Fast Algorithms to Compute Frequency Transformation Matrices for IIR Digital Filters

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Abstract

Frequency transformations derive filters of various types from a filter of low-pass type, and the transformations required laborious hand computation. The present authors have been already proposed the frequency transformation matrices which can replace the hand computation by automatic procedures, but the procedures still require a large number of computation. In this paper, fast algorithms to compute the elements of the frequency transformation matrices are proposed. Properties of those matrices are first investigated, and then some fast algorithms are presented. Finally, the proposed algorithms are compared with those based on the definitions of the matrices.

Key Words: IIR Digital Filters, Frequency Transformation

1 Introduction

The design methods of IIR digital filters have been extensively studied for these thirty years, and they are roughly divided into two categories. One (A) is based on bilinear and frequency transformations from analog filters [2]-[4],[6], and the other (B) is to optimize frequency characteristics to the desired one [1],[5],[8]. The method (A) preserves frequency characteristics of the original analog filters, so we can take advantage of highly developed analog filter design techniques. The method (B) enables more flexible design, but the best design algorithm has not been established yet.

The inherent problem of (A) was that the frequency transformations require laborious hand computation. The direct application of the transformation yields a complicated formula of the target transfer function, which has to be reduced into the form of a rational polynomial by hand computation. The yielded formula becomes more complicated for a transfer function with a higher order that can be used in practice to realize minute characteristics.

The present authors have been already proposed the frequency transformation matrices [7] which can replace the hand computation in the frequency transformations by automatic procedures. However, the procedures still require a large number of computation, which contain a lot of binomial coefficients. In computing the elements of the bilinear transformation matrices [2],[3],[6], a large number of computation is also required. To decrease the number of computation, Bose [2],[3] proposed the fast algorithm to compute the elements of the matrices. Similarly, fast algorithms to compute the elements of the frequency transformation matrices must be studied.

The objective of this paper is to study the fast computation of the elements of the frequency transformation matrices. In Section 2, we investigate significant properties of those matrices. In Section 3, some fast algorithms to compute the elements of the matrices are proposed, and they are compared with the algorithm based on the definition. Finally, Section 4 makes some concluding remarks.

2 Properties of Frequency Transformation Matrices

In this section, significant properties of the frequency transformation matrices are investigated. Hereafter, frequency means angular frequency, and $\left\langle \begin{array}{c} a \\ b \end{array} \right\rangle$ denotes

$$\left\langle \begin{array}{c} a \\ b \end{array} \right\rangle := \begin{cases} 0, & b > a \text{ or } b < 0, \\ \dfrac{a!}{b!(a-b)!}, & \text{otherwise.} \end{cases}$$

Let ω_0 be the cut-off frequency of the original low-pass filter. The transfer function $H_0(z)$ of this filter can be written as

$$H_0(z) = \frac{a_0 + a_1 z^{-1} + \cdots + a_n z^{-n}}{b_0 + b_1 z^{-1} + \cdots + b_n z^{-n}}, \tag{1}$$

of which order of numerator polynomial is the same as that of denominator one, as a result of the bilinear transformation from an analog low-pass filter [7].

2.1 Properties of the Matrix to Design Low-pass Filters

The frequency transformation formula to design the low-pass filter with the cut-off frequency ω_1 is shown in Table 1, and the matrix which corresponds to that transformation was obtained [7] as

$$\boldsymbol{T}_n^{\mathrm{LP}} := \boldsymbol{T}_n^{\mathrm{LP}}(\alpha) = \left[t_{i,j}^{\mathrm{LP},n}(\alpha) \right] \in \boldsymbol{R}^{(n+1)\times(n+1)},$$

where

$$t_{i,j}^{\mathrm{LP},n} := t_{i,j}^{\mathrm{LP},n}(\alpha) = \sum_{k=0}^{i} (-1)^{j-i} \left\langle \begin{array}{c} j \\ i-k \end{array} \right\rangle \left\langle \begin{array}{c} n-j \\ k \end{array} \right\rangle \alpha^{j-i+2k}. \tag{2}$$

The transfer function $H_1(z)$ of the derived low-pass filter can be represented by

$$H_1(z) = \frac{c_0 + c_1 z^{-1} + \cdots + c_n z^{-n}}{d_0 + d_1 z^{-1} + \cdots + d_n z^{-n}}. \tag{3}$$

Here, the relations between the coefficients of (1) and those of (3) are as follows:

$$\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = \boldsymbol{T}_n^{\mathrm{LP}} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}, \quad \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_n \end{bmatrix} = \boldsymbol{T}_n^{\mathrm{LP}} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{bmatrix}. \tag{4}$$

Since the coefficients $\{c_i\}_{i=0}^{n}$ of (3) can be obtained by (4), the numerator polynomial $N_1(z)$ of $H_1(z)$ can be written as

$$N_1(z) = \sum_{i=0}^{n} c_i z^{-i} = \sum_{i=0}^{n}\left(\sum_{j=0}^{n} t_{i,j}^{\mathrm{LP},n} a_j\right) z^{-i} = \sum_{j=0}^{n}\left(\sum_{i=0}^{n} t_{i,j}^{\mathrm{LP},n} z^{-i}\right) a_j. \quad (5)$$

On the other hand, since the transfer function $H_1(z)$ can be obtained as a result of the frequency transformation of $H_0(z)$, $N_1(z)$ can also be written as

$$N_1(z) = \sum_{i=0}^{n} c_i z^{-i} = \left(1-\alpha z^{-1}\right)^n \sum_{j=0}^{n} a_j \left(\frac{z^{-1}-\alpha}{1-\alpha z^{-1}}\right)^j = \sum_{j=0}^{n}\left\{\left(1-\alpha z^{-1}\right)^{n-j}\left(z^{-1}-\alpha\right)^j\right\} a_j. \quad (6)$$

From (5) and (6), we have

$$\left(1-\alpha z^{-1}\right)^{n-j}\left(z^{-1}-\alpha\right)^j = \sum_{i=0}^{n} t_{i,j}^{\mathrm{LP},n} z^{-i}. \quad (7)$$

Proposition 2.1 The first-row elements and the first-column ones of the matrix $\boldsymbol{T}_n^{\mathrm{LP}}$ are as follows.

(i) $$t_{0,j}^{\mathrm{LP},n} = (-\alpha)^j, \quad j = 0,1,\ldots,n, \quad (8)$$

(ii) $$t_{i,0}^{\mathrm{LP},n} = (-\alpha)^i \left\langle \begin{array}{c} n \\ i \end{array} \right\rangle, \quad i = 0,1,\ldots,n. \quad (9)$$

Proof The proofs easily follow from the definition of (2). ■

Proposition 2.2 The elements $\left\{t_{i,j}^{\mathrm{LP},n}\right\}$ of the matrix $\boldsymbol{T}_n^{\mathrm{LP}}$ satisfy the following properties.

(i) $$t_{i,j}^{\mathrm{LP},n} = t_{n-i,n-j}^{\mathrm{LP},n}, \quad i,j = 0,1,\ldots,n, \quad (10)$$

(ii) $$t_{n-i,j}^{\mathrm{LP},n} = (-\alpha)^n \cdot t_{i,j}^{\mathrm{LP},n}\left(\frac{1}{\alpha}\right), \quad i,j = 0,1,\ldots,n, \quad (11)$$

(iii) $$t_{i,j}^{\mathrm{LP},n} = t_{i-1,j-1}^{\mathrm{LP},n} + \alpha \cdot t_{i-1,j}^{\mathrm{LP},n} - \alpha \cdot t_{i,j-1}^{\mathrm{LP},n}, \quad i,j = 0,1,\ldots,n, \quad (12)$$

where

$$t_{i,j}^{\mathrm{LP},n} = 0, \quad i < 0 \text{ or } j < 0 \text{ or } i > n \text{ or } j > n.$$

Proof Proof of (i): Replacing j in (7) by $(n-j)$, we have

$$\left(1-\alpha z^{-1}\right)^{j}\left(z^{-1}-\alpha\right)^{n-j}=\sum_{i=0}^{n} t_{i,n-j}^{\mathrm{LP},n} z^{-i}. \tag{13}$$

Next, replacing z^{-1} in (13) by z, and then multiplying z^{-n} to both sides, we have

$$\left(z^{-1}-\alpha\right)^{j}\left(1-\alpha z^{-1}\right)^{n-j}=\sum_{i=0}^{n} t_{n-i,n-j}^{\mathrm{LP},n} z^{-i}. \tag{14}$$

The left side of (14) is equal to that of (7). This completes the proof.

