}^\ell; \boldsymbol{R}^m)$ such that

$$S_{A(\alpha)+B(\beta)HC(\gamma)}(t)V \subset V \ (t \geq 0)$$

for all $(\alpha,\beta,\gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^q \times \boldsymbol{R}^r$. □

For system $S(\alpha,\beta,\gamma)$ generalized $S(A,B,C)$-invariant subspace V has the property that, if an arbitrary initial state $x(0)$ stays in V, then there exists a measurement feedback $H \in \boldsymbol{B}(\boldsymbol{R}^\ell; \boldsymbol{R}^m)$ which is independent of all $(\alpha,\beta,\gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^q \times \boldsymbol{R}^r$ such that state trajectory $x(t)$ stays in V for all $t \geq 0$.

The following lemma is very useful to prove main results.

Lemma 2.2 [3] Let V be a closed subspace of X, A be an infinitesimal generator with C_0-semigroup $\{S_A(t); t \geq 0\}$ on X, and $Q_1 \in \boldsymbol{B}(X)$.

(i) If $S_{A+Q_1}(t)V \subset V$ for all $t \geq 0$, then $(A+Q_1)(V \cap \mathrm{D}(A)) \subset V$.

(ii) If there exists a $Q_2 \in \boldsymbol{B}(X)$ such that $S_{A+Q_2}(t)V \subset V$ for all $t \geq 0$ and $(Q_1 - Q_2)(V \cap \mathrm{D}(A)) \subset V$, then $S_{A+Q_1}(t)V \subset V$ for all $t \geq 0$. □

For a closed subspace V of X define a subspace R_V of $\boldsymbol{R}^m$ and a linear map Q_V on $\boldsymbol{R}^m$ by $R_V := \bigcap_{i=1}^{q} B_i^{-1}V$, where $B_i^{-1}V := \{u \in \boldsymbol{R}^m | B_i u \in V\}$ and $Q_V := \boldsymbol{R}^m \rightarrow \boldsymbol{R}^m$, a projection map onto R_V along $(R_V)^\perp$, where $(R_V)^\perp$ means the orthogonal complement of R_V, respectively.

The following lemma is used to prove Theorem2.4.

Lemma 2.3 The following two statements are equivalent.

(i) There exists an $F \in \boldsymbol{B}(X; \boldsymbol{R}^m)$ such that $S_{A_0+B_0F}(t)V \subset V \ (t \geq 0)$ and $B_iFV \subset V \ \ (i = 1, \cdots, q)$.

(ii) There exists an $\tilde{F} \in \boldsymbol{B}(X; \boldsymbol{R}^m)$ such that $S_{A_0+B_0Q_V\tilde{F}}(t)V \subset V \ (t \geq 0)$.

Proof. ((i) ⇒ (ii)) Suppose that there exists an $F \in \boldsymbol{B}(X; \boldsymbol{R}^m)$ such that $S_{A_0+B_0F}(t)V \subset V \ (t \geq 0)$ and $B_iFV \subset V \ \ (i = 1, \cdots, q)$. Then, $FV \subset \bigcap_{i=1}^{q} B_i^{-1}V = R_V$. Hence, $Q_VFV = FV$. Thus, $(B_0Q_VF - B_0F)V = \{0\}$. Then, it follows from Lemma 2.2(ii) that $S_{A_0+B_0Q_VF}(t)V \subset V \ (t \geq 0)$. ((ii) ⇒ (i)) Suppose that there exists an $\tilde{F} \in \boldsymbol{B}(X; \boldsymbol{R}^m)$ such that $S_{A_0+B_0Q_V\tilde{F}}(t)V \subset V \ (t \geq 0)$. Define $F := Q_V\tilde{F}$. Then, $S_{A_0+B_0F}(t)V \subset V \ (t \geq 0)$. Further, $B_iFV = B_iQ_V\tilde{F}V \subset B_iR_V \subset V$. This completes the proof. □

The following theorem is the infinite-dimensional version of the results of Bhattacharyya[1].

Theorem 2.4 The following three statements are equivalent.

(i) V is a generalized controlled $S(A,B)$-invariant.

(ii) There exists an $F \in \boldsymbol{B}(X;\boldsymbol{R}^m)$ such that $S_{A_0+B_0F}(t)V \subset V\ (t \geq 0), B_iFV \subset V\ \ (i = 1,\cdots,q)$ and $A_iV \subset V\ (i = 1,\cdots,p)$.

