A Deterministic Approach to Linear Programs with Several Additional Multiplicative Constraints

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Abstract. We consider a global optimization problem of minimizing a linear function subject to p linear multiplicative constraints as well as ordinary linear constraints. We show that this problem can be reduced to $2p$-dimensional reverse convex program, and present an algorithm for solving the resulting problem. Our algorithm converges to a globally optimal solution and yields an ϵ-approximate solution in finite time for any $\epsilon > 0$. We also report some results of computational experiment.

Key words: Global optimization, deterministic approach, reverse convex program, linear multiplicative constraint, ϵ-approximate solution.

1. Introduction

We observed a remarkable progress in the last decade in the field of deterministic algorithms for solving a certain class of global optimization problems. In fact, a variety of nonconvex minimization problems have been successfully solved by exploiting their special structures. The readers are referred to Horst-Pardalos [3] and Konno-Thach-Tuy [8] for the state-of-the-art in the field.

Among the more intensively studied class of problems are what we call low rank nonconvex problems. These problems have the property that the original problem reduces to an easy (usually convex) minimization problem when a few variables are fixed or more generally, when a vector of the form Bx is fixed where x is the original variable and B is a low rank affine mapping. Problems included in this class are low rank nonconvex quadratic programming problems [7, 21], minimum cost production-transportation problems with a low rank concave cost function [12, 20], minimization of a sum and a product of linear fractional functions [9, 10], etc.

A number of highly nonconvex minimization problems can be converted to low rank nonconvex minimization problems by applying appropriate parameterization techniques. For example, convex multiplicative programming problems [5, 6], i.e. minimization of a product of several non-negative valued convex functions, can be reduced to low rank problems by introducing a few auxiliary variables. Also, some class of reverse convex programming problems can be converted to low rank nonconvex programming problems by using newly developed duality theorem [17] in global optimization. Readers can find abundant examples of low rank nonconvex minimization problem in Konno-Thach-Tuy [8], which can be solved in an efficient way by applying outer approximation method, partitioning/branch and bound algorithm and even variants of parametric simplex algorithm.

The purpose of this article is to propose a practical algorithm for solving a linear programming problem with several linear multiplicative constraints, yet another class of reverse convex minimization problems [19]. Problems with linear multiplicative terms in the objective function and/or constraint have been under intensive study in the past several years [5, 10, 11, 13 – 16, 18] (see also Konno-Kuno [6] for a survey on algorithms and applications of these problems). We will propose a divide-and-cut algorithm based upon an outer-approximation and partitioning strategy [4]. It will be shown that small-to-medium scale problems can be solved in a practical amount of time, if the number of linear multiplicative terms in the constraints is less than five.

In section 2, we define the problem and convert it to a master problem which has a low rank nonconvex structure. Section 3 will be devoted to the description of the divide-and-cut algorithm and its convergence properties. Also, we will illustrate the algorithm by using an example in two dimensional space. In Section 4, we will present the results of numerical experiments. Some final remarks are given in Section 5.

2. Master Problem in the $2p$-Dimensional Space

The problem we consider in this paper is a linear program with p additional linear multiplicative constraints:

$$[\mathrm{P}] \quad \left| \begin{array}{ll} \text{minimize} & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} \geq \boldsymbol{b}, \quad \boldsymbol{x} \geq \boldsymbol{0} \\ & (\boldsymbol{d}_{1j}^{\mathrm{T}}\boldsymbol{x} + \delta_{1j})(\boldsymbol{d}_{2j}^{\mathrm{T}}\boldsymbol{x} + \delta_{2j}) \leq 1, \quad j = 1, \ldots, p, \end{array} \right.$$

where $A \in \mathbb{R}^{m\times n}$, $\boldsymbol{b} \in \mathbb{R}^m$, $\boldsymbol{c}$, $\boldsymbol{d}_{ij} \in \mathbb{R}^n$ and $\delta_{ij} \in \mathbb{R}$, $i = 1, 2$, $j = 1, \ldots, p$. We assume for simplicity that the set

$$X = \{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} \geq \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\} \tag{2.1}$$

is bounded and has a nonempty interior, and that

$$\boldsymbol{d}_{ij}^{\mathrm{T}}\boldsymbol{x} + \delta_{ij} \geq 0, \quad \forall \boldsymbol{x} \in X, \quad i = 1, 2,\ j = 1, \ldots, p. \tag{2.2}$$

