Generalized Invariant Subspaces and Robust Disturbance-Rejection Problems with Incomplete-State Feedback

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Abstract

In the so-called geometric approach, some generalized invariant subspaces are investigated, and then robust disturbance-rejection problems with incomplete-state feedback are formulated and their solvability conditions are presented.

Key words : Generalized Invariant Subspaces, Uncertain Systems, Disturbance-Rejection, Incomplete-State Feedback, Geometric Approach

Abbreviated title : Generalized Invariant Subspaces and Robust Disturbance-Rejection

1 Introduction

The so-called geometric approach has been used successfully to study various disturbance-rejection problems (see e.g. [4], [7], [9]). This approach has been extended to uncertain systems, and some of the corresponding disturbance-rejection problems for uncertain systems have been studied (see e.g. [1], [2], [3], [6]). In particular, Bhattacharyya[1] investigated a generalized (A, B)-invariant subspace and studied the disturbance-rejection problem with state feedback for a broader class of uncertain systems. However, from the practical viewpoint the problem of disturbance-rejection by incomplete-state (measurement) feedback, instead of state feedback, becomes more important.

In the present paper, we will first introduce some generalized invariant subspaces, and their properties will be studied. Further, we will formulate the robust disturbance-rejection problems with incomplete-state feedback for uncertain systems which are investigated by Bhattacharyya[1], and their solvability conditions are presented. Finally, an illustrative example will be given.

2 Generalized Invariant Subspaces

Consider the following linear systems defined in an Euclidean space $X := \boldsymbol{R}^n$:

$$S(\alpha,\beta,\gamma) : \begin{cases} \frac{d}{dt}x(t) = A(\alpha)x(t) + B(\beta)u(t), \\ y(t) = C(\gamma)x(t), \end{cases}$$

where $x(t) \in X$, $u(t) \in U := \boldsymbol{R}^m$ and $y(t) \in Y := \boldsymbol{R}^p$ are the state, the input and the incomplete-state (measurement output), respectively. And coefficient matrices $A(\alpha), B(\beta)$ and $C(\gamma)$ have unknown parameters in the sence that

$$\begin{aligned} A(\alpha) &= A_0 + \alpha_1 A_1 + \cdots + \alpha_p A_p := A_0 + \Delta A(\alpha) \in \boldsymbol{R}^{n\times n}, \\ B(\beta) &= B_0 + \beta_1 B_1 + \cdots + \beta_q B_q := B_0 + \Delta B(\beta) \in \boldsymbol{R}^{n\times m}, \\ C(\gamma) &= C_0 + \gamma_1 C_1 + \cdots + \gamma_r C_r := C_0 + \Delta C(\gamma) \in \boldsymbol{R}^{\ell\times n}, \end{aligned}$$

where $\alpha := (\alpha_1, \cdots, \alpha_p) \in \boldsymbol{R}^p$, $\beta := (\beta_1, \cdots, \beta_q) \in \boldsymbol{R}^q$, $\gamma := (\gamma_1, \cdots, \gamma_r) \in \boldsymbol{R}^r$.

In system $S(\alpha,\beta,\gamma)$ (A_0, B_0, C_0) and $(\Delta A(\alpha), \Delta B(\beta), \Delta C(\gamma))$ represents the nominal system model and a specific uncertain perturbation, respectively.

2.1 Generalized (A, B)-invariant Subspaces

In this subsection, a generalized (A, B)-invariant subspace and its properties which are investigated by Bhattachryya[1] are summarized.

Definition 2.1 Let V, Ω $(\subset X)$ be subspaces.

V is said to be a generalized (A, B)-invariant if there exists an $F \in \boldsymbol{R}^{m\times n}$ such that

$$(A(\alpha) + B(\beta)F)V \subset V$$

for all $(\alpha, \beta) \in \boldsymbol{R}^p \times \boldsymbol{R}^q$. Further, define

$\boldsymbol{F}(A, B; V) := \{F \in \boldsymbol{R}^{m\times n} | (A(\alpha) + B(\beta)F)V \subset V$ for all $(\alpha, \beta) \in \boldsymbol{R}^p \times \boldsymbol{R}^q\}$.

$\boldsymbol{V}(A, B; \Omega) := \{V(\subset \Omega) | \ {}^\exists F \in \boldsymbol{R}^{m\times n}$ s.t. $(A(\alpha) + B(\beta)F)V \subset V$ for all $(\alpha, \beta) \in \boldsymbol{R}^p \times \boldsymbol{R}^q\}$. □

For a subspace V of X define a subspace R_V of $\boldsymbol{R}^m$ and a linear map Q_V on $\boldsymbol{R}^m$ by

$R_V := \bigcap_{i=1}^{q} B_i^{-1} V$, where $B_i^{-1} V := \{u \in \boldsymbol{R}^m | B_i u \in V\}$ and

$Q_V := \boldsymbol{R}^m \rightarrow \boldsymbol{R}^m$, a projection map onto R_V along $(R_V)^\perp$, where $(R_V)^\perp$ means an orthogonal complement of R_V.

Lemma 2.2 [1] There exists an $F \in \boldsymbol{R}^{m\times n}$ such that $(A_0 + B_0F)V \subset V$ and $B_iFV \subset V$ $(i = 1,\cdots,q)$ if and only if $A_0V \subset \mathrm{Im}B_0Q_V + V$. □

Lemma 2.3 [1] The following three statements are equivalent.

(i) $V \in \boldsymbol{V}(A, B; \Omega)$.

(ii) There exists an $F \in \boldsymbol{R}^{m\times n}$ such that $(A_0 + B_0F)V \subset V$ and $B_iFV \subset V$ $(i = 1,\cdots,q)$, and $A_iV \subset V \subset \Omega$ $(i = 1,\cdots,p)$.

(iii) $A_0V \subset \mathrm{Im}B_0Q_V + V$ and $A_iV \subset V \subset \Omega$ $(i = 1,\cdots,p)$. □

Lemma 2.4 [1] $\boldsymbol{V}(A, B; \Omega)$ has a unique maximal element $V^*(\Omega)$ which may be calculated from the following algorithm.

Step1. $V_0 := \Omega$.

Step2. $R_k := \bigcap_{i=1}^{q} B_i^{-1}V_k$ $(\subset \boldsymbol{R}^m)$, where $B_i^{-1}V_k := \{u \in \boldsymbol{R}^m | B_iu \in V_k\}$.

Step3. $Q_k := \boldsymbol{R}^m \rightarrow \boldsymbol{R}^m$, a projection map onto R_k along $(R_k)^{\perp}$.

Step4. $B_{ok} := B_0Q_k$.