(iii) There exists an $\tilde{F} \in \boldsymbol{B}(X;\boldsymbol{R}^m)$ such that $S_{A_0+B_0Q_V\tilde{F}}(t)V \subset V\ (t \geq 0)$ and $A_iV \subset V\ (i = 1,\cdots,p)$.

Proof. ((i) $\Rightarrow$ (ii)) Suppose that V is a generalized controlled $S(A,B)$-invariant. Then, there exists an $F \in \boldsymbol{B}(X;\boldsymbol{R}^m)$ such that

$$S_{A(\alpha)+B(\beta)F}(t)V \subset V \quad (t \geq 0) \tag{1}$$

for all $(\alpha,\beta) \in \boldsymbol{R}^p \times \boldsymbol{R}^q$.

First, suppose that $\alpha_1 = \cdots = \alpha_p = \beta_1 = \cdots = \beta_q = 0$ in (1). Then,

$$S_{A_0+B_0F}(t)V \subset V\ (t \geq 0)$$

which with Lemma2.2(i) implies

$$(A_0 + B_0F)(V \cap \mathrm{D}(A_0)) \subset V. \tag{2}$$

Further, suppose that $\alpha_1 = 1$ and $\alpha_2 = \cdots = \alpha_p = \beta_1 = \cdots = \beta_q = 0$ in (1). Then,

$$S_{A_0+A_1+B_0F}(t)V \subset V\ (t \geq 0)$$

which with Lemma2.2(i) implies

$$(A_0 + A_1 + B_0F)(V \cap \mathrm{D}(A_0)) \subset V. \tag{3}$$

Hence, it follows from (2) and (3) that $A_1(V \cap D(A_0)) \subset V$. Since A_1 is a bounded linear operator, $A_1V \subset V$. Similarly, one can prove $A_iV \subset V\ (i = 2,\cdots,p)$.

Next, suppose that $\beta_1 = 1$ and $\alpha_1 = \cdots = \alpha_p = \beta_2 = \cdots = \beta_q = 0$ in (1). Then,

$$(A_0 + (B_0 + B_1)F)(V \cap \mathrm{D}(A_0)) \subset V. \tag{4}$$

Hence, it follows from (2), (4) and boundedness of B_1F that $B_1FV \subset V$. Similarly, one can prove $B_iFV \subset V\ (i = 2,\cdots,q)$.

((ii) $\Rightarrow$ (i)) Suppose that there exists an $F \in \boldsymbol{B}(X;\boldsymbol{R}^m)$ such that

$$S_{A_0+B_0F}(t)V \subset V\ (t \geq 0), B_iFV \subset V\ (i = 1,\cdots,q) \text{ and } A_iV \subset V\ (i = 1,\cdots,p).$$

Then,

$$\begin{aligned}(A(\alpha)+B(\beta)F) &= \{(A_0+\alpha_1A_1+\cdots\alpha_pA_p)+(B_0+\beta_1B_1+\cdots+\beta_qB_q)F\} \\ &= (A_0+B_0F)+\sum_{i=1}^{p}\alpha_iA_i+\sum_{i=1}^{q}\beta_iB_iF.\end{aligned}$$

Since $S_{A_0+B_0F}(t)V\subset V$ and $(\sum_{i=1}^{p}\alpha_iA_i+\sum_{i=1}^{q}\beta_iB_iF)V\subset V$, it follows from Lemma 2.2(ii) that

$$S_{A(\alpha)+B(\beta)F}(t)V\subset V\quad(t\geq 0)$$

for all $(\alpha,\beta)\in \boldsymbol{R}^p\times\boldsymbol{R}^q$, which implies V is a generalized controlled $S(A,B)$-invariant.
((ii) $\Leftrightarrow$ (iii)) The proof follows from Lemma 2.3. □

For a closed subspace V of X let P_V be a linear map on $\boldsymbol{R}^\ell$ satisfying $\mathrm{Ker}P_V=\sum_{i=1}^{r}C_iV$ and $V=\phi\oplus(V\cap\mathrm{Ker}P_VC_0)$ for some subspace ϕ. Since, $C_0\phi\cap\mathrm{Ker}P_V=\{0\}$, we can define a projection map $P_V:\boldsymbol{R}^\ell\to\boldsymbol{R}^\ell$ onto $C_0\phi\oplus\Gamma$ along $\mathrm{Ker}P_V=\sum_{i=1}^{r}C_iV$ for some Γ satisfying $V=\phi\oplus(V\cap\mathrm{Ker}P_VC_0)$.