Under these conditions, $\boldsymbol{d}_{ij}^{\mathrm{T}}\boldsymbol{x}$ attains a minimum and a maximum over the polytope X. Let

$$\left.\begin{array}{l} \ell_{ij} = \min\{\boldsymbol{d}_{ij}^{\mathrm{T}}\boldsymbol{x} \mid \boldsymbol{x} \in X\} + \delta_{ij} \\ u_{ij} = \max\{\boldsymbol{d}_{ij}^{\mathrm{T}}\boldsymbol{x} \mid \boldsymbol{x} \in X\} + \delta_{ij} \end{array}\right\}, \quad i = 1, 2,\ j = 1, \ldots, p. \tag{2.3}$$

Then we have

$$0 \leq \ell_{ij} < u_{ij}, \quad i = 1, 2,\ j = 1, \ldots, p. \tag{2.4}$$

We also see that the jth multiplicative constraint is redundant if $u_{1j}u_{2j} \leq 1$, and that [P] is infeasible if $\ell_{1j}\ell_{2j} > 1$. To exclude these cases, we assume in the sequel that

$$\ell_{1j}\ell_{2j} \leq 1 < u_{1j}u_{2j}, \quad j = 1, \ldots, p. \tag{2.5}$$

Remark. The objective function $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$ also attains a minimum over X at some vertex $\bar{\boldsymbol{x}}$. If $\bar{\boldsymbol{x}}$ happens to satisfy all the multiplicative constraints, we can conclude that $\bar{\boldsymbol{x}}$ is a globally optimal solution to [P], without applying the algorithm presented in the paper. For this trivial case, however, our algorithm can still work. □

Let us introduce $2p$ auxiliary variables ξ_{ij}, $i = 1, 2$, $j = 1, \ldots, p$, and transform [P] into an equivalent problem:

$$\left|\begin{array}{ll} \text{minimize} & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} & \boldsymbol{x} \in X \\ & \left.\begin{array}{l} \boldsymbol{d}_{ij}^{\mathrm{T}}\boldsymbol{x} + \delta_{ij} \leq \xi_{ij}, \quad \xi_{ij} \geq 0, \quad i = 1, 2 \\ \xi_{1j}\xi_{2j} \leq 1 \end{array}\right\} \quad j = 1, \ldots, p. \end{array}\right. \tag{2.6}$$

For any fixed $\boldsymbol{\xi} = (\xi_{11}, \xi_{21}, \ldots, \xi_{1p}, \xi_{2p})^{\mathrm{T}}$, we have a linear program:

$$[\mathrm{P}(\boldsymbol{\xi})] \quad \left|\begin{array}{ll} \text{minimize} & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} & \boldsymbol{x} \in X \\ & D\boldsymbol{x} + \boldsymbol{d} \leq \boldsymbol{\xi}, \end{array}\right.$$

where $D = [\boldsymbol{d}_{11}, \boldsymbol{d}_{21}, \ldots, \boldsymbol{d}_{1p}, \boldsymbol{d}_{2p}]^{\mathrm{T}}$ and $\boldsymbol{d} = (\delta_{11}, \delta_{21}, \ldots, \delta_{1p}, \delta_{2p})^{\mathrm{T}}$. Unless the feasible set is empty, $[\mathrm{P}(\boldsymbol{\xi})]$ has an optimal solution, which we denote by $\boldsymbol{x}^*(\boldsymbol{\xi})$. Let

$$\Omega = \{\boldsymbol{\xi} \in \mathbb{R}^{2p} \mid \exists \boldsymbol{x} \in X,\ \boldsymbol{\xi} \geq D\boldsymbol{x} + \boldsymbol{d}\}, \tag{2.7}$$

and define a function:

$$f(\boldsymbol{\xi}) = \begin{cases} \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}^*(\boldsymbol{\xi}) & \text{if } \boldsymbol{\xi} \in \Omega \\ +\infty & \text{otherwise.} \end{cases} \tag{2.8}$$

Note that Ω is included in the nonnegative orthant of $\mathbb{R}^{2p}$ under condition (2.2). Hence, $f(\boldsymbol{\xi})$ is finite only if $\boldsymbol{\xi} \geq \mathbf{0}$.

Lemma 2.1. *The function* $f : \mathbb{R}^{2p} \to \mathbb{R} \cup \{-\infty, +\infty\}$ *is convex polyhedral, continuous on* Ω *and satisfies*

$$f(\boldsymbol{\xi}^1) \geq f(\boldsymbol{\xi}^2) \quad \textit{if } \boldsymbol{\xi}^1 \leq \boldsymbol{\xi}^2. \tag{2.9}$$

Proof: Both the convex polyhedrality and continuity follow from a well-known result on parametric linear programming (see e.g [2]). The monotonicity (2.9) is obvious if $\boldsymbol{\xi}^1 \notin \Omega$. Otherwise, it is proved by the following relation of inclusion:

$$\emptyset \neq \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} \in X,\ D\boldsymbol{x} + \boldsymbol{d} \leq \boldsymbol{\xi}^1\} \subseteq \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} \in X,\ D\boldsymbol{x} + \boldsymbol{d} \leq \boldsymbol{\xi}^2\}. \qquad \square$$