Step5. $V_{k+1} := V_k \cap A_0^{-1}(\mathrm{Im}B_{0k} + V_k) \cap A_1^{-1}V_k \cap \cdots \cap A_p^{-1}V_k$.

Step6. $V^*(\Omega) := V_n$. □

Example 2.5 Consider the following matrices and a subspace given by

$$A(\alpha) = \begin{bmatrix} 1+\alpha_1+\alpha_2 & \alpha_2 & 0 & -\alpha_1-\alpha_2 \\ 0 & \alpha_2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & \alpha_2 & 1 & 1 \end{bmatrix}, \quad B(\beta) = \begin{bmatrix} 0 & \beta_2 \\ 0 & 0 \\ 0 & 0 \\ \beta_1 & 1 \end{bmatrix}, \text{ and } \Omega = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Then, we have

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \text{ and } B_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

From the algorithm of Lemma 2.4 we obtain

$$V_0 = \Omega, \quad R_0 = \mathrm{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \quad Q_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_{00} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } V_1 = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Further, we obtain

$$R_1 = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\},\quad Q_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\quad B_{01} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},\ \text{and } V_2 = V_1 = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\}.$$

Thus, V_2 is a maximal element of $\boldsymbol{V}(A, B; \Omega)$. □

2.2 Generalized (C, A)-invariant Subspaces

In this subsection, a new concept of generalized (C, A)-invariant subspace is introduced and its properties are investigated.

Definition 2.6 Let V, $\mathcal{E}$ $(\subset X)$ be subspaces.

V is said to be a generalized (C, A)-invariant if there exists a $G \in \boldsymbol{R}^{n\times\ell}$ such that

$$(A(\alpha) + GC(\gamma))V \subset V$$

for all $(\alpha, \gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^r$. Further, define

$\boldsymbol{G}(C, A; V) := \{G \in \boldsymbol{R}^{n\times\ell} \mid (A(\alpha) + GC(\gamma))V \subset V \text{ for all } (\alpha, \gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^r\}$.

$\boldsymbol{V}(C, A; \mathcal{E}) := \{V\ (\supset \mathcal{E}) \mid {}^{\exists}G \in \boldsymbol{R}^{n\times\ell} \text{ s.t. } (A(\alpha) + GC(\gamma))V \subset V \text{ for all } (\alpha, \gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^r\}$. □

For a subspace V of X let P_V be a linear map on $\boldsymbol{R}^\ell$ satisfying $\text{Ker}P_V = \sum_{i=1}^{r} C_i V$ and $V = \phi \oplus (V \cap \text{Ker}P_V C_0)$ for some subspace ϕ. Since, $C_0\phi \cap \text{Ker}P_V = \{0\}$, we can define a projection map $P_V : \boldsymbol{R}^\ell \to \boldsymbol{R}^\ell$ onto $C_0\phi \oplus \Gamma$ along $\text{Ker}P_V = \sum_{i=1}^{r} C_i V$ for some Γ satisfying $V = \phi \oplus (V \cap \text{Ker}P_V C_0)$.

Concerning the above invariance, the following lemma and theorem can be obtained.

Lemma 2.7 There exists a $G \in \boldsymbol{R}^{n\times\ell}$ such that $(A_0 + GC_0)V \subset V$ and $GC_iV \subset V$ $(i = 1, \cdots, r)$ if and only if $A_0(V \cap \text{Ker}P_V C_0) \subset V$.

(Proof) (Necessity) Suppose that there exists a $G \in \boldsymbol{R}^{n\times\ell}$ such that $(A_0 + GC_0)V \subset V$ and $GC_iV \subset V$ $(i = 1, \cdots, r)$. Let v be an arbitrary element of $(V \cap \text{Ker}P_V C_0)$. Then, $C_0 v \in \text{Ker}P_V = \sum_{i=1}^{r} C_i V$ which imply $C_0 v = \sum_{i=1}^{r} C_i v_i$ for some $v_i \in V$. Hence, $GC_0 v = \sum_{i=1}^{r} GC_i v_i \in \sum_{i=1}^{r} GC_i V \subset V$. Thus, $A_0 v = (A_0 + GC_0)v - GC_0 v \in V + V = V$.

(Sufficiency) Suppose that $A_0(V\cap \mathrm{Ker}P_VC_0) \subset V$. Noticing that $V = \phi \oplus (V \cap \mathrm{Ker}P_VC_0)$, choose a basis $\{v_1, \cdots, v_a\}$ from $(V\cap\mathrm{Ker}P_VC_0)$ and a basis $\{v_{a+1}, \cdots, v_b\}$ from ϕ. Define $w_{i-a} := C_0v_i$ $(i = a+1, \cdots, b)$. Then, a set $\{w_1, \cdots, w_{b-a}\}$ is linearly independent. In fact, $\alpha_1 w_1 + \cdots + \alpha_{b-a}w_{b-a} = 0$ implies $\alpha_1 v_{a+1}+\cdots+\alpha_{b-a}v_b \in \mathrm{Ker}C_0 \subset \mathrm{Ker}P_VC_0$. Further, $\alpha_1 v_{a+1}+\cdots+\alpha_{b-a}v_b \in \phi$. Hence, $\alpha_1 v_{a+1} + \cdots + \alpha_{b-a}v_b \in \phi\cap(V\cap\mathrm{Ker}P_VC_0) = \{0\}$. Thus, $\alpha_1 = \cdots = \alpha_{b-a} = 0$.
Now, noticing that $Y = \boldsymbol{R}^\ell = C_0\phi \oplus \Gamma \oplus \mathrm{Ker}P_V$ and a set $\{w_1, \cdots, w_{b-a}\}$ is a basis of $C_0\phi$, choose a basis $\{w_{b-a+1}, \cdots, w_l\}$ from $\Gamma \oplus \mathrm{Ker}P_V$. Define a linear map G from $\boldsymbol{R}^\ell$ to $\boldsymbol{R}^n$ such that

$$Gw_i := \begin{cases} -A_0v_{a+i} & (i = 1, \cdots, b-a) \\ 0 & (i = b-a+1, \cdots, \ell). \end{cases}$$

In order to prove $(A_0 + GC_0)V \subset V$ let an arbitrary element $x(= x_1 + x_2) \in V = \phi \oplus (V \cap \mathrm{Ker}P_VC_0)$ such that $x_1 \in \phi$ and $x_2 \in (V \cap \mathrm{Ker}P_VC_0)$. Then,

$$\begin{aligned}(A_0 + GC_0)x &= (A_0 + GC_0)x_1 + (A_0 + GC_0)x_2 \\ &= \sum_{i=a+1}^{b} \alpha_i(A_0 + GC_0)v_i + A_0x_2 + GC_0x_2 \\ &= A_0x_2 \quad (\text{by } C_0(V \cap \mathrm{Ker}P_VC_0) \subset \mathrm{Ker}P_V \subset \mathrm{Ker}G) \\ &\in V\end{aligned}$$

for some α_i which imply $(A_0 + GC_0)V \subset V$. Moreover, since $C_iV \subset \mathrm{Ker}P_V \subset \mathrm{Ker}G$ we obtain $GC_iV = \{0\} \subset V$ for all $i = 1, \cdots, r$. This completes the proof. □

Theorem 2.8 The following three statements are equivalent.