The following lemma is used to prove Theorem2.6.

Lemma 2.5 The following two statements are equivalent.

(i) There exists a $G\in\boldsymbol{B}(\boldsymbol{R}^\ell;X)$ such that $S_{A_0+GC_0}(t)V\subset V$ $(t\geq 0)$ and $GC_iV\subset V$ $(i=1,\cdots,r)$.

(ii) There exists a $\tilde{G}\in\boldsymbol{B}(\boldsymbol{R}^\ell;X)$ such that $S_{A_0+\tilde{G}P_VC_0}(t)V\subset V$ $(t\geq 0)$.

Proof. ((i) $\Rightarrow$ (ii)) Suppose that there exists a $G\in\boldsymbol{B}(\boldsymbol{R}^\ell;X)$ such that $S_{A_0+GC_0}(t)V\subset V$ $(t\geq 0)$ and $GC_iV\subset V$ $(i=1,\cdots,r)$. Then, from Lemma 2.2(ii) it suffices to show $(GP_VC_0-GC_0)V\subset V$. Choose an arbitrary element $x\in V$. Then, x can be decomposed as $x=y+z$ $(y\in\phi, z\in(V\cap\mathrm{Ker}P_VC_0))$. Since $P_VC_0x=P_VC_0y=C_0y$,

$$\begin{aligned}(GP_VC_0-GC_0)x &= GC_0y-GC_0y-GC_0z \\ &= -GC_0z \\ &\in G(\mathrm{Ker}P_V) \\ &= \sum_{i=1}^{r}GC_iV \\ &\subset V.\end{aligned}$$

Thus, $S_{A_0+GP_VC_0}(t)V \subset V$ $(t \geq 0)$.

((ii) $\Rightarrow$ (i)) Suppose that there exists a $\tilde{G} \in \boldsymbol{B}(\boldsymbol{R}^\ell; X)$ such that $S_{A_0+\tilde{G}P_VC_0}(t)V \subset V$ $(t \geq 0)$. Define $G := \tilde{G}P_V$. Then, $S_{A_0+GC_0}(t)V \subset V$ $(t \geq 0)$. Further, $GC_iV = \tilde{G}P_VC_iV \subset \tilde{G}P_V(\mathrm{Ker}P_V) = \{0\} \subset V$. This completes the proof. □

Theorem 2.6 The following three statements are equivalent.

(i) V is a generalized conditioned $S(C, A)$-invariant.

(ii) There exists a $G \in \boldsymbol{B}(\boldsymbol{R}^\ell; X)$ such that $S_{A_0+GC_0}(t)V \subset V$ $(t \geq 0)$, $GC_iV \subset V$ $(i = 1, \cdots, r)$ and $A_iV \subset V$ $(i = 1, \cdots, p)$.

(iii) There exists a $\tilde{G} \in \boldsymbol{B}(\boldsymbol{R}^\ell; X)$ such that $S_{A_0+\tilde{G}P_VC_0}(t)V \subset V$ $(t \geq 0)$ and $A_iV \subset V$ $(i = 1, \cdots, p)$.

Proof. ((i) $\Rightarrow$ (ii)) Suppose that V is a generalized conditioned $S(C, A)$-invariant. Then, there exists a $G \in \boldsymbol{B}(\boldsymbol{R}^\ell; X)$ such that

$$S_{A(\alpha)+GC(\gamma)}(t)V \subset V \quad (t \geq 0) \tag{5}$$

for all $(\alpha, \gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^r$.

First, suppose that $\alpha_1 = \cdots = \alpha_p = \gamma_1 = \cdots = \gamma_r = 0$ in (5). Then,

$$S_{A_0+GC_0}(t)V \subset V \ (t \geq 0)$$

which with Lemma2.2(i) implies

$$(A_0 + GC_0)(V \cap \mathrm{D}(A_0)) \subset V. \tag{6}$$

Further, suppose that $\alpha_1 = 1$ and $\alpha_2 = \cdots = \alpha_p = \gamma_1 = \cdots = \gamma_r = 0$ in (5). Then,

$$S_{A_0+A_1+GC_0}(t)V \subset V \ (t \geq 0)$$

which with Lemma2.2(i) implies

$$(A_0 + A_1 + GC_0)(V \cap \mathrm{D}(A_0)) \subset V. \tag{7}$$

Hence, it follows from (6) and (7) that $A_1(V \cap \mathrm{D}(A_0)) \subset V$. Since A_1 is a bounded linear operator, $A_1V \subset V$. Similarly, one can prove $A_iV \subset V$ $(i = 2, \cdots, p)$.