Proof of (ii): Replacing α in (7) by $1/\alpha$, and then multiplying $(-\alpha)^n$ to both sides, we have

$$\left(1-\alpha z^{-1}\right)^{j}\left(z^{-1}-\alpha\right)^{n-j}=\sum_{i=0}^{n}(-\alpha)^{n}\cdot\left\{t_{i,j}^{\mathrm{LP},n}\left(\frac{1}{\alpha}\right)\right\} z^{-i}. \tag{15}$$

The left side of (15) is equal to that of (13). Therefore, (11) holds.

Proof of (iii): Replacing j in (7) by $(j-1)$, we have

$$\left(1-\alpha z^{-1}\right)^{n-j+1}\left(z^{-1}-\alpha\right)^{j-1}=\sum_{i=0}^{n} t_{i,j-1}^{\mathrm{LP},n} z^{-i}. \tag{16}$$

From (7) and (16), we have

$$(1-\alpha z^{-1})\sum_{i=0}^{n} t_{i,j}^{\mathrm{LP},n} z^{-i}=(z^{-1}-\alpha)\sum_{i=0}^{n} t_{i,j-1}^{\mathrm{LP},n} z^{-i}. \tag{17}$$

Using (8)-(10), (17) can be rewritten as follows.

$$\sum_{i=1}^{n}\left(t_{i,j}^{\mathrm{LP},n}-t_{i-1,j-1}^{\mathrm{LP},n}-\alpha\cdot t_{i-1,j}^{\mathrm{LP},n}+\alpha\cdot t_{i,j-1}^{\mathrm{LP},n}\right) z^{-i}=0.$$

Therefore, the following equation holds:

$$t_{i,j}^{\mathrm{LP},n}-t_{i-1,j-1}^{\mathrm{LP},n}-\alpha\cdot t_{i-1,j}^{\mathrm{LP},n}+\alpha\cdot t_{i,j-1}^{\mathrm{LP},n}=0,$$

which is equal to (12). ■

Proposition 2.3 The relations between the elements $\left\{t_{i,j}^{\mathrm{LP},n}\right\}$ of $\boldsymbol{T}_n^{\mathrm{LP}}$ and $\left\{t_{i,j}^{\mathrm{LP},n-1}\right\}$ of $\boldsymbol{T}_{n-1}^{\mathrm{LP}}$ are as follows.

(i) $$t_{i,j}^{\mathrm{LP},n}= t_{i,j}^{\mathrm{LP},n-1}-\alpha\cdot t_{i-1,j}^{\mathrm{LP},n-1},\quad i,j=0,1,\ldots,n, \tag{18}$$

(ii) $$t_{i,j}^{\mathrm{LP},n}= t_{i-1,j-1}^{\mathrm{LP},n-1}-\alpha\cdot t_{i,j-1}^{\mathrm{LP},n-1},\quad i,j=0,1,\ldots,n. \tag{19}$$

Proof Proof of (i): Replacing n in (7) by $(n-1)$, we have

$$\left(1-\alpha z^{-1}\right)^{n-1-j}\left(z^{-1}-\alpha\right)^{j}=\sum_{i=0}^{n-1} t_{i,j}^{n-1} z^{-i}. \tag{20}$$

From (7) and (20), we have

$$\begin{aligned}
\sum_{i=0}^{n} t_{i,j}^{\mathrm{LP},n} z^{-i} &= \left(1-\alpha z^{-1}\right)\sum_{i=0}^{n-1} t_{i,j}^{n-1} z^{-i} \\
&= \sum_{i=0}^{n-1} t_{i,j}^{n-1} z^{-i} - \alpha \sum_{i=1}^{n} t_{i-1,j}^{n-1} z^{-i} \\
&= \sum_{i=0}^{n} \left(t_{i,j}^{n-1} - \alpha \cdot t_{i-1,j}^{n-1}\right) z^{-i}.
\end{aligned}$$

Therefore, (18) holds.

Proof of (ii): Clearly (19) holds true from (12) and (18). ■

Proposition 2.4 The multiplication of the frequency transformation matrices $\boldsymbol{T}_n^{\mathrm{LP}}(\alpha)$ and $\boldsymbol{T}_n^{\mathrm{LP}}(-\alpha)$ becomes to $(1-\alpha^2)^n$-times identity matrix, that is,

$$\boldsymbol{T}_n^{\mathrm{LP}}(\alpha)\cdot\boldsymbol{T}_n^{\mathrm{LP}}(-\alpha)=\boldsymbol{T}_n^{\mathrm{LP}}(-\alpha)\cdot\boldsymbol{T}_n^{\mathrm{LP}}(\alpha)=(1-\alpha^2)^n\boldsymbol{I}_{n+1}, \tag{21}$$

where $\boldsymbol{I}_{n+1}$ is an identity matrix of $(n+1)\times(n+1)$.

Proof The matrix $\boldsymbol{T}_n^{\mathrm{LP}}(\alpha)$ describes the transformation from the coefficients of the original low-pass filter (with the cut-off frequency ω_0) into those of the derived low-pass filter (with the cut-off frequency ω_1).

Here, we consider the inverse transformation from the derived low-pass filter into the original. The transformation parameter α' of this case is given as

$$\alpha'=\frac{\sin\left(\dfrac{\omega_1-\omega_0}{2}\right)T}{\sin\left(\dfrac{\omega_1+\omega_0}{2}\right)T}=-\frac{\sin\left(\dfrac{\omega_0-\omega_1}{2}\right)T}{\sin\left(\dfrac{\omega_0+\omega_1}{2}\right)T}=-\alpha,$$

which means that the inverse transformation matrix is given by $\boldsymbol{T}_n^{\mathrm{LP}}(-\alpha)$. Hence, there exists $p\in\boldsymbol{R}$ which satisfy

$$\boldsymbol{T}_n^{\mathrm{LP}}(\alpha)\cdot\boldsymbol{T}_n^{\mathrm{LP}}(-\alpha)=\boldsymbol{T}_n^{\mathrm{LP}}(-\alpha)\cdot\boldsymbol{T}_n^{\mathrm{LP}}(\alpha)=p\cdot\boldsymbol{I}_{n+1}. \tag{22}$$

The parameter p in (22) can be obtained by the multiplication of the first-row elements of $\boldsymbol{T}_n^{\mathrm{LP}}(\alpha)$ and the first-column ones of $\boldsymbol{T}_n^{\mathrm{LP}}(-\alpha)$, i.e.,

$$\begin{aligned} p &= \sum_{i=0}^{n} \left\{ t_{0,i}^{\mathrm{LP},n}(\alpha) \cdot t_{i,0}^{\mathrm{LP},n}(-\alpha) \right\} = \sum_{i=0}^{n} (-\alpha)^i \cdot \alpha^i \left\langle \begin{array}{c} n \\ i \end{array} \right\rangle \\ &= \sum_{i=0}^{n} (-\alpha^2)^i \left\langle \begin{array}{c} n \\ i \end{array} \right\rangle = (1-\alpha^2)^n. \end{aligned}$$

Therefore, (21) holds. ∎

2.2 Properties of the Matrix to Design High-pass Filters

The frequency transformation formula to design the high-pass filter with the cut-off frequency ω_1 is shown in Table 1, and the matrix which corresponds to that transformation was obtained [7] as

$$\boldsymbol{T}_n^{\mathrm{HP}} := \boldsymbol{T}_n^{\mathrm{HP}}(\tilde{\alpha}) = \left[t_{i,j}^{\mathrm{HP},n}(\tilde{\alpha}) \right] \in \boldsymbol{R}^{(n+1)\times(n+1)},$$

where

$$\begin{aligned} t_{i,j}^{\mathrm{HP},n} &:= t_{i,j}^{\mathrm{HP},n}(\tilde{\alpha}) \\ &= \sum_{k=0}^{i} (-1)^j \left\langle \begin{array}{c} j \\ i-k \end{array} \right\rangle \left\langle \begin{array}{c} n-j \\ k \end{array} \right\rangle \tilde{\alpha}^{j-i+2k} \\ &= (-1)^i \cdot t_{i,j}^{\mathrm{LP},n}(\tilde{\alpha}). \end{aligned}$$

Here, the following three propositions can be obtained in the same manner as Subsection 2.1.