(i) $V \in \boldsymbol{V}(C, A; \mathcal{E})$.

(ii) There exists a $G \in \boldsymbol{R}^{n\times\ell}$ such that $(A_0 + GC_0)V \subset V$ and $GC_iV \subset V$ $(i = 1, \cdots, r)$, and $A_iV \subset V$ $(i = 1, \cdots, p)$ and $\mathcal{E} \subset V$.

(iii) $A_0(V\cap \mathrm{Ker}P_VC_0) \subset V$, $A_iV \subset V$ $(i = 1, \cdots, p)$ and $\mathcal{E} \subset V$.

(Proof) ((i)⇒ (ii)) Suppose that $V \in \boldsymbol{V}(C, A; \mathcal{E})$. Then, there exists a $G \in \boldsymbol{R}^{n\times\ell}$ such that

$$(A(\alpha) + GC(\gamma))V \subset V \tag{1}$$

for all $(\alpha, \gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^r$ and $\mathcal{E} \subset V$. Now, choose $\alpha_1 = \cdots = \alpha_p = 0$ and $\gamma_1 = \cdots = \gamma_r = 0$ in (2). Then,

$$(A_0 + GC_0)V \subset V.$$

Further, choose $\alpha_1 = 1, \alpha_2 = \cdots = \alpha_p = 0$ and $\gamma_1 = \cdots = \gamma_r = 0$ in (2). Then

$$(A_0 + A_1 + GC_0)V \subset V$$

which imply $A_1 V \subset V$. Similarly, it can be easily proved $A_i V \subset V$ $(i = 1, \cdots, p)$. On the other hand, choose $\alpha_1 = 1, \alpha_2 = \cdots = \alpha_p = 0$ and $\gamma_1 = 1, \gamma_2 = \cdots = \gamma_r = 0$ in (2). Then,

$$(A_0 + A_1 + G(C_0 + C_1))V \subset V$$

which imply $GC_1 V \subset V$. Similarly, it can be easily proved $GC_i V \subset V$ $(i = 1, \cdots, r)$.
((ii)$\Rightarrow$ (i)) Suppose that there exists a $G \in \boldsymbol{R}^{n\times\ell}$ such that $(A_0 + GC_0)V \subset V$ and $GC_i V \subset V$ $(i = 1, \cdots, r)$, and $A_i V \subset V$ $(i = 1, \cdots, p)$ and $\mathcal{E} \subset V$. Let an arbitrary element $x \in V$. Then,

$$\begin{aligned} (A(\alpha) + GC(\gamma))x &= (A_0 + \alpha_1 A_1 + \cdots + \alpha_p A_p + GC_0 + \gamma_1 GC_1 + \cdots + \gamma_r GC_r)x \\ &= (A_0 + GC_0)x + (\alpha_1 A_1 + \cdots + \alpha_p A_p)x + (\gamma_1 GC_1 + \cdots + \gamma_r GC_r)x \\ &\in V, \end{aligned}$$

for all $(\alpha, \gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^r$ which imply $V \in \boldsymbol{V}(C, A; \mathcal{E})$.
((ii)$\Leftrightarrow$ (iii)) The proof is obvious from Lemma 2.7. □

The following lemma guarantees the existence of a unique minimal element of $\boldsymbol{V}(C, A; \mathcal{E})$.

Lemma 2.9 $\boldsymbol{V}(C, A; \mathcal{E})$ is closed under the operation of subspace intersection, that is, $\boldsymbol{V}(C, A; \mathcal{E})$ has a unique minimal element $V_*(\mathcal{E})$.
(Proof) Let V_1, V_2 be subspaces of $\boldsymbol{V}(C, A; \mathcal{E})$. Then, from Theorem 2.8 we obtain

$$A_0(V_1 \cap \operatorname{Ker} P_{V_1} C_0) \subset V_1,\ A_i V_1 \subset V_1\ (i = 1, \cdots, p),\ \mathcal{E} \subset V_1,$$

$$A_0(V_2 \cap \operatorname{Ker} P_{V_2} C_0) \subset V_2,\ A_i V_2 \subset V_2\ (i = 1, \cdots, p) \text{ and } \mathcal{E} \subset V_2,$$

where $P_{V_j} : \boldsymbol{R}^\ell \to \boldsymbol{R}^\ell$ be the projection map onto $\Lambda_{V_j} := C_0\phi_j \oplus \Gamma_j$ along $\operatorname{Ker} P_{V_j} = \sum_{i=1}^{r} C_i V_j$ for some subspace Γ_j satisfying $V_j = \phi_j \oplus (V \cap \operatorname{Ker} P_{V_j} C_0)$ $(j = 1, 2)$. Since $\operatorname{Ker} P_{(V_1 \cap V_2)} \subset (\operatorname{Ker} P_{V_1} \cap \operatorname{Ker} P_{V_2})$, one can define the projection map $P_{V_1 \cap V_2} : \boldsymbol{R}^\ell \to \boldsymbol{R}^\ell$ onto $\Lambda_{(V_1 \cap V_2)} := C_0 \phi_{12} \oplus \Gamma_{12}$ along $\operatorname{Ker} P_{(V_1 \cap V_2)} = \sum_{i=1}^{r} C_i(V_1 \cap V_2)$ satisfying $\Lambda_{V_j} \subset \Lambda_{(V_1 \cap V_2)}$ $(j = 1, 2)$, where $(V_1 \cap V_2) = \phi_{12} \oplus ((V_1 \cap V_2) \cap \operatorname{Ker} P_{(V_1 \cap V_2)} C_0)$.
In order to prove $(V_1 \cap V_2) \in \boldsymbol{V}(C, A; \mathcal{E})$, from Theorem 2.8 it suffices to show that

$$A_0(V_1 \cap V_2 \cap \operatorname{Ker} P_{(V_1 \cap V_2)} C_0) \subset (V_1 \cap V_2), \tag{2}$$

$$A_i(V_1 \cap V_2) \subset (V_1 \cap V_2)\ (i = 1, \cdots, p) \text{ and } \mathcal{E} \subset (V_1 \cap V_2). \tag{3}$$

Since (3) is clear, we prove (2). Now, the following inclusion holds.