Next, suppose that $\gamma_1 = 1$ and $\alpha_1 = \cdots = \alpha_p = \gamma_2 = \cdots = \gamma_r = 0$ in (5). Then,

$$(A_0 + G(C_0 + C_1))(V \cap \mathrm{D}(A_0)) \subset V. \tag{8}$$

Hence, it follows from (6), (8) and boundedness of GC_1 that $GC_1V \subset V$. Similarly, one can prove $GC_iV \subset V$ $(i = 2, \cdots, r)$.

((ii) $\Rightarrow$ (i)) Suppose that there exists a $G \in \boldsymbol{B}(\boldsymbol{R}^\ell; X)$ such that

$$S_{A_0+GC_0}(t)V \subset V\ (t \geq 0), GC_iV \subset V\ (i = 1, \cdots, r) \text{ and } A_iV \subset V\ (i = 1, \cdots, p).$$

Then,

$$\begin{aligned}(A(\alpha) + GC(\gamma)) &= \{(A_0 + \alpha_1 A_1 + \cdots \alpha_p A_p) + G(C_0 + \gamma_1 C_1 + \cdots + \gamma_r C_r)\} \\ &= (A_0 + GC_0) + \sum_{i=1}^{p} \alpha_i A_i + \sum_{i=1}^{r} \gamma_i C_i.\end{aligned}$$

Since $S_{A_0+GC_0}(t)V \subset V$ and $(\sum_{i=1}^{p} \alpha_i A_i + \sum_{i=1}^{r} \gamma_i C_i)V \subset V$, it follows from Lemma 2.2(ii) that

$$S_{A(\alpha)+GC(\gamma)}(t)V \subset V \quad (t \geq 0)$$

for all $(\alpha, \gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^r$, which implies V is a generalized conditined $S(C, A)$-invariant.

((ii) $\Leftrightarrow$ (iii)) The proof follows from Lemma 2.5. □

The following theorem is an interesting result.

Theorem 2.7 The following three statements are equivalent.

(i) V is a generalized $S(A, B, C)$-invariant.

(ii) There exists an $H \in \boldsymbol{B}(\boldsymbol{R}^\ell; \boldsymbol{R}^m)$ such that $S_{A_0+B_0HC_0}(t)V \subset V$ $(t \geq 0)$, $B_iHC_jV \subset V$ $(i = 0, \cdots, q, j = 0, \cdots, r; (i,j) \neq (0,0))$ and $A_iV \subset V$ $(i = 1, \cdots, p)$.

(iii) There exists a $K \in \boldsymbol{B}(\boldsymbol{R}^\ell; \boldsymbol{R}^m)$ such that $S_{A_0+B_0Q_VKP_VC_0}(t)V \subset V$ $(t \geq 0)$ and $A_iV \subset V$ $(i = 1, \cdots, p)$.

Proof. ((i) $\Rightarrow$ (ii)) Suppose that V is a generalized $S(A, B, C)$-invariant. Then, there exists an $H \in \boldsymbol{B}(\boldsymbol{R}^\ell; \boldsymbol{R}^m)$ such that

$$S_{A(\alpha)+B(\beta)HC(\gamma)}(t)V \subset V \quad (t \geq 0) \tag{9}$$

for all $(\alpha, \beta, \gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^q \times \boldsymbol{R}^r$.

First, suppose that $\alpha_1 = \cdots = \alpha_p = \beta_1 = \cdots = \beta_q = \gamma_1 = \cdots = \gamma_r = 0$ in (9). Then,

$$S_{A_0+B_0HC_0}(t)V \subset V\ (t \geq 0)$$

which with Lemma2.2(i) implies

$$(A_0 + B_0HC_0)(V \cap \mathrm{D}(A_0)) \subset V. \tag{10}$$

Further, suppose that $\alpha_1 = 1$ and $\alpha_2 = \cdots = \alpha_p = \beta_1 = \cdots = \beta_q = \gamma_1 = \cdots = \gamma_r = 0$ in (9). Then,

$$S_{A_0+A_1+B_0HC_0}(t)V \subset V \ (t \geq 0)$$

which with Lemma2.2(i) implies

$$(A_0 + A_1 + B_0HC_0)(V \cap \mathrm{D}(A_0)) \subset V. \tag{11}$$

Hence, it follows from (10) and (11) that $A_1(V\cap \mathrm{D}(A_0)) \subset V$. Since A_1 is a bounded linear operator, $A_1V \subset V$. Similarly, one can prove $A_iV \subset V$ $(i = 2, \cdots, p)$.