Proposition 2.5 The first-row elements and the first-column ones of the matrix $\boldsymbol{T}_n^{\mathrm{HP},n}$ are as follows.

$$\begin{aligned} &\text{(i)} & t_{0,j}^{\mathrm{HP},n} &= (-\tilde{\alpha})^j, \quad j = 0,1,\ldots,n, \\ &\text{(ii)} & t_{i,0}^{\mathrm{HP},n} &= \tilde{\alpha}^i \left\langle \begin{array}{c} n \\ i \end{array} \right\rangle, \quad i = 0,1,\ldots,n. \end{aligned}$$

∎

Proposition 2.6 The elements $\left\{ t_{i,j}^{\mathrm{HP},n} \right\}$ of the matrix $\boldsymbol{T}_n^{\mathrm{HP}}$ satisfy the following properties.

$$\begin{aligned} &\text{(ii)} & t_{i,j}^{\mathrm{HP},n} &= t_{n-i,n-j}^{\mathrm{HP},n}, \quad i,j = 0,1,\ldots,n, \\ &\text{(ii)} & t_{n-i,j}^{\mathrm{HP},n} &= (-\tilde{\alpha})^n \cdot t_{i,j}^{\mathrm{HP},n}\left(\frac{1}{\tilde{\alpha}}\right), \quad i,j = 0,1,\ldots,n, \\ &\text{(iii)} & t_{i,j}^{\mathrm{HP},n} &= -\left(t_{i-1,j-1}^{\mathrm{HP},n} + \tilde{\alpha} \cdot t_{i-1,j}^{\mathrm{HP},n} + \tilde{\alpha} \cdot t_{i,j-1}^{\mathrm{HP},n} \right), \quad i,j = 0,1,\ldots,n, \end{aligned}$$

where

$$t_{i,j}^{\mathrm{HP},n} = 0, \quad i < 0 \text{ or } j < 0 \text{ or } i > n \text{ or } j > n. \qquad \blacksquare$$

Proposition 2.7 The relations between the elements $\left\{t_{i,j}^{\mathrm{HP},n}\right\}$ of $\boldsymbol{T}_n^{\mathrm{HP}}$ and $\left\{t_{i,j}^{\mathrm{HP},n-1}\right\}$ of $\boldsymbol{T}_{n-1}^{\mathrm{HP}}$ are as follows.

$$\begin{aligned}
&\text{(i)} \qquad t_{i,j}^{\mathrm{HP},n} = t_{i,j}^{\mathrm{HP},n-1} + \tilde{\alpha} \cdot t_{i-1,j}^{\mathrm{HP},n-1}, \quad i,j = 0,1,\ldots,n,\\
&\text{(ii)} \qquad t_{i,j}^{\mathrm{HP},n} = t_{i-1,j-1}^{\mathrm{HP},n-1} + \tilde{\alpha} \cdot t_{i,j-1}^{\mathrm{HP},n-1}, \quad i,j = 0,1,\ldots,n.
\end{aligned} \qquad \blacksquare$$

The matrix $\boldsymbol{T}_n^{\mathrm{HP}}$ also have the following property.

Proposition 2.8 A square of the matrix $\boldsymbol{T}_n^{\mathrm{HP}}(\tilde{\alpha})$ becomes to $(1-\tilde{\alpha}^2)^n$-times identity matrix, that is,

$$\boldsymbol{T}_n^{\mathrm{HP}}(\tilde{\alpha}) \cdot \boldsymbol{T}_n^{\mathrm{HP}}(\tilde{\alpha}) = (1-\tilde{\alpha}^2)^n \boldsymbol{I}_{n+1}. \tag{23}$$

Proof The matrix $\boldsymbol{T}_n^{\mathrm{HP}}(\tilde{\alpha})$ describes the transformation from the coefficients of the original low-pass filter (with the cut-off frequency ω_0) into those of the derived high-pass filter (with the cut-off frequency ω_1).

Here, we consider the inverse transformation from the derived high-pass filter into the original low-pass filter. The transformation parameter $\tilde{\alpha}'$ of this case is given as

$$\tilde{\alpha}' = -\frac{\cos\left(\dfrac{\omega_1+\omega_0}{2}\right)T}{\cos\left(\dfrac{\omega_1-\omega_0}{2}\right)T} = -\frac{\cos\left(\dfrac{\omega_0+\omega_1}{2}\right)T}{\cos\left(\dfrac{\omega_0-\omega_1}{2}\right)T} = \tilde{\alpha},$$

which means that the inverse transformation matrix is also given by $\boldsymbol{T}_n^{\mathrm{HP}}(\tilde{\alpha})$. $\blacksquare$

2.3 Properties of the Matrix to Design Band-pass Filters

The frequency transformation formula to design the band-pass filter with the cut-off frequencies ω_1, ω_2 $(\omega_1 < \omega_2)$ is shown in Table 1, and the matrix $\boldsymbol{T}_n^{\mathrm{BP}}$ which corresponds to that transformation was obtained [7] as

$$\boldsymbol{T}_n^{\mathrm{BP}} := \left[t_{i,j}^{\mathrm{BP},n}(u,v)\right] \in \boldsymbol{R}^{(2n+1)\times(n+1)},$$

where

$$
\begin{aligned}
t_{i,j}^{\mathrm{BP},n} &:= t_{i,j}^{\mathrm{BP},n}(u,v) = \sum_{\ell=0}^{i}(-1)^j \beta_{i-\ell,j}\gamma_{\ell,j}, && (24)\\
\beta_{\ell,j} &= \sum_{k=0}^{\lfloor \ell/2 \rfloor} \left\langle \begin{array}{c} j \\ \ell-k \end{array} \right\rangle \left\langle \begin{array}{c} \ell-k \\ k \end{array} \right\rangle u^{j-\ell+k} v^{\ell-2k}, && (25)\\
\gamma_{\ell,j} &= \sum_{k=0}^{\lfloor \ell/2 \rfloor} \left\langle \begin{array}{c} n-j \\ \ell-k \end{array} \right\rangle \left\langle \begin{array}{c} \ell-k \\ k \end{array} \right\rangle u^{k} v^{\ell-2k}. && (26)
\end{aligned}
$$

In (25) and (26), $\lfloor \ell/2 \rfloor$ denotes the maximum integer not exceeding $\ell/2$. The transfer function $H_2(z)$ of the derived filter can be written as

$$
H_2(z) = \frac{e_0 + e_1 z^{-1} + \cdots + e_n z^{-n} + \cdots + e_{2n} z^{-2n}}{f_0 + f_1 z^{-1} + \cdots + f_n z^{-n} + \cdots + f_{2n} z^{-2n}}. \tag{27}
$$

Here, the relations between the coefficients of (1) and those of (27) are as follows:

$$
\begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ e_n \\ \vdots \\ e_{2n} \end{bmatrix} = \boldsymbol{T}_n^{\mathrm{BP}} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}, \quad \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \\ \vdots \\ f_{2n} \end{bmatrix} = \boldsymbol{T}_n^{\mathrm{BP}} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{bmatrix}. \tag{28}
$$

Since the coefficients $\{e_i\}_{i=0}^{2n}$ of (27) can be obtained by (28), the numerator polynomial $N_2(z)$ of $H_2(z)$ can be written as

$$
\begin{aligned}
N_2(z) &= \sum_{i=0}^{2n} e_i z^{-i} \\
&= \sum_{i=0}^{2n} \left(\sum_{j=0}^{n} t_{i,j}^{\mathrm{BP},n} a_j \right) z^{-i} \\
&= \sum_{j=0}^{n} \left(\sum_{i=0}^{2n} t_{i,j}^{\mathrm{BP},n} z^{-i} \right) a_j. && (29)
\end{aligned}
$$

On the other hand, since the transfer function $H_2(z)$ can be obtained as a result of the frequency transformation of $H_0(z)$, $N_2(z)$ can also be written as

$$
\begin{aligned}
N_2(z) &= \sum_{i=0}^{2n} e_i z^{-i} \\
&= \left(uz^{-2} + vz^{-1} + 1\right)^n \sum_{j=0}^{n} a_j \left(-\frac{z^{-2} + vz^{-1} + u}{uz^{-2} + vz^{-1} + 1} \right)^j \\
&= \sum_{j=0}^{n} \left\{ (-1)^j \left(uz^{-2} + vz^{-1} + 1\right)^{n-j} \left(z^{-2} + vz^{-1} + u\right)^j \right\} a_j. && (30)
\end{aligned}
$$

From (29) and (30), we have

$$(-1)^j \left(uz^{-2} + vz^{-1} + 1\right)^{n-j} \left(z^{-2} + vz^{-1} + u\right)^j = \sum_{i=0}^{2n} t_{i,j}^{\mathrm{BP},n} z^{-i}. \tag{31}$$

Proposition 2.9 The first-row elements and the first-column ones of the matrix $\boldsymbol{T}_n^{\mathrm{BP}}$ are as follows.