$$\mathrm{Ker} P_{(V_1 \cap V_2)} C_0 \subset \{\mathrm{Ker} P_{V_1} C_0 \cap \mathrm{Ker} P_{V_2} C_0\}.$$

In fact, let x be an arbitrary element of $\mathrm{Ker}P_{(V_1\cap V_2)}C_0$. Then, $C_0 x \in \mathrm{Ker}P_{(V_1\cap V_2)} \subset \{ \mathrm{Ker}P_{V_1} \cap \mathrm{Ker}P_{V_2}\}$. Hence, $x \in \{ \mathrm{Ker}P_{V_1} C_0 \cap \mathrm{Ker}P_{V_2} C_0\}$. Then,

$$\begin{aligned} A_0(V_1 \cap V_2 \cap \mathrm{Ker}P_{(V_1\cap V_2)}C_0) &\subset A_0(V_1 \cap V_2 \cap \mathrm{Ker}P_{V_1} C_0 \cap \mathrm{Ker}P_{V_2} C_0) \\ &\subset A_0(V_1 \cap \mathrm{Ker}P_{V_1} C_0) \cap A_0(V_2 \cap \mathrm{Ker}P_{V_2} C_0) \\ &\subset (V_1 \cap V_2), \end{aligned}$$

which imply (2). Thus, $V_1 \cap V_2 \in \boldsymbol{V}(C, A; \mathcal{E})$, which means that $\boldsymbol{V}(C, A; \mathcal{E})$ has a unique minimal element. This completes the proof. □

For the computation of $V_*(\mathcal{E})$ we have the following algorithm.

Theorem 2.10 A unique minimal element $V_*(\mathcal{E})$ of $\boldsymbol{V}(C, A; \mathcal{E})$ can be calculated by the following algorithm.

Step1. $V_0 := \mathcal{E}$.

Step2. $P_k := \boldsymbol{R}^\ell \to \boldsymbol{R}^\ell$, a projection map onto $C_0\Omega_k \oplus \Gamma_k$ along $\mathrm{Ker}P_k$ for some Γ_k such that $\mathrm{Ker}P_k = \sum_{i=1}^{r} C_i V_k$ and $V_k = \Omega_k \oplus (V_k \cap \mathrm{Ker}P_k C_0)$.

Step3. $C_{ok} := P_k C_0$.

Step4. $V_{k+1} := V_k + A_0(\mathrm{Ker}C_{0k} \cap V_k) + A_1 V_k + \cdots + A_p V_k$.

Step5. $V_*(\mathcal{E}) := V_n$.

(Proof) We first observe that the sequence $\{V_k\}$ is nondecreasing : clearly $V_0 \subset V_1$, and if $V_{k-1} \subset V_k$, then, noticing that $\mathrm{Ker}C_{ok} \supset \mathrm{Ker}C_{ok-1}$,

$$\begin{aligned} V_{k+1} &= V_k + A_0(\mathrm{Ker}C_{ok} \cap V_k) + A_1 V_k + \cdots + A_p V_k \\ &\supset V_{k-1} + A_0(\mathrm{Ker}C_{ok-1} \cap V_{k-1}) + A_1 V_{k-1} + \cdots + A_p V_{k-1} \\ &= V_k. \end{aligned}$$

Thus, there exists an m such that $V_k = V_m$ for all $k \geq m$. Now, let V be an arbitrary element of $\boldsymbol{V}(C, A; \mathcal{E})$. Then, from Theorem 2.8 we have

$$A_0(V \cap \mathrm{Ker} P_V C_0) \subset V,\ A_i V \subset V\ (i = 1, \cdots, p) \text{ and } \mathcal{E} \subset V.$$

Then, clearly $V_0(= \mathcal{E}) \subset V$, and if $V_{k-1} \subset V$, then

$$\begin{aligned} V_k &= V_{k-1} + A_0(\mathrm{Ker} C_{0k-1} \cap V_{k-1}) + A_1 V_{k-1} + \cdots + A_p V_{k-1} \\ &\subset V + A_0(\mathrm{Ker} C_{0k-1} \cap V) + A_1 V + \cdots + A_p V \\ &= V + A_0(\mathrm{Ker} C_{0k-1} \cap V). \end{aligned}$$

Now, since $\mathrm{Ker} P_{k-1} \subset \mathrm{Ker} P_V$ we have $A_0(\ \mathrm{Ker} C_{0k-1} \cap V) \subset V$. Hence, $V_k \subset V$ for all $k \geq 0$ which imply $V_m(= V_{m+1}) \subset V$, that is

$$V_m = V_m + A_0(\mathrm{Ker} P_m C_0 \cap V_m) + A_1 V_m + \cdots + A_p V_m.$$

Hence, we have

$$A_0(\mathrm{Ker} P_m C_0 \cap V_m) \subset V_m, A_i V_m \subset V_m \quad (i = 1, \cdots, p) \text{ and } \mathcal{E}\ (= V_0) \subset V_m.$$

It follws from Theorem 2.8 that $V_m \in \boldsymbol{V}(C, A; \mathcal{E})$. Thus, subspace V_m is a minimal element of $\boldsymbol{V}(C, A; \mathcal{E})$. This completes the proof. □

Example 2.11 Consider $A(\alpha)$ of Example 2.5, $C(\gamma)$ given by

$$C(\gamma) = \begin{bmatrix} \gamma_1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \text{ and } \mathcal{E} = \mathrm{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Then, we have

$$C_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

From the algorithm of Theorem 2.10 we obtain

$$V_0 = \mathcal{E},\ P_0 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \ \Omega_0 = \mathrm{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}, \ \Gamma_0 = \{\mathbf{0}\},\ C_{00} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \text{ and } V_1 = V_0.$$

Thus, V_1 is a minimal element of $\boldsymbol{V}(C, A; \mathcal{E})$. □

2.3 Generalized (A, B, C)-invariant Subspaces

Definition 2.12 Let $V\ (\subset X)$ be subspace.

V is said to be a generalized (A, B, C)-invariant if there exists an $H \in \boldsymbol{R}^{m \times \ell}$ such that

$$(A(\alpha) + B(\beta) H C(\gamma)) V \subset V$$

for all $(\alpha,\beta,\gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^q \times \boldsymbol{R}^r$. Further, define

$\boldsymbol{H}(A,B,C;V) := \{H \in \boldsymbol{R}^{m\times\ell} \mid (A(\alpha)+B(\beta)HC(\gamma))V \subset V$ for all $(\alpha,\beta,\gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^q \times \boldsymbol{R}^r\}$,

$\boldsymbol{V}(A,B,C) := \{V \mid {}^{\exists}H \in \boldsymbol{R}^{m\times\ell}$ s.t. $(A(\alpha)+B(\beta)HC(\gamma))V \subset V$ for all $(\alpha,\beta,\gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^q \times \boldsymbol{R}^r\}$.