Next, suppose that $\beta_1 = 1$ and $\alpha_1 = \cdots = \alpha_p = \beta_2 = \cdots = \beta_q = \gamma_1 = \cdots = \gamma_r = 0$ in (9). Then,

$$(A_0 + (B_0 + B_1)HC_0)(V \cap \mathrm{D}(A_0)) \subset V. \tag{12}$$

Hence, it follows from (10), (12) and boundedness of B_1HC_0 that $B_1HC_0V \subset V$.
Further, suppose that $\gamma_1 = 1$ and $\alpha_1 = \cdots = \alpha_p = \beta_1 = \cdots = \beta_q = \gamma_2 = \cdots = \gamma_r = 0$ in (9). Then,

$$(A_0 + B_0H(C_0 + C_1))(V \cap \mathrm{D}(A_0)) \subset V. \tag{13}$$

From (10), (13) and boundedness of B_0HC_1, one obtain $B_0HC_1V \subset V$. Similarly, one can prove $B_iHC_jV \subset V$ $(i = 1, \cdots, q, j = 1, \cdots, r)$.
((ii) $\Rightarrow$ (i)) Suppose that there exists an $H \in \boldsymbol{B}(\boldsymbol{R}^\ell; \boldsymbol{R}^m)$ such that $S_{A_0+B_0HC_0}(t)V \subset V$ $(t \geq 0), B_iHC_jV \subset V$ $(i = 0, \cdots, q, j = 0, \cdots, r; (i,j) \neq (0,0))$ and $A_iV \subset V$ $(i = 1, \cdots, p)$. Then,

$$\begin{aligned}
&(A(\alpha) + B(\beta)HC(\gamma)) \\
&\quad = \{(A_0 + B_0HC_0) + \sum_{i=1}^{p} \alpha_i A_i + (\sum_{i=1}^{q} \beta_i B_i) H (\sum_{i=1}^{r} \gamma_i C_i)\} \\
&\quad = (A_0 + B_0HC_0) + \sum_{i=1}^{p} \alpha_i A_i + \sum_{i=1}^{r} \gamma_i B_0HC_i + \sum_{i=1}^{q} \beta_i B_i HC_0 + \sum_{i=1}^{q} \sum_{j=1}^{r} \beta_i \gamma_j B_i HC_j.
\end{aligned}$$

Since $S_{A_0+B_0HC_0}(t)V \subset V$ and $(\sum_{i=1}^{p} \alpha_i A_i + \sum_{i=1}^{r} \gamma_i B_0HC_i + \sum_{i=1}^{q} \beta_i B_i HC_0 + \sum_{i=1}^{q} \sum_{j=1}^{r} \beta_i \gamma_j B_i HC_j)V \subset V$, it follows from Lemma 2.2(ii) that