(i) $$t_{0,j}^{\mathrm{BP},n} = (-u)^j, \quad j = 0, 1, \ldots, n, \tag{32}$$

(ii) $$t_{i,0}^{\mathrm{BP},n} = \sum_{k=0}^{i} \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle \left\langle \begin{matrix} k \\ i-k \end{matrix} \right\rangle u^{i-k} v^{2k-i}, \quad i = 0, 1, \ldots, 2n. \tag{33}$$

Proof The proofs of (32) and (33) easily follow from (24) and (31), respectively. ■

Proposition 2.10 The elements $\left\{t_{i,j}^{\mathrm{BP},n}\right\}$ of the matrix $\boldsymbol{T}_n^{\mathrm{BP}}$ satisfy the following properties.

(i) $$t_{i,j}^{\mathrm{BP},n} = (-1)^n \cdot t_{2n-i,n-j}^{\mathrm{BP},n}, \quad i = 0, 1, \ldots, 2n \text{ and } j = 0, 1, \ldots, n, \tag{34}$$

(ii) $$t_{n-i,j}^{\mathrm{BP},n} = (-u)^n \cdot t_{i,j}^{\mathrm{BP},n}\left(\frac{1}{u}, \frac{v}{u}\right), \quad i = 0, 1, \ldots, 2n \text{ and } j = 0, 1, \ldots, n, \tag{35}$$

(iii) $$t_{i,j}^{\mathrm{BP},n} = -\left\{t_{i-2,j-1}^{\mathrm{BP},n} + v\left(t_{i-1,j-1}^{\mathrm{BP},n} + t_{i-1,j}^{\mathrm{BP},n}\right) + u\left(t_{i,j-1}^{\mathrm{BP},n} + t_{i-2,j}^{\mathrm{BP},n}\right)\right\},$$
$$i = 0, 1, \ldots, 2n \text{ and } j = 0, 1, \ldots, n, \tag{36}$$

where

$$t_{i,j}^{\mathrm{BP},n} = 0, \quad i < 0 \text{ or } j < 0 \text{ or } i > 2n \text{ or } j > n.$$

Proof Proof of (i): Replacing j in (31) by $(n-j)$, we have

$$(-1)^{n-j} \left(uz^{-2} + vz^{-1} + 1\right)^j \left(z^{-2} + vz^{-1} + u\right)^{n-j} = \sum_{i=0}^{2n} t_{i,n-j}^{\mathrm{BP},n} z^{-i}. \tag{37}$$

Next, replacing z^{-1} in (37) by z, and then multiplying $(-z)^{-n}$ to both sides, we have

$$(-1)^j \left(z^{-2} + vz^{-1} + u\right)^j \left(uz^{-2} + vz^{-1} + 1\right)^{n-j} = \sum_{i=0}^{2n} \left\{(-1)^n \, t_{n-i,n-j}^{\mathrm{BP},n}\right\} z^{-i}. \tag{38}$$

The left side of (38) is equal to that of (31). Therefore, (34) holds.

Proof of (ii): Replacing u and v in (31) by $1/u$ and $1/v$ respectively, and then multiplying $(-u)^n$ to both sides, we have

$$(-1)^{n-j} \left(z^{-2} + vz^{-1} + u\right)^{n-j} \left(uz^{-2} + vz^{-1} + 1\right)^j = \sum_{i=0}^{2n} (-u)^n \cdot \left\{t_{i,j}^{\mathrm{BP},n}\left(\frac{1}{u}, \frac{v}{u}\right)\right\} z^{-i}. \tag{39}$$

The left side of (39) is equal to that of (37). This completes the proof.

Proof of (iii): Replacing j in (31) by $(j-1)$, we have

$$(-1)^{j-1}\left(uz^{-2}+vz^{-1}+1\right)^{n-j+1}\left(z^{-2}+vz^{-1}+u\right)^{j-1}=\sum_{i=0}^{2n}t_{i,j-1}^{\mathrm{BP},n}z^{-i}. \tag{40}$$

From (31) and (40), we have

$$\left(uz^{-2}+vz^{-1}+1\right)\sum_{i=0}^{2n}t_{i,j}^{\mathrm{BP},n}z^{-i}=-\left(z^{-2}+vz^{-1}+u\right)\sum_{i=0}^{2n}t_{i,j-1}^{\mathrm{BP},n}z^{-i}. \tag{41}$$

Using (32)-(34), (41) can be rewritten as follows.

$$\sum_{i=1}^{n}\left\{t_{i,j}^{\mathrm{BP},n}+t_{i-2,j-1}^{\mathrm{BP},n}+v\left(t_{i-1,j-1}^{\mathrm{BP},n}+t_{i-1,j}^{\mathrm{BP},n}\right)+u\left(t_{i,j-1}^{\mathrm{BP},n}+t_{i-2,j}^{\mathrm{BP},n}\right)\right\}z^{-i}=0.$$

Therefore, the following equation holds:

$$t_{i,j}^{\mathrm{BP},n}+t_{i-2,j-1}^{\mathrm{BP},n}+v\left(t_{i-1,j-1}^{\mathrm{BP},n}+t_{i-1,j}^{\mathrm{BP},n}\right)+u\left(t_{i,j-1}^{\mathrm{BP},n}+t_{i-2,j}^{\mathrm{BP},n}\right)=0,$$

which is equal to (36). ■

Proposition 2.11 The relations between the elements $\left\{t_{i,j}^{\mathrm{BP},n}\right\}$ of $\boldsymbol{T}_n^{\mathrm{BP}}$ and $\left\{t_{i,j}^{\mathrm{BP},n-1}\right\}$ of $\boldsymbol{T}_{n-1}^{\mathrm{BP}}$ are as follows.

$$\text{(i)}\qquad t_{i,j}^{\mathrm{BP},n}=u\cdot t_{i-2,j}^{\mathrm{BP},n-1}+v\cdot t_{i-1,j}^{\mathrm{BP},n-1}+t_{i,j}^{\mathrm{BP},n-1},$$
$$i=0,1,\ldots,2n \text{ and } j=0,1,\ldots,n, \tag{42}$$

$$\text{(ii)}\qquad t_{i,j}^{\mathrm{BP},n}=-\left(u\cdot t_{i,j}^{\mathrm{BP},n-1}+v\cdot t_{i-1,j}^{\mathrm{BP},n-1}+t_{i-2,j}^{\mathrm{BP},n-1}\right),$$
$$i=0,1,\ldots,2n \text{ and } j=0,1,\ldots,n. \tag{43}$$

Proof Proof of (i): Replacing n in (31) by $(n-1)$, we have

$$(-1)^{j}\left(uz^{-2}+vz^{-1}+1\right)^{n-1-j}\left(z^{-2}+vz^{-1}+u\right)^{j}=\sum_{i=0}^{2n-2}t_{i,j}^{\mathrm{BP},n-1}z^{-i}. \tag{44}$$

From (31) and (44), we have

$$\begin{aligned}
\sum_{i=0}^{2n}t_{i,j}^{\mathrm{BP},n}z^{-i} &= \left(uz^{-2}+vz^{-1}+1\right)\sum_{i=0}^{2n-2}t_{i,j}^{\mathrm{BP},n-1}z^{-i}\\
&= u\sum_{i=0}^{2n-2}t_{i,j}^{\mathrm{BP},n-1}z^{-(i+2)}+v\sum_{i=0}^{2n-2}t_{i,j}^{\mathrm{BP},n-1}z^{-(i+1)}+\sum_{i=0}^{2n-2}t_{i,j}^{\mathrm{BP},n-1}z^{-i}\\
&= u\sum_{i=2}^{2n}t_{i-2,j}^{\mathrm{BP},n-1}z^{-i}+v\sum_{i=1}^{2n-1}t_{i-1,j}^{\mathrm{BP},n-1}z^{-i}+\sum_{i=0}^{2n-2}t_{i,j}^{\mathrm{BP},n-1}z^{-i}\\
&= \sum_{i=0}^{2n}\left(u\cdot t_{i-2,j}^{\mathrm{BP},n-1}+v\cdot t_{i-1,j}^{\mathrm{BP},n-1}+t_{i,j}^{\mathrm{BP},n-1}\right)z^{-i}.
\end{aligned}$$

Therefore, (42) holds.