□

The following two lemmas are used to prove Lemma 2.15.

Lemma 2.13 [5] Let $V, W (\subset X)$ be subspaces. Then, there exist subspaces X_0 and X_1 such that $V = X_1 \oplus (V \cap W), X = X_0 \oplus W$ and $X_1 \subset X_0$. □

Lemma 2.14 [8] Let $F \in \boldsymbol{R}^{m\times n}$ and $T \in \boldsymbol{R}^{\ell\times n}$. Then, there exists a $K \in \boldsymbol{R}^{m\times\ell}$ such that $F = KT$ if and only if $\mathrm{Ker}T \subset \mathrm{Ker}F$. □

Lemma 2.15 There exists a $K \in \boldsymbol{R}^{m\times\ell}$ such that $(A_0 + B_0Q_VKP_VC_0)V \subset V$ if and only if $A_0V \subset \mathrm{Im}B_0Q_V + V$ and $A_0(V \cap \mathrm{Ker}P_VC_0) \subset V$.

(Proof) (Necessity) Suppose that there exists a $K \in \boldsymbol{R}^{m\times\ell}$ such that $(A_0+B_0Q_VKP_VC_0)V \subset V$. Let v be an arbitrary element of V. Then, $w := (A_0 + B_0Q_VKP_VC_0)v \in V$. Hence, $A_0v = w - B_0Q_VKP_VC_0v \in V + \mathrm{Im}B_0Q_V$ which imply $A_0V \subset V + \mathrm{Im}B_0Q_V$. Further, let v be an arbitrary element of $V \cap \mathrm{Ker}P_VC_0$. Then, we have $A_0v = (A_0 + B_0Q_VKP_VC_0)v \in V$ which imply $A_0(V \cap \mathrm{Ker}P_VC_0) \subset V$.

(Sufficiency) Suppose that $A_0V \subset \mathrm{Im}B_0Q_V + V$ and $A_0(V \cap \mathrm{Ker}P_VC_0) \subset V$. Then, from Lemma 2.13 there exists subspaces Ω and X_0 such that $V = \Omega \oplus (V \cap \mathrm{Ker}P_VC_0)$, $X = X_0 \oplus \mathrm{Ker}P_VC_0$ and $\Omega \subset X_0$. Now, let $L : X \to X$, be a projection map onto X_0 along $\mathrm{Ker}P_VC_0$. Since $A_0V \subset \mathrm{Im}B_0Q_V + V$, there exists an $H \in \boldsymbol{R}^{m\times n}$ such that $(A_0 + B_0Q_VH)V \subset V$ (see e.g. [9]). Define $H_0 := HL$. Then, clearly $\mathrm{Ker}P_VC_0 \subset \mathrm{Ker}H_0$. Hence, from Lemma 2.14 there exists a $K \in \boldsymbol{R}^{m\times\ell}$ such that $H_0 = KP_VC_0$. Thus, we have

$$\begin{aligned}(A_0 + B_0Q_VKP_VC_0)V &= (A_0 + B_0Q_VH_0)V \qquad (4)\\ &= (A_0 + B_0Q_VH_0)\Omega + (A_0 + B_0Q_VH_0)(V \cap \mathrm{Ker}P_VC_0)\\ &= (A_0 + B_0Q_VHL)\Omega + (A_0 + B_0Q_VKP_VC_0)(V \cap \mathrm{Ker}P_VC_0)\\ &= (A_0 + B_0Q_VH)\Omega + A_0(V \cap \mathrm{Ker}P_VC_0)\\ &\subset V + V\\ &= V.\end{aligned}$$

This completes the proof. □

Using this lemma, we have the following theorem.

Theorem 2.16 There exists an $H \in \boldsymbol{R}^{m\times\ell}$ such that $(A(\alpha)+B(\beta)HC(\gamma))V \subset V$ for all $(\alpha,\beta,\gamma) \in \boldsymbol{R}^p\times\boldsymbol{R}^q\times\boldsymbol{R}^r$ if and only if there exists a $K \in \boldsymbol{R}^{m\times\ell}$ such that $(A_0+B_0Q_VKP_VC_0)V \subset V$ and $A_iV \subset V$ $(i=1,\cdots,p)$.

(Proof) (Necessity) Suppose that there exists an $H \in \boldsymbol{R}^{m\times\ell}$ such that $(A(\alpha)+B(\beta)HC(\gamma))V \subset V$ for all $(\alpha,\beta,\gamma) \in \boldsymbol{R}^p\times\boldsymbol{R}^q\times\boldsymbol{R}^r$. Then, clearly, $V \in \boldsymbol{V}(A,B;X)\cap\boldsymbol{V}(C,A;\{\boldsymbol{0}\})$. By virture of Lemma 2.3 and Theorem 2.8 we have

$$A_0V \subset \mathrm{Im}B_0Q_V+V,\ A_0(V\cap\mathrm{Ker}P_VC_0)\subset V \text{ and } A_iV\subset V\ (i=1,\cdots,p).$$

Thus, it follows from Lemma 2.15 that there exists a $K \in \boldsymbol{R}^{m\times\ell}$ such that

$$(A_0+B_0Q_VKP_VC_0)V \subset V.$$

(Sufficiency) Suppose that there exists a $K \in \boldsymbol{R}^{m\times\ell}$ such that

$$(A_0+B_0Q_VKP_VC_0)V \subset V \text{ and } A_iV\subset V\ (i=1,\cdots,p).$$

Let x be an arbitrary element of V. Then, noticing that $\mathrm{Im}Q_V \subset R_V = \bigcap_{i=1}^{q} B_i^{-1}V$,

$$\begin{aligned}
&(A(\alpha)+B(\beta)Q_VKP_VC(\gamma))x\\
&= (A_0+\alpha_1A_1+\cdots+\alpha_pA_p+(B_0+\beta_1B_1+\cdots+\beta_qB_q)Q_VKP_V(C_0+\gamma_1C_1+\cdots+\gamma_rC_r))x\\
&= (A_0+B_0Q_VKP_VC_0)x+\alpha_1A_1x+\cdots+\alpha_pA_px\\
&\qquad +(\beta_1B_1+\cdots+\beta_qB_q)Q_VKP_V(C_0+\gamma_1C_1+\cdots+\gamma_rC_r)x\\
&\in V.
\end{aligned}$$

Define $H := Q_VKP_V$. Then, we have

$$(A(\alpha)+B(\beta)HC(\gamma))V \subset V$$

for all $(\alpha,\beta,\gamma) \in \boldsymbol{R}^p\times\boldsymbol{R}^q\times\boldsymbol{R}^r$. □

Concerning the three generalized invariant subspaces, the following corollary can be obtained.