Using the function f, we can rewrite (2.6) as follows:

$$\text{[MP]} \quad \left| \begin{array}{ll} \text{minimize} & f(\boldsymbol{\xi}) \\ \text{subject to} & \xi_{1j}\xi_{2j} \leq 1, \quad j = 1, \ldots, p, \end{array} \right.$$

which we call the *master problem* of [P]. The above argument is then summarized into the following:

Lemma 2.2. *If [MP] has an optimal solution $\boldsymbol{\xi}^*$ such that $f(\boldsymbol{\xi}^*) < +\infty$, any optimal solution $\boldsymbol{x}^*(\boldsymbol{\xi}^*)$ to [P($\boldsymbol{\xi}^*$)] solves the original problem [P]. Otherwise, [P] is infeasible.*

3. Divide-and-Cut Algorithm for the Master Problem

As seen in the previous section, we can solve [P] by solving the master problem [MP]. Assuming for a while that the optimal value of [MP] is finite, let us observe some properties of its optimal solution $\boldsymbol{\xi}^*$.

The following is an immediate consequence of Lemma 2.1:

Lemma 3.1. *There exists an optimal solution $\boldsymbol{\xi}^*$ to [MP] such that*

$$\xi^*_{1j}\xi^*_{2j} = 1, \quad j = 1, \ldots, p. \tag{3.1}$$

Proof: Follows from the monotonicity (2.9) of the objective function f. $\square$

Note that Lemma 3.1 does not imply that $(\boldsymbol{d}_{1j}^{\mathrm{T}}\boldsymbol{x}^*(\boldsymbol{\xi}^*) + \delta_{1j})(\boldsymbol{d}_{2j}^{\mathrm{T}}\boldsymbol{x}^*(\boldsymbol{\xi}^*) + \delta_{2j}) = 1$ for every j. Since [P($\boldsymbol{\xi}^*$)] involves no equality constraints, $\boldsymbol{x}^*(\boldsymbol{\xi}^*)$ might satisfy some of the multiplicative constraints of [P] with inequality. Let

$$\underline{\xi}_{1j} = \max\{\ell_{1j}, 1/u_{2j}\}, \quad \underline{\xi}_{2j} = \max\{\ell_{2j}, 1/u_{1j}\}, \quad j = 1, \ldots, p, \tag{3.2}$$

where ℓ_{ij}'s and u_{ij}'s are given by (2.3). It follows from (2.4) that $\underline{\xi}_{ij}$'s are all positive numbers.

Lemma 3.2. *Among optimal solutions to [MP] satisfying (3.1) exists a $\boldsymbol{\xi}^*$ such that*

$$\xi^*_{ij} \geq \underline{\xi}_{ij}, \quad i = 1, 2,\ j = 1, \ldots, p. \tag{3.3}$$

Proof: Any optimal solution $\boldsymbol{\xi}^*$ to [MP] is a point in Ω, and hence

$$\exists \boldsymbol{x} \in X, \ \ \xi^*_{ij} \geq \boldsymbol{d}^{\mathrm{T}}_{ij}\boldsymbol{x} + \delta_{ij} \geq \ell_{ij}, \quad i = 1, 2, \ j = 1, \dots, p. \tag{3.4}$$

Choose an arbitrary optimal solution $\boldsymbol{\xi}^\circ$ satisfying (3.1) and suppose $\ell_{1k} \leq \xi^\circ_{1k} < 1/u_{2k}$ for some k. Then $\xi^\circ_{2k} > u_{2k}$ and the constraint $\boldsymbol{d}^{\mathrm{T}}_{2k}\boldsymbol{x} + \delta_{2k} \leq \xi^\circ_{2k}$ is redundant in [P($\boldsymbol{\xi}^\circ$)]. Therefore, letting

$$\xi'_{1j} = \xi^\circ_{1j} \ \text{ for each } j, \quad \xi'_{2j} = \begin{cases} u_{2k} & \text{if } j = k \\ \xi^\circ_{2j} & \text{otherwise,} \end{cases}$$

we have $f(\boldsymbol{\xi}') = f(\boldsymbol{\xi}^\circ)$. Moreover, letting

$$\xi''_{1j} = \begin{cases} 1/u_{2k} & \text{if } j = k \\ \xi'_{1j} & \text{otherwise,} \end{cases} \qquad \xi''_{2j} = \xi'_{2j} \ \text{ for each } j,$$

we have $f(\boldsymbol{\xi}'') \leq f(\boldsymbol{\xi}')$ by the monotonicity (2.9). The resulting $\boldsymbol{\xi}''$ is thus optimal and satisfies $(\xi''_{1k}, \xi''_{2k}) \geq (1/u_{2k}, 1/u_{1k})$ under condition (2.5), as well as satisfying $\xi''_{1k}\xi''_{2k} = 1$. In this way, $\boldsymbol{\xi}^\circ$ can yields an optimal solution $\boldsymbol{\xi}^*$ satisfying (3.1) and

$$\xi^*_{1j} \geq 1/u_{2j}, \quad \xi^*_{2j} \geq 1/u_{1j}, \quad j = 1, \dots, p.$$