Proof of (ii): Clearly (43) is true from (36) and (42). ■

2.4 Properties of the Matrix to Design Band-stop Filters

The frequency transformation formula to design the band-pass filter with the cut-off frequencies ω_1, ω_2 ($\omega_1 < \omega_2$) is shown in Table 1, and the matrix which corresponds to this transformation was obtained [7] as

$$
\begin{aligned}
\boldsymbol{T}_n^{\mathrm{BS}} &:= \left[t_{ij}^{\mathrm{BS},n}(\tilde{u},\tilde{v})\right] \in \boldsymbol{R}^{(2n+1)\times(n+1)} \\
&= \left.\boldsymbol{T}_n^{\mathrm{BP}}\right|_{u=\tilde{u},v=\tilde{v}} \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & \ddots & \\ 0 & & & (-1)^n \end{bmatrix},
\end{aligned}
$$

where

$$
\begin{aligned}
t_{i,j}^{\mathrm{BS},n} &:= t_{i,j}^{\mathrm{BS},n}(\tilde{u},\tilde{v}) \\
&= (-1)^j \cdot t_{i,j}^{\mathrm{BP},n}(\tilde{u},\tilde{v}) \\
&= \left.\sum_{\ell=0,}^{i} \beta_{i-\ell,j}\gamma_{\ell,j}\right|_{u=\tilde{u},v=\tilde{v}} .
\end{aligned}
$$

Then, the following three propositions can be obtained in the same manner as Subsection 2.3.

Proposition 2.12 The first-row elements and the first-column ones of the matrix $\boldsymbol{T}_n^{\mathrm{BS},n}$ can be written as follows.

(i) $t_{0,j}^{\mathrm{BS},n} = \tilde{u}^j, \quad j = 0, 1, \ldots, n,$

(ii) $t_{i,0}^{\mathrm{BS},n} = \displaystyle\sum_{k=0}^{i} \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle \left\langle \begin{matrix} k \\ i-k \end{matrix} \right\rangle \tilde{u}^{i-k}\tilde{v}^{2k-i}, \quad i = 0, 1, \ldots, 2n.$ ■

Proposition 2.13 The elements $\left\{t_{i,j}^{\mathrm{BS},n}\right\}$ of the matrix $\boldsymbol{T}_n^{\mathrm{BS}}$ satisfy the following properties.

(i) $t_{i,j}^{\mathrm{BS},n} = t_{2n-i,n-j}^{\mathrm{BS},n}, \quad i = 0, 1, \ldots, 2n$ and $j = 0, 1, \ldots, n.$

(ii) $t_{n-i,j}^{\mathrm{BS},n} = \tilde{u}^n \cdot t_{i,j}^{\mathrm{BS},n}\left(\dfrac{1}{\tilde{u}}, \dfrac{\tilde{v}}{\tilde{u}}\right), \quad i = 0, 1, \ldots, 2n$ and $j = 0, 1, \ldots, n.$

(iii) $t_{i,j}^{\mathrm{BS},n} = t_{i-2,j-1}^{\mathrm{BS},n} + \tilde{v}\left(t_{i-1,j-1}^{\mathrm{BS},n} - t_{i-1,j}^{\mathrm{BS},n}\right) + \tilde{u}\left(t_{i,j-1}^{\mathrm{BS},n} - t_{i-2,j}^{\mathrm{BS},n}\right),$

$i = 0, 1, \ldots, 2n$ and $j = 0, 1, \ldots, n,$

where

$$t_{i,j}^{\mathrm{BS},n} = 0, \quad i < 0 \text{ or } j < 0 \text{ or } i > 2n \text{ or } j > n. \qquad \blacksquare$$

Proposition 2.14 The relations between the elements $\left\{t_{i,j}^{\mathrm{BS},n}\right\}$ of $\boldsymbol{T}_n^{\mathrm{BS}}$ and $\left\{t_{i,j}^{\mathrm{BS},n-1}\right\}$ of $\boldsymbol{T}_{n-1}^{\mathrm{BS}}$ are as follows.

$$\text{(i)} \qquad t_{i,j}^{\mathrm{BS},n} = \tilde{u} \cdot t_{i-2,j}^{\mathrm{BS},n-1} + \tilde{v} \cdot t_{i-1,j}^{\mathrm{BS},n-1} + t_{i,j}^{\mathrm{BS},n-1},$$
$$i = 0, 1, \ldots, 2n \text{ and } j = 0, 1, \ldots, n,$$

$$\text{(ii)} \qquad t_{i,j}^{\mathrm{BS},n} = -\left(\tilde{u} \cdot t_{i,j}^{\mathrm{BS},n-1} + \tilde{v} \cdot t_{i-1,j}^{\mathrm{BS},n-1} + t_{i-2,j}^{\mathrm{BS},n-1}\right),$$
$$i = 0, 1, \ldots, 2n \text{ and } j = 0, 1, \ldots, n, \qquad \blacksquare$$

3 Fast Algorithms to Compute Frequency Transformation Matrices

In this section, some fast algorithms to compute the elements of the frequency transformation matrices are first proposed, and then the algorithms are compared with those based on the definitions by the number of addition and multiplication. Based on the comparison, the best algorithms will be chosen.

3.1 Fast Design of Low-pass Filters

To compute all the elements of the matrix $\boldsymbol{T}_n^{\mathrm{LP}}$ to design low-pass filters, the following two algorithms can be considered.

Algorithm 3.1 All the elements $\left\{t_{i,j}^{\mathrm{LP},n}\right\}$ of the matrix $\boldsymbol{T}_n^{\mathrm{LP}}$ are computed by the following five steps.

[a] The element $t_{0,0}^{\mathrm{LP},n}$ is one for any n.

[b] From (8), the following relation holds.

$$t_{0,j}^{\mathrm{LP},n} = (-\alpha) \cdot t_{0,j-1}^{\mathrm{LP},n}, \quad j = 1, 2, \ldots, n. \tag{45}$$

Based on the relation (45), the first-row elements $\left\{t_{0,j}^{\mathrm{LP},n}\right\}_{j=1}^{n}$ are determined.

[c] From (9), the following relation holds.

$$t_{i,0}^{\mathrm{LP},n} = t_{0,i}^{\mathrm{LP},n} \left\langle \begin{array}{c} n \\ i \end{array} \right\rangle, \quad i = 1, 2, \ldots, n. \tag{46}$$

Based on the relation (46), the first-column elements $\left\{t_{i,0}^{\mathrm{LP},n}\right\}_{i=1}^{\lfloor n/2 \rfloor}$ are determined.

[d] Based on (12), the elements $\left\{t_{i,j}^{\mathrm{LP},n}\right\}_{i=1}^{\lfloor n/2 \rfloor}$ are determined for $j = 1, 2, \ldots, n$.

[e] Based on (10), the elements $\left\{t_{i,j}^{\mathrm{LP},n}\right\}_{i=\lfloor n/2 \rfloor+1}^{n}$ are determined for $j = 0, 1, \ldots, n$. ■

Algorithm 3.2 The elements $\left\{t_{i,j}^{\mathrm{LP},n}\right\}$ of the matrix $\boldsymbol{T}_n^{\mathrm{LP}}$ can also be computed by the following six steps.

[a] Same to [a] of Algorithm 3.1.

[b] Same to [b] of Algorithm 3.1.