Corollary 2.17 V is generalized (A,B,C)-invariant subspace (i.e., $V \in \boldsymbol{V}(A,B,C)$) if and only if V is generalized (A,B)-invariant and generalized (C,A)-invariant subspace (i.e., $V \in \boldsymbol{V}(A,B;X)\cap\boldsymbol{V}(C,A;\boldsymbol{0})$).

(Proof) The proof follows from Lemmas 2.3, 2.15 and Theorems 2.8, 2.16. □

3 Robust Disturbance-Rejection

In this section, we will study the robust disturbance-rejection problem. The linear control system to be considered is given by

$$S(\alpha,\beta,\gamma,\delta,\sigma,\tau):\begin{cases}\dfrac{d}{dt}x(t)=A(\alpha)x(t)+B(\beta)u(t)+E(\sigma)\xi(t)+S(\tau)\eta(t),\\ y(t)=C(\gamma)x(t),\\ z(t)=D(\delta)x(t)\end{cases}$$

with $x(t)\in X:=\boldsymbol{R}^n$, $u(t)\in\boldsymbol{R}^m$, $y(t)\in\boldsymbol{R}^\ell$, $z(t)\in\boldsymbol{R}^\mu$ are the state, the input, the incomplete-state (measurement output) and the controlled output, respectively, $\xi(t)\in\boldsymbol{R}^\eta$ is a disturbance which can be measured by controller and $\eta(t)\in\boldsymbol{R}^\rho$ is also a disturbance which can not be measured by controller. It is assumed that coefficient matrices have the following unknown parameters.

$$\begin{aligned}
A(\alpha) &= A_0+\alpha_1A_1+\cdots+\alpha_pA_p := A_0+\Delta A(\alpha)\in\boldsymbol{R}^{n\times n},\\
B(\beta) &= B_0+\beta_1B_1+\cdots+\beta_qB_q := B_0+\Delta B(\beta)\in\boldsymbol{R}^{n\times m},\\
C(\gamma) &= C_0+\gamma_1C_1+\cdots+\gamma_rC_r := C_0+\Delta C(\gamma)\in\boldsymbol{R}^{\ell\times n},\\
D(\delta) &= D_0+\delta_1D_1+\cdots+\delta_sD_s := D_0+\Delta D(\delta)\in\boldsymbol{R}^{\mu\times n},\\
E(\sigma) &= E_0+\sigma_1E_1+\cdots+\sigma_tE_t := E_0+\Delta D(\sigma)\in\boldsymbol{R}^{n\times\eta},\\
S(\tau) &= S_0+\tau_1S_1+\cdots+\tau_uS_u := S_0+\Delta S(\tau)\in\boldsymbol{R}^{n\times\rho},
\end{aligned}$$

where

$$\begin{aligned}
&\alpha:=(\alpha_1,\cdots,\alpha_p)\in\boldsymbol{R}^p,\ \beta:=(\beta_1,\cdots,\beta_q)\in\boldsymbol{R}^q,\ \gamma:=(\gamma_1,\cdots,\gamma_r)\in\boldsymbol{R}^r,\\
&\delta:=(\delta_1,\cdots,\delta_s)\in\boldsymbol{R}^s,\ \sigma:=(\sigma_1,\cdots,\sigma_t)\in\boldsymbol{R}^t,\ \tau:=(\tau_1,\cdots,\tau_u)\in\boldsymbol{R}^u.
\end{aligned}$$

In System $S(\alpha,\beta,\gamma,\delta,\sigma,\tau)$ $(A_0,B_0,C_0,D_0,E_0,S_0)$ and $(\Delta A(\alpha)$, $\Delta B(\beta)$, $\Delta C(\gamma)$, $\Delta D(\delta)$, $\Delta E(\sigma)$, $\Delta S(\tau))$ represent the nominal system model and a specific uncertain perturbation, respectively. Now, we apply to system $S(\alpha,\beta,\gamma,\delta,\sigma,\tau)$ an incomplete-state feedback and measurable disturbances of the form

$$u(t)=Hy(t)+R\eta(t)$$

where $H\in\boldsymbol{R}^{m\times\ell}, R\in\boldsymbol{R}^{m\times\rho}$. Then, the closed-loop system is given by

$$S_{H,R}(\alpha,\beta,\gamma,\delta,\sigma,\tau):\begin{cases}\dfrac{d}{dt}x(t)=(A(\alpha)+B(\beta)HC(\gamma))x(t)+(B(\beta)R+S(\tau))\eta(t)+E(\sigma)\xi(t),\\ z(t)=D(\delta)x(t).\end{cases}$$

The robust disturbance-Rejection problem with incomplete-state feedback and measurable disturbances for system $S(\alpha,\beta,\gamma,\delta,\sigma,\tau)$ is stated as follows: Given matrices $A_i, B_i, C_i, D_i, E_i, S_i$ for system $S(\alpha,\beta,\gamma,\delta,\sigma,\tau)$, find if possible an incomplete-state feedback gain $H \in \boldsymbol{R}^{m\times\ell}$ and $R \in \boldsymbol{R}^{m\times p}$ such that the closed-loop system $S_{H,R}(\alpha,\beta,\gamma,\delta,\sigma,\tau)$ rejects the disturbances ξ and η from the controlled output z for all parameters $(\alpha,\beta,\gamma,\delta,\sigma,\tau) \in \boldsymbol{R}^p \times \boldsymbol{R}^q \times \boldsymbol{R}^r \times \boldsymbol{R}^s \times \boldsymbol{R}^t \times \boldsymbol{R}^u$. This problem can be rephrased as follows.

Problem 3.1 Given matrices $A_i, B_i, C_i, D_i, E_i, S_i$ for system $S(\alpha,\beta,\gamma,\delta,\sigma,\tau)$, find if possible an incomplete-state feedback gain $H \in \boldsymbol{R}^{m\times\ell}$ and $R \in \boldsymbol{R}^{m\times p}$ such that

$$\begin{aligned}
&\langle A(\alpha)+B(\beta)HC(\gamma)|\mathrm{Im}(B(\beta)R+S(\tau))+\mathrm{Im}E(\sigma)\rangle \\
:=\ &\sum_{k=1}^{n}(A(\alpha)+B(\beta)HC(\gamma))^{k-1}(\mathrm{Im}(B(\beta)R+S(\tau))+\mathrm{Im}E(\sigma)) \\
\subset\ &\mathrm{Ker}D(\delta)
\end{aligned}$$

for all parameters $(\alpha,\beta,\gamma,\delta,\sigma,\tau) \in \boldsymbol{R}^p \times \boldsymbol{R}^q \times \boldsymbol{R}^r \times \boldsymbol{R}^s \times \boldsymbol{R}^t \times \boldsymbol{R}^u$. □

Remark 3.2 If $C(\gamma) = I_n$ (identity matrix), then Problem 3.1 reduces to the robust disturbance-rejection problem with state feedback and measurable disturbances (RDRPSFMD). If $R = 0$ and $S(\tau) = 0$, then Problem 3.1 reduces to the robust disturbance-rejection problem with incomplete-state feedback (RDRPISF). If $C(\gamma) = I_n$, $R = 0$ and $S(\tau) = 0$, then Problem 3.1 reduces to the robust disturbance-rejection problem with state feedback (RDRPSF) formulated by Bhattachryya[1]. □

The following two lemmas can be used to prove Lemma 3.5.