$$S_{A(\alpha)+B(\beta)HC(\gamma)}(t)V \subset V \quad (t \geq 0)$$

for all $(\alpha,\beta,\gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^q \times \boldsymbol{R}^r$, which implies V is a generalized $S(A,B,C)$-invariant.
((ii) $\Rightarrow$ (iii)) Suppose that there exists an $H \in \boldsymbol{B}(\boldsymbol{R}^\ell;\boldsymbol{R}^m)$ such that $S_{A_0+B_0HC_0}(t)V \subset V$ $(t \geq 0)$, $B_iHC_jV \subset V$ $(i=0,\cdots,q, j=0,\cdots,r; (i,j) \neq (0,0))$ and $A_iV \subset V$ $(i=1,\cdots,p)$. Then, it follows from Lemmas 2.3 and 2.5 that there exists an $\tilde{F} \in \boldsymbol{B}(X;\boldsymbol{R}^m)$ such that $S_{A_0+B_0Q_V\tilde{F}}(t)V \subset V$ $(t \geq 0)$ and there exists a $\tilde{G} \in \boldsymbol{B}(\boldsymbol{R}^\ell;X)$ such that $S_{A_0+\tilde{G}P_VC_0}(t)V \subset V$ $(t \geq 0)$. From Remark in [8](p.106) there exists a $K \in \boldsymbol{B}(\boldsymbol{R}^\ell;\boldsymbol{R}^m)$ such that $S_{A_0+B_0Q_VKP_VC_0}(t)V \subset V$ $(t \geq 0)$.
((iii) $\Rightarrow$ (ii)) Suppose that there exists a $K \in \boldsymbol{B}(\boldsymbol{R}^\ell;\boldsymbol{R}^m)$ such that $S_{A_0+B_0Q_VKP_VC_0}(t)V \subset V$ $(t \geq 0)$ and $A_iV \subset V$ $(i=1,\cdots,p)$. Define $H := Q_VKP_V$. Then, $S_{A_0+B_0HC_0}(t)V \subset V$ $(t \geq 0)$. Further, one obtain $B_iHC_jV = B_iQ_VKP_VC_jV \subset B_i\mathrm{Im}Q_V \subset B_iR_V \subset V$ $(i=0,\cdots,q, j=0,\cdots,r; (i,j) \neq (0,0))$. This completes the proof. □

Corollary 2.8 V is a generalized $S(A,B,C)$-invariant if and only if V is a generalized controlled $S(A,B)$-invariant and a generalized conditioned $S(C,A)$-invariant.
Proof. The proof follows from Theorems 2.4, 2.6 and 2.7. □

It is interesting to note from Theorems 2.4, 2.6 and 2.7 that generalized invariant subspaces are connected with the invariances of a finite number of conditions. So, we can check whether a given subspace V is a generalized invariant or not.

3 An Application to Robust Disturbance-Rejection

In this section, the infinite-dimensional version of disturbance-rejection problems for uncertain systems which were investigated by Bhattacharyya[1] are studied.

Consider the following uncertain system $S(\alpha,\beta,\gamma,\delta,\sigma)$ defined in a Hilbert space X.

$$S(\alpha,\beta,\gamma,\delta,\sigma): \begin{cases} \dfrac{d}{dt}x(t) = A(\alpha)x(t) + B(\beta)u(t) + E(\sigma)\xi(t), \\ y(t) = C(\gamma)x(t), \\ z(t) = D(\delta)x(t) \end{cases}$$

where $x(t) \in X$, $u(t) \in U := \boldsymbol{R}^m$, $y(t) \in Y := \boldsymbol{R}^\ell$, $z(t) \in Z := \boldsymbol{R}^\mu$ and $\xi \in L_1^{loc}((0,\infty);Q)$ are the state, the input, the measurement output, the controlled output and the disturbance which is a Hilbert space Q valued locally integrable function, respectively. It is assumed that coefficient operators have the following unknown parameters.

$$\begin{aligned} A(\alpha) &= A_0 + \alpha_1A_1 + \cdots + \alpha_pA_p := A_0 + \Delta A(\alpha), \\ B(\beta) &= B_0 + \beta_1B_1 + \cdots + \beta_qB_q := B_0 + \Delta B(\beta), \end{aligned}$$

$$
\begin{aligned}
C(\gamma) &= C_0 + \gamma_1 C_1 + \cdots + \gamma_r C_r := C_0 + \Delta C(\gamma), \\
D(\delta) &= D_0 + \delta_1 D_1 + \cdots + \delta_s D_s := D_0 + \Delta D(\delta), \\
E(\sigma) &= E_0 + \sigma_1 E_1 + \cdots + \sigma_t E_t := E_0 + \Delta D(\sigma),
\end{aligned}
$$

where A_i, B_i, C_i are the same as system $S(\alpha,\beta,\gamma)$ in Section 2, $D_i \in \boldsymbol{B}(X;\boldsymbol{R}^\mu)$, $E_i \in \boldsymbol{B}(Q;X)$ and $\alpha := (\alpha_1,\cdots,\alpha_p) \in \boldsymbol{R}^p$, $\beta := (\beta_1,\cdots,\beta_q) \in \boldsymbol{R}^q$, $\gamma := (\gamma_1,\cdots,\gamma_r) \in \boldsymbol{R}^r$, $\delta := (\delta_1,\cdots,\delta_s) \in \boldsymbol{R}^s$, $\sigma := (\sigma_1,\cdots,\sigma_t) \in \boldsymbol{R}^t$.