[c] Same to [c] of Algorithm 3.1.

[d] Based on (18), the elements $\left\{t_{i,j}^{\mathrm{LP},n}\right\}_{i=1}^{\lfloor n/2 \rfloor}$ are determined for $j = 1, 2, \ldots, n-1$. In this case, the elements of the $(n-1)$th-order matrix $\boldsymbol{T}_{n-1}^{\mathrm{LP}}$ are required. They are computed recursively.

[e] Based on (19), the $(n+1)$th-column elements $\left\{t_{i,n}^{\mathrm{LP},n}\right\}_{i=1}^{\lfloor n/2 \rfloor}$ are determined.

[f] Same to [e] of Algorithm 3.1. ■

In this paper, three algorithms are compared by the number of computations of addition (including subtraction) and multiplication (including division), respectively. The three algorithms to be compared are as follows.

(a) Algorithm based on the definition (2).

(b) Algorithm 3.1.

(c) Algorithm 3.2.

Figures 1 and 2 respectively show the number of addition and multiplication to compute all the elements of the matrix $\boldsymbol{T}_n^{\mathrm{LP}}$ for the above three algorithms. Comparing (a) with (b) and (c), the latter two can compute the elements of the matrix $\boldsymbol{T}_n^{\mathrm{LP}}$ with much fewer operation than (a). Comparing (b) with (c), (b) seems to work as same as (c) in the case $n \leq 3$, and better than (c) in the case $n \geq 4$. Therefore, the algorithm (b) can be considered as the best algorithm to compute the elements of the matrix $\boldsymbol{T}_n^{\mathrm{LP}}$.

Similarly to Algorithm 3.1, all the elements of the matrix $\boldsymbol{T}_n^{\mathrm{HP}}$ to design high-pass filters can be quickly computed.

3.2 Fast Design of Band-pass Filters

To compute all the elements of the matrix $\boldsymbol{T}_n^{\mathrm{BP}}$ to design band-pass filters, the following two algorithms can also be considered.

Algorithm 3.3 All the elements $\left\{t_{i,j}^{\mathrm{BP},n}\right\}$ of the matrix $\boldsymbol{T}_n^{\mathrm{BP}}$ are computed by the following five steps.

[a] The first-row first-column element $t_{0,0}^{\mathrm{BP},n}$ is one for any n.

[b] From (32), the following relation holds.

$$t_{0,j}^{\mathrm{BP},n} = (-u) \cdot t_{0,j-1}^{\mathrm{BP},n}, \quad j = 1, 2, \ldots, n. \tag{47}$$

Based on the relation (47), the first-row elements $\left\{t_{0,j}^{\mathrm{BP},n}\right\}_{j=1}^{n}$ are determined.

[c] Based on (33), the first-column elements $\left\{t_{i,0}^{\mathrm{BP},n}\right\}_{i=1}^{n}$ are determined.

[d] Based on (36), the elements $\left\{t_{i,j}^{\mathrm{BP},n}\right\}_{i=1}^{n}$ are determined for $j = 1, 2, \ldots, n$.

[e] Based on (34), the elements $\left\{t_{i,j}^{\mathrm{BP},n}\right\}_{i=n+1}^{2n}$ are determined for $j = 0, 1, \ldots, n$. ■

Algorithm 3.4 The elements $\left\{t_{i,j}^{\mathrm{BP},n}\right\}$ of the matrix $\boldsymbol{T}_n^{\mathrm{BP}}$ can also be computed by the following six steps.

[a] Same to [a] of Algorithm 3.3.

[b] Same to [b] of Algorithm 3.3.

[c] Same to [c] of Algorithm 3.3.

[d] Based on (42), the elements $\left\{t_{i,j}^{\mathrm{BP},n}\right\}_{i=1}^{n}$ are determined for $j = 1, 2, \ldots, n-1$. In this case, the elements of the $(n-1)$th-order matrix $\boldsymbol{T}_{n-1}^{\mathrm{BP}}$ must be computed. They are computed recursively.

[e] Based on (43), the $(n+1)$th-column elements $\left\{t_{i,n}^{\mathrm{BP},n}\right\}_{i=1}^{n}$ are determined.

[f] Same to [e] of Algorithm 3.3. ■

As the analogy with Subsection 3.1, three algorithms are compared by the number of computations of addition (including subtraction) and multiplication (including division), respectively. The three algorithms to be compared are as follows.

(a) Algorithm based on the definition (24).

(b) Algorithm 3.3.

(c) Algorithm 3.4.

Figures 3 and 4 respectively show the number of additions and multiplications to compute all the elements of the matrix $\boldsymbol{T}_n^{\mathrm{BP}}$ for the above three algorithms. Comparing (a) with (b) and (c), the latter two can compute the elements of the matrix $\boldsymbol{T}_n^{\mathrm{BP}}$ with much fewer operation than (a). Comparing (b) with (c), (b) seems to work better than the algorithm 3.4 for any n. Therefore, the algorithm (b) can considered as the best algorithm to compute the elements of the matrix $\boldsymbol{T}_n^{\mathrm{BP}}$, also in this case.

Similarly to Algorithm 3.3, all the elements of the matrix $\boldsymbol{T}_n^{\mathrm{BS}}$ to design band-pass filters can be quickly computed.

4 Concluding Remarks

In this paper, some properties of the frequency transformation matrices were investigated, and the fast algorithms to compute the elements of the matrices were also presented. The proposed Algorithms 3.1 (for low-pass filter) and 3.3 (for band-pass filter) can compute all the elements of the matrices with much fewer operation than the algorithms based on the definitions. Similar algorithms to design high-pass and band-pass filters can also be considered.

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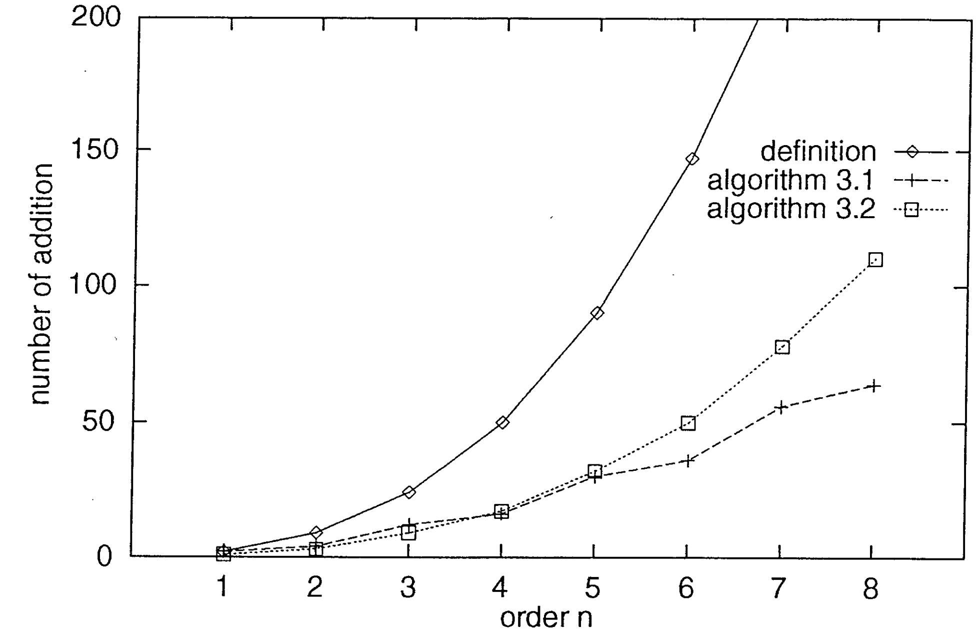
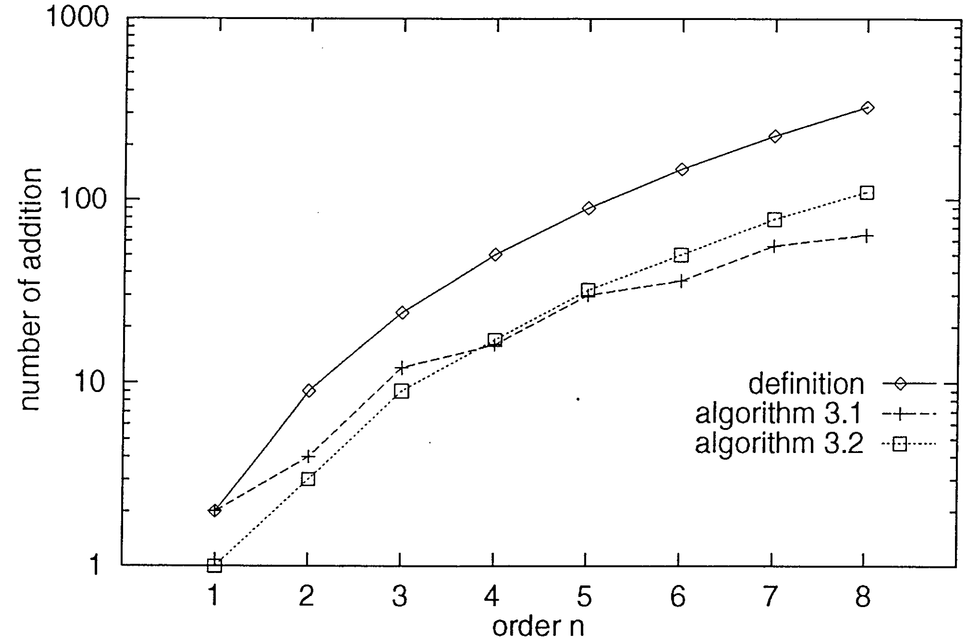
Table 1: Transformation formulae of frequency transformations for IIR digital filters