This, together with (3.4), proves (3.3). □

Let

$$\boldsymbol{s}_j = (\underline{\xi}_{1j}, 1/\underline{\xi}_{1j})^{\mathrm{T}}, \quad \boldsymbol{t}_j = (1/\underline{\xi}_{2j}, \underline{\xi}_{2j})^{\mathrm{T}}, \quad j = 1, \dots, p, \tag{3.5}$$

and let

$$\Phi_0 = H(\boldsymbol{s}_1, \boldsymbol{t}_1) \times \cdots \times H(\boldsymbol{s}_p, \boldsymbol{t}_p), \tag{3.6}$$

where $H(\boldsymbol{s}_j, \boldsymbol{t}_j)$ denotes the convex hull of $\{\boldsymbol{0}, \boldsymbol{s}_j, \boldsymbol{t}_j\} \subset \mathbb{R}^2$. It follows from (3.2) and (3.5) that $s_{1j} \leq t_{1j}$ and $s_{2j} \geq t_{2j}$ for each j under conditions (2.4) and (2.5), where $\boldsymbol{s}_j = (s_{1j}, s_{2j})^{\mathrm{T}}$ and $\boldsymbol{t}_j = (t_{1j}, t_{2j})^{\mathrm{T}}$. The following lemma claims that we have only to search the polytope Φ_0 for an optimal solution $\boldsymbol{\xi}^*$ to [MP]:

Lemma 3.3. *The polytope Φ_0 contains an optimal solution $\boldsymbol{\xi}^*$ to [MP] if it exists.*

Proof: From Lemma 3.2, we can suppose

$$\boldsymbol{\xi}^*_j = (\xi^*_{1j}, \xi^*_{2j})^{\mathrm{T}} \in \{\boldsymbol{\xi}_j \in \mathbb{R}^2 \mid \boldsymbol{\xi}_j \geq \underline{\boldsymbol{\xi}}_j\} \cap \{\boldsymbol{\xi}_j \in \mathbb{R}^2 \mid \xi_{1j}\xi_{2j} = 1\}, \quad j = 1, \dots, p,$$

where $\underline{\boldsymbol{\xi}}_j = (\underline{\xi}_{1j}, \underline{\xi}_{2j})^{\mathrm{T}}$. It is easy to see that there is a constant $\alpha > 0$ such that $\alpha\boldsymbol{\xi}^*_j \in \operatorname{conv}\{\boldsymbol{s}_j, \boldsymbol{t}_j\}$. Since $\xi_{1j}\xi_{2j}$ is a quasiconcave function [1, 5], we have

$$\alpha^2\xi^*_{1j}\xi^*_{2j} \geq \min\{\underline{\xi}_{1j}/\underline{\xi}_{1j}, \underline{\xi}_{2j}/\underline{\xi}_{2j}\} = 1,$$

which implies that $\alpha \geq 1$. Hence,

$$\boldsymbol{\xi}_j^* \in \mathrm{conv}\{\underline{\boldsymbol{\xi}}_j, \boldsymbol{s}_j, \boldsymbol{t}_j\} \subset H(\boldsymbol{s}_j, \boldsymbol{t}_j). \qquad \square$$

From Lemma 3.3, we see that [MP] is equivalent to

$$\left| \begin{array}{ll} \text{minimize} & f(\boldsymbol{\xi}) \\ \text{subject to} & \boldsymbol{\xi} \in \Xi \cap \Phi_0, \end{array} \right. \tag{3.7}$$

where

$$\Xi = \{\boldsymbol{\xi} \in \mathbb{R}^{2p} \mid \xi_{1j}\xi_{2j} \leq 1, \; j = 1, \ldots, p\}. \tag{3.8}$$

To solve the master problem reduced to (3.7), we generate a sequence of subsets Φ_k's in $\mathbb{R}^{2p}$ such that

$$\Phi_0 \supset \Phi_1 \supset \cdots \supset \Phi_k \supset \cdots \supseteq \Xi \cap \Phi_0. \tag{3.9}$$