Lemma 3.3 There exists an $R \in \boldsymbol{R}^{m\times p}$ such that $\mathrm{Im}(B_0R + S_0) \subset V$ and $\mathrm{Im}B_iR \subset V$ $(i = 1,\cdots,q)$ if and only if $\mathrm{Im}S_0 \subset V + \mathrm{Im}B_0Q_V$.

(Proof) (Necessity) Suppose that there exists an $R \in \boldsymbol{R}^{m\times p}$ such that $\mathrm{Im}(B_0R + S_0) \subset V$ and $\mathrm{Im}B_iR \subset V$ $(i = 1,\cdots,q)$. Let $x = (S_0y)$ be an arbitrary element of $\mathrm{Im}S_0$. Since $Ry \in R_V = \bigcap_{i=1}^{q} B_i^{-1}V$, $Q_VRy = Ry$. Thus, we have

$$\begin{aligned}
x &= S_0y \\
&= (B_0R+S_0)y - B_0Ry \\
&= (B_0R+S_0)y - B_0Q_VRy
\end{aligned}$$

$$\begin{aligned} &\in \mathrm{Im}(B_0R+S_0)+\mathrm{Im}(B_0Q_V) \\ &\in V+\mathrm{Im}B_0Q_V. \end{aligned}$$

(Sufficiency) Suppose that $\mathrm{Im}S_0 \subset V+\mathrm{Im}B_0Q_V$. Let $\{v_1,\cdots,v_k\}$ be a basis of $\mathrm{Im}S_0$. Then, there exists a set $\{\xi_1,\cdots,\xi_k\}$ such that $v_i = S_0\xi_i$. Then, it can be easily proved $\{\xi_1,\cdots,\xi_k\}$ is linearly independent. Now, choose a basis $\{\xi_{k+1},\cdots,\xi_\rho\}$ from $\mathrm{Ker}S_0$ such that $\boldsymbol{R}^\rho$ = span $\{\xi_1,\cdots,\xi_k\}\oplus\mathrm{Ker}S_0$. Since $v_i \in V+\mathrm{Im}B_0Q_V$, there exists $x_i \in V$ and $r_i \in R_V$ such that $v_i = x_i + B_0r_i$. Define a linear map $R:\boldsymbol{R}^\rho \to \boldsymbol{R}^m$ such that

$$R\xi_i = \begin{cases} -r_i & (i=1,\cdots,k) \\ 0 & (i=k+1,\cdots,\rho). \end{cases}$$

Then, for each $i=1,\cdots,k$

$$\begin{aligned} (B_0R+S_0)\xi_i &= B_0R\xi_i + S_0\xi_i \\ &= B_0R\xi_i + x_i + B_0r_i \\ &= B_0R\xi_i + x_i + B_0(-R\xi_i) \\ &= x_i \\ &\in V. \end{aligned}$$

Further, for each $i=k+1,\cdots,\rho$

$$(B_0R+S_0)\xi_i = 0 \in V.$$

Thus, $\mathrm{Im}(B_0R+S_0) \subset V$. Moreover, from the definition of R we have $\mathrm{Im}B_iR \subset V$ $(i=1,\cdots,q)$. This completes the proof. □

Lemma 3.4 There exists an $R \in \boldsymbol{R}^{m\times\rho}$ such that $\mathrm{Im}(B(\beta)R+S(\tau)) \subset V$ for all $(\beta,\tau) \in \boldsymbol{R}^q \times \boldsymbol{R}^u$ if and only if there exists an $R \in \boldsymbol{R}^{m\times\rho}$ such that $\mathrm{Im}(B_0R+S_0) \subset V$, $\mathrm{Im}B_iR \subset V$ $(i=1,\cdots,q)$ and $\mathrm{Im}S_i \subset V$ $(i=1,\cdots,u)$.

(Proof) (Necessity) Suppose that there exists an $R \in \boldsymbol{R}^{m\times\rho}$ such that $\mathrm{Im}(B(\beta)R+S(\tau)) \subset V$ for all $(\beta,\tau) \in \boldsymbol{R}^q \times \boldsymbol{R}^u$. Then, clearly $\mathrm{Im}(B_0R+S_0) \subset V$ and $\mathrm{Im}((B_0+B_1)R+S_0) \subset V$. Thus, $\mathrm{Im}(B_1R) \subset V$. Similarly, it can be easily obtained $\mathrm{Im}(B_iR) \subset V$ for all $i=1,\cdots,q$. Further, since $\mathrm{Im}(B_0R+S_0+S_1) \subset V$, we obtain $\mathrm{Im}S_1 \subset V$. Similarly, it can be easily obtained $\mathrm{Im}S_i \subset V$ for all $i=1,\cdots,u$.

(Sufficiency) Suppose that there exists an $R \in \boldsymbol{R}^{m\times p}$ such that $\mathrm{Im}(B_0R+S_0) \subset V$, $\mathrm{Im}B_iR \subset V$ $(i=1,\cdots,q)$ and $\mathrm{Im}S_i \subset V$ $(i=1,\cdots,u)$. Let y $(=(B(\beta)R+S(\tau))x)$ be an arbitrary element of $\mathrm{Im}(B(\beta)R+S(\tau))$. Then, we have

$$\begin{aligned} y &= (B(\beta)R+S(\tau))x \\ &= \{(B_0+\beta_1B_1+\cdots+\beta_qB_q)R+(S_0+\tau_1S_1+\cdots+\tau_uS_u)\}x \\ &= (B_0R+S_0)x+\beta_1B_1Rx+\cdots+\beta_qB_qRx+\tau_1S_1x+\cdots+\tau_uS_ux \\ &\in V, \end{aligned}$$

which imply $\mathrm{Im}(B(\beta)R+S(\tau)) \subset V$ for all $(\beta,\tau) \in \boldsymbol{R}^q \times \boldsymbol{R}^u$. □

The following lemma can be used to prove Theorem 3.7.