Filter Type / Cut-off Frequency	Transformation Formulae / Parameters
Low-pass	$z^{-1} = \dfrac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$
ω_1	$\alpha = \dfrac{\sin\left(\dfrac{\omega_0 - \omega_1}{2}\right)T}{\sin\left(\dfrac{\omega_0 + \omega_1}{2}\right)T}$
High-pass	$z^{-1} = -\dfrac{z^{-1} + \tilde{\alpha}}{1 + \tilde{\alpha} z^{-1}}$
ω_1	$\tilde{\alpha} = -\dfrac{\cos\left(\dfrac{\omega_0 + \omega_1}{2}\right)T}{\cos\left(\dfrac{\omega_0 - \omega_1}{2}\right)T}$
Band-pass	$z^{-1} = -\dfrac{z^{-2} + vz^{-1} + u}{uz^{-2} + vz^{-1} + 1}$
$\omega_1, \omega_2 \ (\omega_1 < \omega_2)$	$u = \dfrac{k-1}{k+1}, \quad v = -\dfrac{2\alpha k}{k+1},$ $k = \cot\left(\dfrac{\omega_2 - \omega_1}{2}\right)T \tan\dfrac{\omega_0}{2}T,$ $\alpha = \dfrac{\cos\left(\dfrac{\omega_2 + \omega_1}{2}\right)T}{\cos\left(\dfrac{\omega_2 - \omega_1}{2}\right)T}$
Band-stop	$z^{-1} = \dfrac{z^{-2} + \tilde{v}z^{-1} + \tilde{u}}{\tilde{u}z^{-2} + \tilde{v}z^{-1} + 1}$
$\omega_1, \omega_2 \ (\omega_1 < \omega_2)$	$\tilde{u} = -\dfrac{\tilde{k}-1}{\tilde{k}+1}, \quad \tilde{v} = -\dfrac{2\tilde{\alpha}}{\tilde{k}+1},$ $\tilde{k} = \tan\left(\dfrac{\omega_2 - \omega_1}{2}\right)T \tan\dfrac{\omega_0}{2}T,$ $\tilde{\alpha} = \dfrac{\cos\left(\dfrac{\omega_2 + \omega_1}{2}\right)T}{\cos\left(\dfrac{\omega_2 - \omega_1}{2}\right)T}$

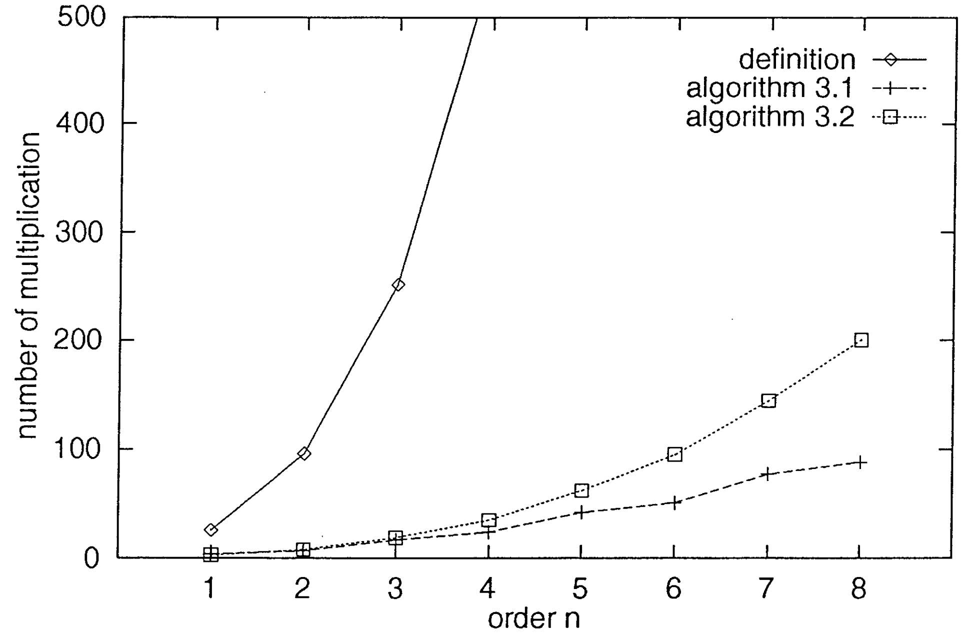


(a) linear scale

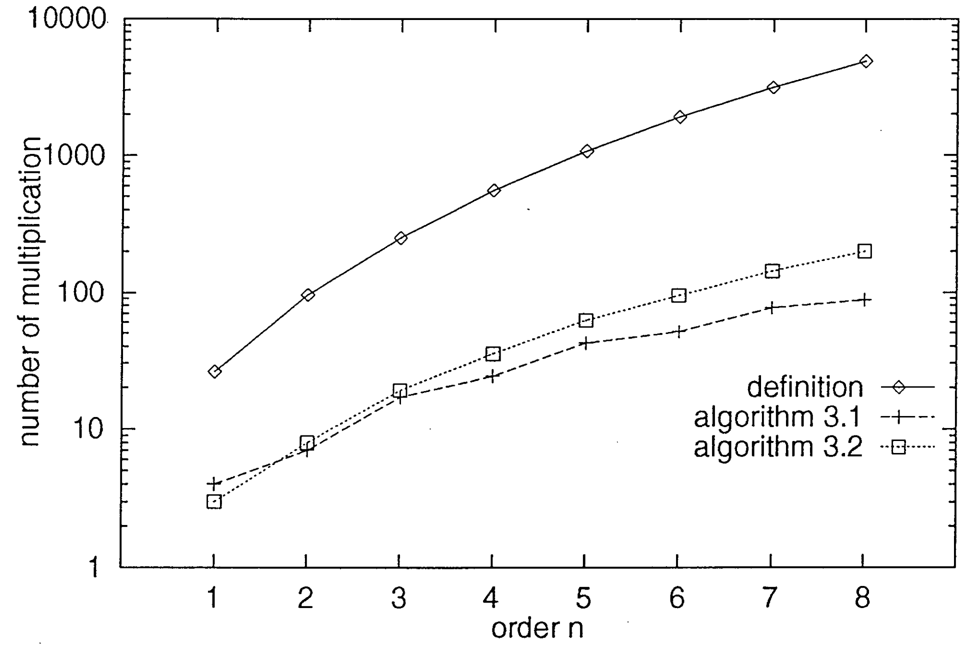


(b) linear-to-log scale

Figure 1: Comparison of the number of additions to compute all the elements of the matrix T_n^{LP}