For each k we compute $\boldsymbol{\xi}^k \in \arg\min\{f(\boldsymbol{\xi}) \mid \boldsymbol{\xi} \in \Phi_k\}$. If $\boldsymbol{\xi}^k$ happens to be a point in Ξ, then $\boldsymbol{\xi}^k$ is optimal to [MP] and hence [P($\boldsymbol{\xi}^k$)] provides an optimal solution $\boldsymbol{x}^*(\boldsymbol{\xi}^k)$ to [P]. Otherwise, we construct the next relaxation Φ_{k+1} by discarding some portion of Φ_k containing $\boldsymbol{\xi}^k$ but no points in Ξ. In this process, the main difficulty is that the usual cutting plane procedures cannot be used because Ξ is not a convex set. In the rest of this section, however, we will show that it is possible to overcome this difficulty if we exploit a special structure possessed by (3.7). Namely, the feasible set $\Xi \cap \Phi_0$ can be expressed as follows in terms of the orthogonal product of p subsets in $\mathbb{R}^2$:

$$\Xi \cap \Phi_0 = \Xi_1 \times \cdots \times \Xi_p, \tag{3.10}$$

where

$$\Xi_j = H(\boldsymbol{s}_j, \boldsymbol{t}_j) \cap \{\boldsymbol{\xi}_j \in \mathbb{R}^2 \mid \xi_{1j}\xi_{2j} \leq 1\}, \quad j = 1, \ldots, p. \tag{3.11}$$

Due to this structure, we have only to approximate each Ξ_j in the two-dimensional subspace to generate Φ_k's in the whole space.

3.1. Approximation of the Feasible Set

Let us consider the initial relaxed problem of [MP]:

$[\overline{\text{MP}}_0]$ minimize$\{f(\boldsymbol{\xi}) \mid \boldsymbol{\xi} \in \Phi_0\}$.

Since $[\overline{\text{MP}}_0]$ is equivalent to a linear program with $n + 2p$ variables:

$$\left| \begin{array}{ll} \text{minimize} & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} & \boldsymbol{x} \in X \\ & \left. \begin{array}{l} \boldsymbol{d}_{ij}^{\mathrm{T}}\boldsymbol{x} + \delta_{ij} \leq s_{ij}\lambda_j + t_{ij}\mu_j, \quad i = 1, 2 \\ \lambda_j + \mu_j \leq 1, \quad \lambda_j, \; \mu_j \geq 0 \end{array} \right\} \; j = 1, \ldots, p, \end{array} \right. \tag{3.12}$$

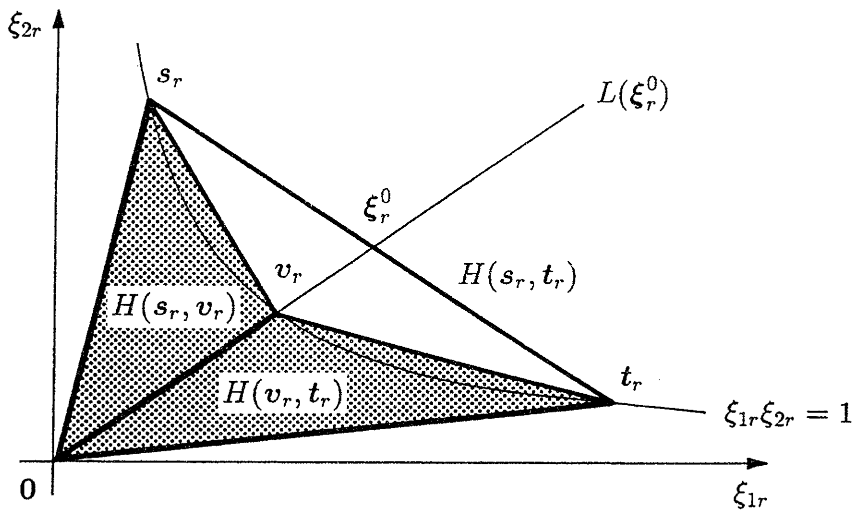

Figure 3.1. Removal of $\boldsymbol{\xi}_r^0$ from $H(\boldsymbol{s}_r, \boldsymbol{t}_r)$.

we can solve it efficiently using available algorithms such as the simplex method. If (3.12) is infeasible, then [P] is also infeasible. Let us suppose that (3.12) is feasible and has an optimal solution $(\boldsymbol{x}^0, \boldsymbol{\lambda}^0, \boldsymbol{\mu}^0)$, where $\boldsymbol{\lambda}^0 = (\lambda_1^0, \ldots, \lambda_p^0)^{\mathrm{T}}$ and $\boldsymbol{\mu}^0 = (\mu_1^0, \ldots, \mu_p^0)^{\mathrm{T}}$. Then $(\boldsymbol{\lambda}^0, \boldsymbol{\mu}^0)$ provides an optimal solution $\boldsymbol{\xi}^0$ to $[\overline{\mathrm{MP}}_0]$ in the form:

$$\xi_{ij}^0 = s_{ij}\lambda_j^0 + t_{ij}\mu_j^0, \quad i = 1, 2,\ j = 1, \ldots, p. \tag{3.13}$$

Unless $\boldsymbol{\xi}^0$ is a point of Ξ, we have to discard some portion containing $\boldsymbol{\xi}^0$ from Φ_0. This can be done in the following way.