Lemma 3.5 There exists an $R \in \boldsymbol{R}^{m\times p}$ such that $\mathrm{Im}(B(\beta)R+S(\tau)) \subset V$ for all $(\beta,\tau) \in \boldsymbol{R}^q \times \boldsymbol{R}^u$ if and only if $\mathrm{Im}S_0 \subset V + \mathrm{Im}B_0Q_V$ and $\mathrm{Im}S_i \subset V$ $(i=1,\cdots,u)$.
(Proof) The proof follows from Lemmas 3.3 and 3.4. □

Theorem 3.6 If there exists a subspace V such that $V \in \boldsymbol{V}\left(A,B;\bigcap_{i=0}^{s}\mathrm{Ker}D_i\right) \cap \boldsymbol{V}\left(C,A;\sum_{i=0}^{t}\mathrm{Im}E_i\right)$, then the RDRPISF is solvable.
(Proof) Suppose that there exists a subspace V such that the stated above condition is satisfied. Then, it follows from Corollary 2.17 that there exists an $H \in \boldsymbol{R}^{m\times \ell}$ such that

$$(A(\alpha)+B(\beta)HC(\gamma))V \subset V$$

for all $(\alpha,\beta,\gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^q \times \boldsymbol{R}^r$ and $\sum_{i=0}^{t}\mathrm{Im}E_i \subset V \subset \bigcap_{i=0}^{s}\mathrm{Ker}D_i$. Further, it can be easily obtained that $\mathrm{Im}E(\sigma) \subset \sum_{i=0}^{t}\mathrm{Im}E_i$ for all $\sigma \in \boldsymbol{R}^t$ and $\bigcap_{i=0}^{s}\mathrm{Ker}D_i \subset \mathrm{Ker}D(\delta)$ for all $\delta \in \boldsymbol{R}^s$. Then, we have

$$\begin{aligned} < A(\alpha)+B(\beta)HC(\gamma) \mid \mathrm{Im}E(\sigma) > &\subset\ < A(\alpha)+B(\beta)HC(\gamma) \mid \sum_{i=0}^{t}\mathrm{Im}E_i > \\ &\subset\ < A(\alpha)+B(\beta)HC(\gamma) \mid V > \\ &=\ V \\ &\subset\ \bigcap_{i=0}^{s}\mathrm{Ker}D_i \\ &\subset\ \mathrm{Ker}D(\delta) \end{aligned}$$

for all parameters $\alpha,\beta,\gamma,\delta,\sigma$. Thus, RDRPISF is solvable. □

Theorem 3.7 If there exists a subspace V such that $V \in \boldsymbol{V}\left(A, B; \bigcap_{i=0}^{s} \mathrm{Ker} D_i\right) \cap \boldsymbol{V}\left(C, A; \sum_{i=0}^{t} \mathrm{Im} E_i\right)$, $\mathrm{Im} S_0 \subset V + \mathrm{Im} B_0 Q_V$ and $\mathrm{Im} S_i \subset V$ $(i = 1, \cdots, u)$, then Problem 3.1 is solvable.

(Proof) Suppose that there exists a subspace V such that the stated above conditions are satisfied. Then, it follows from Corollary 2.17 that there exists an $H \in \boldsymbol{R}^{m \times \ell}$ such that

$$(A(\alpha) + B(\beta) H C(\gamma)) V \subset V$$

for all $(\alpha, \beta, \gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^q \times \boldsymbol{R}^r$ and $\sum_{i=0}^{t} \mathrm{Im} E_i \subset V \subset \bigcap_{i=0}^{s} \mathrm{Ker} D_i$. Further, it follows from Lemma 3.5 that there exists an $R \in \boldsymbol{R}^{m \times \rho}$ such that

$$\mathrm{Im}(B(\beta) R + S(\tau)) \subset V$$

for all $(\beta, \tau) \in \boldsymbol{R}^q \times \boldsymbol{R}^u$. Then, we have

$$\begin{aligned} < A(\alpha) + B(\beta) H C(\gamma) \mid \mathrm{Im}(B(\beta) R + S(\tau)) + \mathrm{Im} E(\sigma) > \quad & \subset \quad < A(\alpha) + B(\beta) H C(\gamma) \mid V > \\ & \subset \quad V \\ & \subset \quad \mathrm{Ker} D(\delta) \end{aligned}$$

for all parameters $\alpha, \beta, \gamma, \delta, \sigma, \tau$. Thus, Problem 3.1 is solvable. □

4 An Illustrative Example

Consider the following systems given by

$$A(\alpha) = \begin{bmatrix} 1 + \alpha_1 + \alpha_2 & \alpha_2 & 0 & -\alpha_1 - \alpha_2 \\ 0 & \alpha_2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & \alpha_2 & 1 & 1 \end{bmatrix}, \quad B(\beta) = \begin{bmatrix} 0 & \beta_2 \\ 0 & 0 \\ 0 & 0 \\ \beta_1 & 1 \end{bmatrix},$$

$$C(\gamma) = \begin{bmatrix} \gamma_1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \quad D(\delta) = \begin{bmatrix} 0 & 0 & 1 + \delta_1 & \delta_2 \end{bmatrix},$$

$$E(\sigma) = \begin{bmatrix} 1 + \sigma_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad S(\tau) = \begin{bmatrix} \tau_1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then, we have

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, D_0 = D_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix},$$

$$E_0 = E_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, S_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, S_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Now, since $\Omega = \bigcap_{i=0}^{2} \mathrm{Ker} D_i$ and $\mathcal{E} = \sum_{i=0}^{1} \mathrm{Im} E_i$, it follows from Examples 2.5 and 2.11 that

$$V := \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \in \boldsymbol{V}\left(A, B; \bigcap_{i=0}^{2} \mathrm{Ker} D_i\right) \cap \boldsymbol{V}\left(C, A; \sum_{i=0}^{1} \mathrm{Im} E_i\right).$$

Thus, from Theorem 3.6 RDRPISF is solvable with incomplete-state feedback gain $H = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$. Further, since

$$\mathrm{Im} S_0 \subset V + \mathrm{Im} B_0 Q_V \quad \text{and} \quad \mathrm{Im} S_1 \subset V,$$

From Theorem 3.7 Problem 3.1 with incomplete-state feedback gain $H = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ and measurable disturbances gain $R = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ is solvable.

5 Concluding Remarks

In this paper, two concepts of generalized (C, A)-invariant subspce and generalized (A, B, C)-invarinat subspace were introduced, and then their properties were investigated in the so-called geometric approach. Further, the relationship between these two invariant subspaces and generalized (A, B)-invariant subspace which was studied by Bhattacharyya [1] was investigated. Finally, using these concepts the robust disturbance-rejection problems with incomplete-state feedback were formulated, and then their solvability conditions were presented.

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