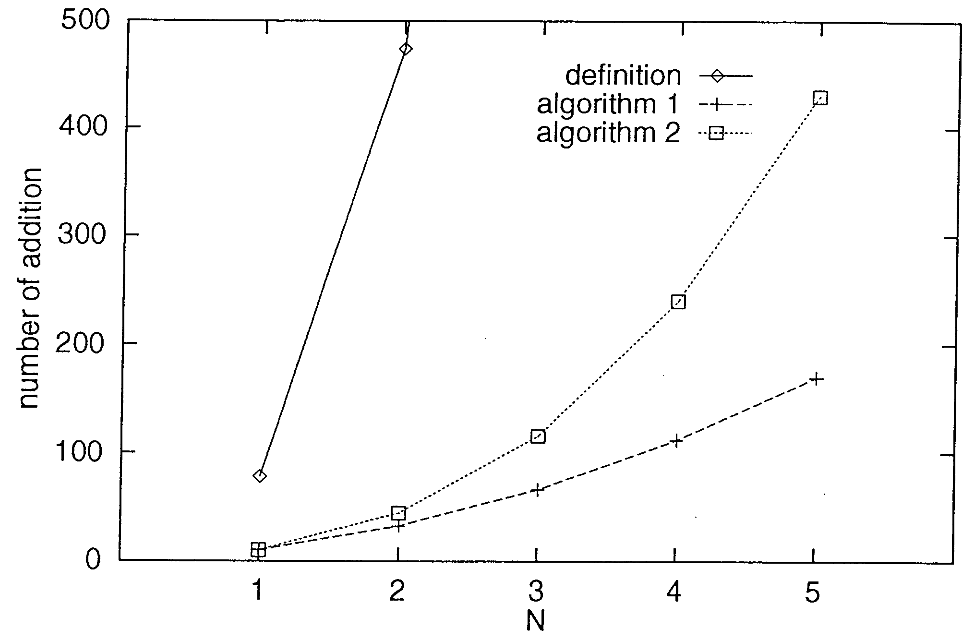


(a) linear scale



(b) linear-to-log scale

Figure 2: Comparison of the number of multiplications to compute all the elements of the matrix $\boldsymbol{T}_n^{\mathrm{LP}}$



(a) linear scale

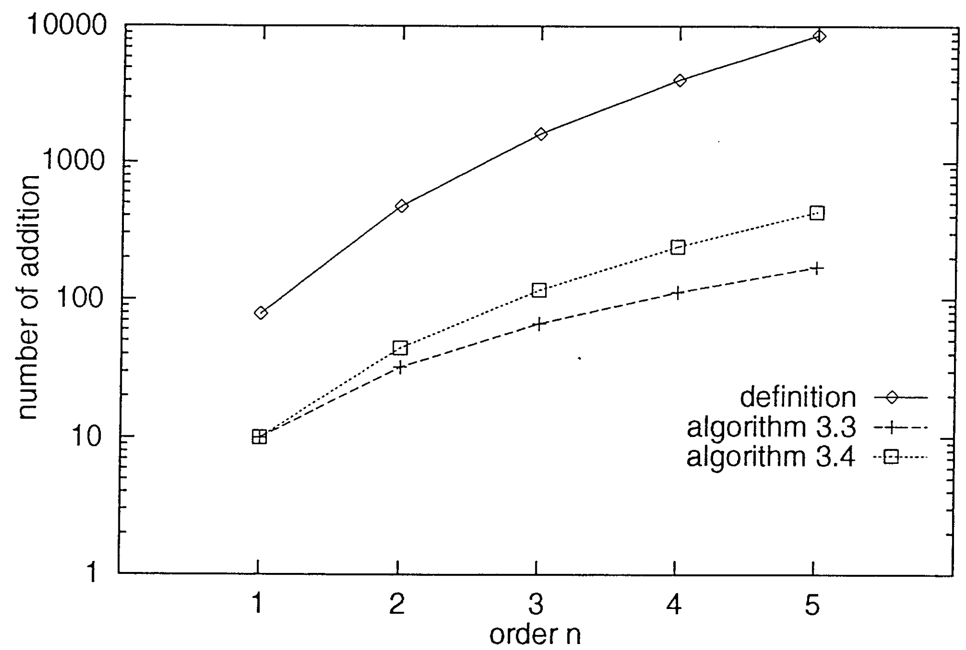
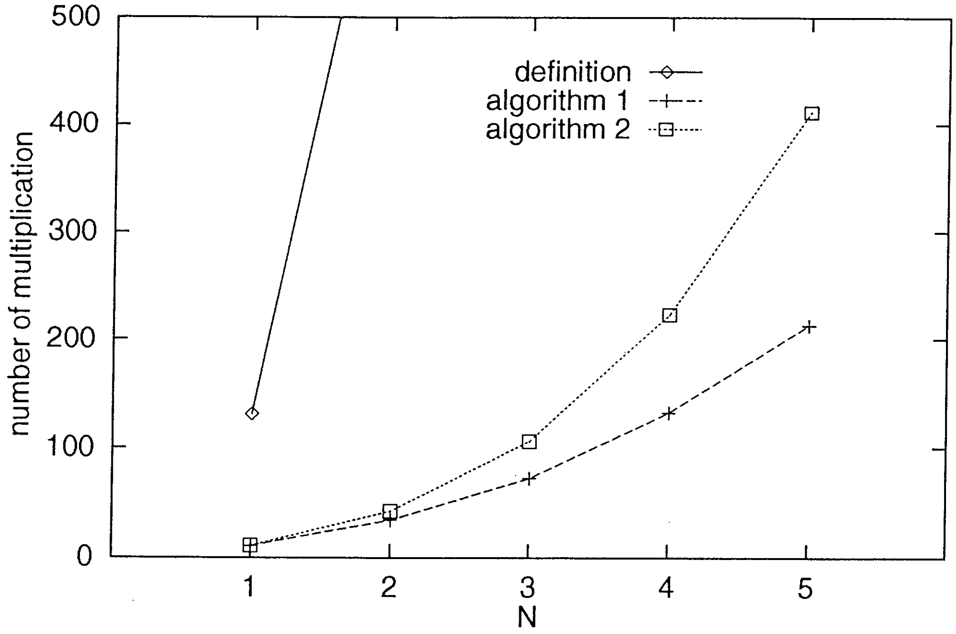


Figure 3: Comparison of the number of additions to compute all the elements of the matrix T_n^{BP}



(a) linear scale

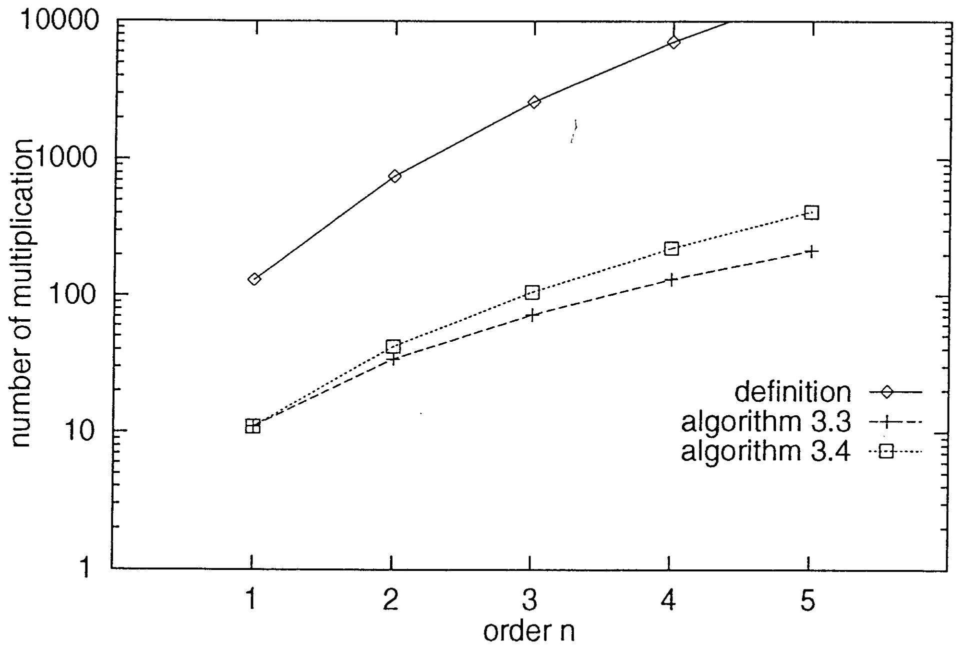


Figure 4: Comparison of the number of multiplications to compute all the elements of the matrix T_n^{BP}