Let

$$r \in \arg\max\{\xi_{1j}^0 \xi_{2j}^0 \mid j = 1, \ldots, p\}, \tag{3.14}$$

and consider $H(\boldsymbol{s}_r, \boldsymbol{t}_r)$ in the ξ_{1r}-ξ_{2r} plane (see Figure 3.1). Let us denote by $L(\boldsymbol{\xi}_r^0)$ the half-line emanating from the origin to $\boldsymbol{\xi}_r^0$, and by $\boldsymbol{v}_r = (v_{1r}, v_{2r})$ the intersection of $L(\boldsymbol{\xi}_r^0)$ and the curve $\xi_{1r}\xi_{2r} = 1$, i.e.

$$v_{1r} = \sqrt{\xi_{1r}^0/\xi_{2r}^0}, \quad v_{2r} = \sqrt{\xi_{2r}^0/\xi_{1r}^0}. \tag{3.15}$$

Between $\boldsymbol{\xi}_r^0$ and $\boldsymbol{v}_r$, both lying on $L(\boldsymbol{\xi}_r^0)$, there is a relation $\xi_{1r}^0\xi_{2r}^0 > v_{1r}v_{2r} = 1$ as long as $\boldsymbol{\xi}^0 \notin \Xi$. Hence, we can remove $\boldsymbol{\xi}_r^0$, as shown in Figure 3.1, by replacing $H(\boldsymbol{s}_r, \boldsymbol{t}_r)$ by the union of two simplices $H(\boldsymbol{s}_r, \boldsymbol{v}_r)$ and $H(\boldsymbol{v}_r, \boldsymbol{t}_r)$. In the whole space, this operation leads to

$$\begin{aligned}\Phi_1 = H(\boldsymbol{s}_1, \boldsymbol{t}_1) \times \cdots \times H(\boldsymbol{s}_{r-1}, \boldsymbol{t}_{r-1}) \times (H(\boldsymbol{s}_r, \boldsymbol{v}_r) \cup H(\boldsymbol{v}_r, \boldsymbol{t}_r)) \times \\ H(\boldsymbol{s}_{r+1}, \boldsymbol{t}_{r+1}) \times \cdots \times H(\boldsymbol{s}_p, \boldsymbol{t}_p).\end{aligned} \tag{3.16}$$

Lemma 3.4. *If an optimal solution* $\boldsymbol{\xi}^0$ *to* $[\overline{MP}_0]$ *is not a point in* Ξ, *then*

$$\boldsymbol{\xi}^0 \notin \Phi_1, \quad \Xi \cap \Phi_0 \subset \Phi_1. \tag{3.17}$$

Proof: Obvious from the definition of Φ_1. □

From Lemma 3.4, we have an alternative relaxed problem of [MP]:

$[\overline{\mathrm{MP}}_1]$ minimize$\{f(\boldsymbol{\xi}) \mid \boldsymbol{\xi} \in \Phi_1\}$.

This problem, however, cannot be solved directly as $[\overline{\mathrm{MP}}_0]$ can, because its feasible set Φ_1 is not a convex set. We then divide Φ_1 into two subsets:

$$\begin{aligned}\Psi_1 = H(\boldsymbol{s}_1, \boldsymbol{t}_1) \times \cdots \times H(\boldsymbol{s}_{r-1}, \boldsymbol{t}_{r-1}) \times H(\boldsymbol{s}_r, \boldsymbol{v}_r) \times \\ H(\boldsymbol{s}_{r+1}, \boldsymbol{t}_{r+1}) \times \cdots \times H(\boldsymbol{s}_p, \boldsymbol{t}_p),\end{aligned} \tag{3.18}$$

$$\begin{aligned}\Psi_2 = H(\boldsymbol{s}_1, \boldsymbol{t}_1) \times \cdots \times H(\boldsymbol{s}_{r-1}, \boldsymbol{t}_{r-1}) \times H(\boldsymbol{v}_r, \boldsymbol{t}_r) \times \\ H(\boldsymbol{s}_{r+1}, \boldsymbol{t}_{r+1}) \times \cdots \times H(\boldsymbol{s}_p, \boldsymbol{t}_p).\end{aligned} \tag{3.19}$$

For $i = 1, 2$, taking Ψ_i as a feasible set, we define a subproblem of $[\overline{\mathrm{MP}}_1]$:

$[\overline{\mathrm{P}}_i]$ minimize$\{f(\boldsymbol{\xi}) \mid \boldsymbol{\xi} \in \Psi_i\}$.