In System $S(\alpha,\beta,\gamma,\delta,\sigma)$ (A_0,B_0,C_0,D_0,E_0) and $(\Delta A(\alpha)$, $\Delta B(\beta)$, $\Delta C(\gamma)$, $\Delta D(\delta)$,$\Delta E(\sigma))$ represent the nominal system model and a specific uncertain perturbation, respectively.

Now, apply to system $S(\alpha,\beta,\gamma,\delta,\sigma)$ a measurement feedback of the form:

$$
u(t) = Hy(t)
$$

where $H \in \boldsymbol{B}(\boldsymbol{R}^\ell;\boldsymbol{R}^m)$. Then, the resulting closed-loop system is given as

$$
S_H(\alpha,\beta,\gamma,\delta,\sigma) : \begin{cases} \dfrac{d}{dt}x(t) = (A(\alpha) + B(\beta)HC(\gamma))x(t) + E(\sigma)\xi(t), \\ z(t) = D(\delta)x(t). \end{cases}
$$

Our robust disturbance-rejection problem with measurement feedback for system $S(\alpha,\beta,\gamma,\delta,\sigma)$ is stated as follows: Given operators A_i, B_i, C_i, D_i, E_i for system $S(\alpha,\beta,\gamma,\delta,\sigma)$, find if possible a measurement feedback gain $H \in \boldsymbol{B}(\boldsymbol{R}^\ell;\boldsymbol{R}^m)$ such that the closed-loop system $S_H(\alpha,\beta,\gamma,\delta,\sigma)$ rejects the disturbances ξ from the controlled output z for all parameters $(\alpha,\beta,\gamma,\delta,\sigma) \in \boldsymbol{R}^p \times \boldsymbol{R}^q \times \boldsymbol{R}^r \times \boldsymbol{R}^s \times \boldsymbol{R}^t$. To achieve this control requirement we must solve the following problem : Given operators A_i, B_i, C_i, D_i, E_i for system $S(\alpha,\beta,\gamma,\delta,\sigma)$, find if possible a measurement feedback gain $H \in \boldsymbol{B}(\boldsymbol{R}^\ell;\boldsymbol{R}^m)$ such that

$$
D(\delta)\int_0^t S_{A(\alpha)+B(\beta)HC(\gamma)}(t-\tau)d\tau = 0 \ (t \geq 0) \text{ for all } \xi \in L_1^{loc}(0,\infty;Q)
$$

or equivalently

$$
< S_{A(\alpha)+B(\beta)HC(\gamma)}(\cdot)|\mathrm{Im}E(\sigma) > := \overline{L\left(\bigcup_{t\geq 0} S_{A(\alpha)+B(\beta)HC(\gamma)}(t)(\mathrm{Im}E(\sigma))\right)} \subset \mathrm{Ker}D(\delta)
$$

for all parameters $(\alpha,\beta,\gamma,\delta,\sigma) \in \boldsymbol{R}^p \times \boldsymbol{R}^q \times \boldsymbol{R}^r \times \boldsymbol{R}^s \times \boldsymbol{R}^t$, where $L(\Omega)$ and overbar mean the linear subspace generated by the set Ω and the closure in X, respectively.

This problem can be rephrased as follows.

Problem 3.1 (Robust Disturbance-Rejection Problem with Measurement Feedback (RDRPMF)) Given operators A_i, B_i, C_i, D_i, E_i for system $S(\alpha, \beta, \gamma, \delta, \sigma)$, find if possible a measurement feedback gain $H \in \boldsymbol{B}(\boldsymbol{R}^\ell; \boldsymbol{R}^m)$ such that

$$< S_{A(\alpha)+B(\beta)HC(\gamma)}(\cdot)|\mathrm{Im}E(\sigma) > \subset \mathrm{Ker}D(\delta)$$

for all parameters $(\alpha, \beta, \gamma, \delta, \sigma) \in \boldsymbol{R}^p \times \boldsymbol{R}^q \times \boldsymbol{R}^r \times \boldsymbol{R}^s \times \boldsymbol{R}^t$. □

Remark 3.2 If $C(\gamma) = I$ (identity operator), then Problem 3.1 reduces to the robust disturbance-rejection problem with state feedback (RDRPSF). □

The following theorem gives solvability conditions for RDRPMF to be solvable.