Since $\Phi_1 = \Psi_1 \cup \Psi_2$, either $[\overline{\mathrm{P}}_1]$ or $[\overline{\mathrm{P}}_2]$ provides an optimal solution $\boldsymbol{\xi}^1$ to $[\overline{\mathrm{MP}}_1]$. Here we should note that $[\overline{\mathrm{P}}_i]$ is a relaxed problem of

$[\mathrm{P}_i]$ minimize$\{f(\boldsymbol{\xi}) \mid \boldsymbol{\xi} \in \Psi_i \cap \Xi\}$,

which is just the same form as (3.7). Therefore, by applying the above procedure recursively to $[\mathrm{P}_i]$'s, we can find a globally optimal solution $\boldsymbol{\xi}^*$ to [MP] whenever it exists.

3.2. Algorithm

We are now ready to present the algorithm for solving [MP]. Let $\epsilon \geq 0$ denote a given tolerance.

Algorithm DC

Step 0. Construct $\Phi_0 = H(\boldsymbol{s}_1, \boldsymbol{t}_1) \times \cdots \times H(\boldsymbol{s}_p, \boldsymbol{t}_p)$ according to (2.3), (3.2), (3.5) and (3.6), and solve the initial relaxed problem $[\overline{\mathrm{MP}}_0]$. If $[\overline{\mathrm{MP}}_0]$ has an optimal solution $\boldsymbol{\xi}^0$, then $\mathcal{P} = \{(\Phi_0, \boldsymbol{\xi}^0)\}$. Otherwise, let $\mathcal{P} = \emptyset$. Go to Step 1 with $k = 0$.

Step 1. If $\mathcal{P} = \emptyset$, then stop — [MP] is infeasible. Otherwise, select a pair $(\Psi', \boldsymbol{\xi}')$ with the least $f(\boldsymbol{\xi}')$ in $\mathcal{P}$ and let $\mathcal{P} = \mathcal{P} \setminus \{(\Psi', \boldsymbol{\xi}')\}$. Let $(\Psi_k, \boldsymbol{\xi}^k) = (\Psi', \boldsymbol{\xi}')$ and

$[\overline{\mathrm{P}}_k]$ minimize$\{f(\boldsymbol{\xi}) \mid \boldsymbol{\xi} \in \Psi_k \equiv H(\boldsymbol{s}_1^k, \boldsymbol{t}_1^k) \times \cdots \times H((\boldsymbol{s}_p^k, \boldsymbol{t}_p^k)\}$.

Step 2. If the optimal solution $\boldsymbol{\xi}^k$ to $[\overline{\mathrm{P}}_k]$ satisfies

$$\max\{\xi_{1j}\xi_{2j} \mid j = 1, \ldots, p\} \leq 1 + \epsilon,$$

then $\boldsymbol{\xi}^\epsilon = \boldsymbol{\xi}^k$ and stop. Otherwise, let r be the smallest index in $\arg\max\{\xi^k_{1j}\xi^k_{2j} \mid j = 1, \ldots, p\}$.

Step 3. Let $\boldsymbol{v}^k_r = \left(\sqrt{\xi^k_{1r}/\xi^k_{2r}}, \sqrt{\xi^k_{2r}/\xi^k_{1r}}\right)^{\mathrm{T}}$ and

$$\begin{aligned}\Psi_{1k} = H(\boldsymbol{s}^k_1, \boldsymbol{t}^k_1) \times \cdots \times H(\boldsymbol{s}^k_{r-1}, \boldsymbol{t}^k_{r-1}) \times H(\boldsymbol{s}^k_r, \boldsymbol{v}^k_r) \times \\ H(\boldsymbol{s}^k_{r+1}, \boldsymbol{t}^k_{r+1}) \times \cdots \times H(\boldsymbol{s}^k_p, \boldsymbol{t}^k_p), \\ \Psi_{2k} = H(\boldsymbol{s}^k_1, \boldsymbol{t}^k_1) \times \cdots \times H(\boldsymbol{s}^k_{r-1}, \boldsymbol{t}^k_{r-1}) \times H(\boldsymbol{v}^k_r, \boldsymbol{t}^k_r) \times \\ H(\boldsymbol{s}^k_{r+1}, \boldsymbol{t}^k_{r+1}) \times \cdots \times H(\boldsymbol{s}^k_p, \boldsymbol{t}^k_p).\end{aligned}$$

For $i = 1, 2$, do the following: Solve

$[\overline{\mathrm{P}}_{ik}]$ minimize$\{f(\boldsymbol{\xi}) \mid \boldsymbol{\xi} \in \Psi_{ik}\}$.