Theorem 3.3 If there exists a generalized $S(A, B, C)$-invariant subspace V such that

$$\sum_{i=0}^{t} \mathrm{Im}E_i \subset V \subset \bigcap_{i=0}^{s} \mathrm{Ker}D_i,$$

then the RDRPMF is solvable.

Proof. Suppose that there exists a generalized $S(A, B, C)$-invariant subspace V such that

$$\sum_{i=0}^{t} \mathrm{Im}E_i \subset V \subset \bigcap_{i=0}^{s} \mathrm{Ker}D_i.$$

Then,

$$S_{A(\alpha)+B(\beta)HC(\gamma)}(t)V \subset V \ (t \geq 0)$$

for all $(\alpha, \beta, \gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^q \times \boldsymbol{R}^r$. Hence,

$$\begin{aligned} < S_{A(\alpha)+B(\beta)HC(\gamma)}(\cdot) \mid \mathrm{Im}E(\sigma) > \ &\subset \ < S_{A(\alpha)+B(\beta)HC(\gamma)}(\cdot) \mid \sum_{i=0}^{t} \mathrm{Im}E_i > \\ &\subset \ < S_{A(\alpha)+B(\beta)C(\gamma)}(\cdot) \mid V > \\ &= \ V \\ &\subset \ \bigcap_{i=0}^{s} \mathrm{Ker}D_i \\ &\subset \ \mathrm{Ker}D(\delta) \end{aligned}$$

for all parameters $(\alpha, \beta, \gamma, \delta, \sigma) \in \boldsymbol{R}^p \times \boldsymbol{R}^q \times \boldsymbol{R}^r \times \boldsymbol{R}^s \times \boldsymbol{R}^t$. Thus, RDRPMF is solvable. □

The next corollary follows from Theorem 3.3.

Corollary 3.4 Assume that $C(\gamma) = I$. Then, if there exists a generalized controlled $S(A, B)$-invariant subspace V such that

$$\sum_{i=0}^{t} \mathrm{Im}E_i \subset V \subset \bigcap_{i=0}^{s} \mathrm{Ker}D_i,$$

then the RDRPSF is solvable. □

It remarks that Theorems 2.4 and 2.6 play important role to check whether a given subspace is a generalized invariant or not.

4 Concluding Remarks

In this paper, some generalized invariant subspces for infinite-dimensional systems were introduced, and then their properties were investigated. Especially, it is interesting that Theorems 2.4, 2.6 and 2.7 say infinitely many conditions are equivalent to finite number of conditions. Further, the infinite-dimensional version of the disturbance-rejection problems for uncertain systems which were investigated by Bhattacharyya[1] were formulated, and then their solvability conditions were presented.

References

[1] S. P. Bhattacharyya, *Generalized Controllability, (A, B)-invariant Subspaces and Parameter Invariant Control*, SIAM J. on Alg. Disc. Meth., Vol. 4, No.4 (1983), pp. 529–533.

[2] G. Conte and A. M. Perdon, *Robust Disturbance Decoupling Problem for Parameter Dependent Families of Linear Systems*, Automatica, Vol. 29, No.2 (1993), pp.475–478.

[3] R.F.Curtain, *(C, A, B)-pairs in infinite dimensions*, Systems & Control Letters, Vol.5, pp. 59–65, (1984).

[4] B. K. Ghosh, *A geometric approach to simultaneous system design: parameter insensitive disturbance decoupling by state and output feedback*, Modeling, Identification and Robust Control, C.I.Byrnes and A.Lindquist(ed.), North-Holland (1986), pp. 476–484.

[5] N. Otsuka et al. *Parameter insensitive disturbance-rejection problem with incomplete-state feedback*, IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences, Vol. E78-A (1995), pp. 1589–1594.

[6] N. Otsuka and H. Inaba, *Parameter-insensitive disturbance-rejection for infinite-dimensional systems*, IMA Journal of Mathematical Control & Information,Vol.14, No.4 (1997), pp. 401–413.

[7] N. Otsuka and H. Inaba, *A note on robust disturbance-rejection problems for infinite-dimensional systems*, Systems & Control Letters, to appear.

[8] H.Zwart, *Geometric Theory for Infinite Dimensional Systems*, Lecture Notes in Control and Information Sciences, Springer-Verlag, 1989.