If $[\overline{\mathrm{P}}_{ik}]$ has an optimal solution $\boldsymbol{\xi}^{ik}$, then $\mathcal{P} = \mathcal{P} \cup \{(\Psi_{ik}, \boldsymbol{\xi}^{ik})\}$.

Step 4. Let $k = k + 1$ and return to Step 1. □

Letting

$$\Phi_k = \bigcup\{\Psi \mid (\Psi, \boldsymbol{\xi}) \in \mathcal{P}\} \tag{3.20}$$

at the beginning of the kth iteration, we see that the sequence, $\Phi_0, \Phi_1, \ldots$, satisfies the relation (3.9). Moreover, $\boldsymbol{\xi}^k$ is an optimal solution to the kth relaxed problem:

$[\overline{\mathrm{MP}}_k]$ minimize$\{f(\boldsymbol{\xi}) \mid \boldsymbol{\xi} \in \Phi_k\}$,

since we choose $(\Psi_k, \boldsymbol{\xi}^k)$ with the least $f(\boldsymbol{\xi}^k)$ from $\mathcal{P}$ every iteration.

If this algorithm terminates, either [MP] is found to be infeasible at Step 1 or an approximate solution $\boldsymbol{\xi}^\epsilon$ is obtained at Step 2. In the latter case, though $\boldsymbol{\xi}^\epsilon$ might not be feasible to [MP], it satisfies the ϵ-feasibility:

$$\xi^\epsilon_{1j}\xi^\epsilon_{2j} \leq 1 + \epsilon, \quad j = 1, \ldots, p, \quad \boldsymbol{\xi}^\epsilon \in \Phi_0, \tag{3.21}$$

and also provides a lower bound of the globally optimal value:

$$f(\boldsymbol{\xi}^\epsilon) \leq f(\boldsymbol{\xi}), \quad \forall \boldsymbol{\xi} \in \Xi \cap \Phi_0. \tag{3.22}$$

It is easily seen that $\boldsymbol{x}^*(\boldsymbol{\xi}^\epsilon)$ provided by $[\mathrm{P}(\boldsymbol{\xi}^\epsilon)]$ has the similar properties for the original problem [P].

Theorem 3.5. *If $\epsilon > 0$, Algorithm DC terminates after finitely many iterations. If $\epsilon = 0$, then every accumulation point of the sequence $\{\boldsymbol{\xi}^k \mid k = 0, 1, \ldots\}$ is a globally optimal solution to [MP].*

Proof: Suppose DC is infinite. Since $\boldsymbol{\xi}^k$'s are generated in the compact set $\Phi_0 \cap \Omega$, there exists a subsequence which converges to a point $\bar{\boldsymbol{\xi}} \in \Phi_0 \cap \Omega$. Let $r \in \arg\max\{\bar{\xi}_{1j}\bar{\xi}_{2j} \mid j = 1, \ldots, p\}$ and take $\{\boldsymbol{\xi}^{k_q} \mid q = 0, 1, \ldots\}$ from the convergent sequence so that the smallest index of $\arg\max\{\xi_{1j}^{k_q}\xi_{2j}^{k_q} \mid j = 1, \ldots, p\}$ is equal to r. Then an infinite sequence is generated in the ξ_{1r}-ξ_{2r} plane as follows:

$$H(\boldsymbol{s}_r^{k_0}, \boldsymbol{t}_r^{k_0}) \supset H(\boldsymbol{s}_r^{k_1}, \boldsymbol{t}_r^{k_1}) \supset \cdots.$$

Note that $\boldsymbol{s}_r^{k_q} = (s_{1r}^{k_q}, s_{2r}^{k_q})^{\mathrm{T}}$ and $\boldsymbol{t}_r^{k_q} = (t_{1r}^{k_q}, t_{2r}^{k_q})^{\mathrm{T}}$ satisfy

$$s_{1r}^0 \le s_{1r}^{k_q} < t_{1r}^{k_q} \le t_{1r}^0, \quad t_{2r}^0 \le t_{2r}^{k_q} < s_{2r}^{k_q} \le s_{2r}^0 \tag{3.23}$$

for each q. Otherwise, $\xi_{1r}^{k_q}\xi_{2r}^{k_q} \le 1$ holds and DC terminates.

Now assume the contrary to the assertion, i.e. there exists some constant $\sigma > \epsilon$ such that

$$\xi_{1r}^{k_q}\xi_{2r}^{k_q} \ge 1 + \sigma \quad \text{for every } q. \tag{3.24}$$

Let us define

$$h_{k_q}(\boldsymbol{\xi}) = (1/s_{1r}^{k_q} - 1/t_{1r}^{k_q})(\xi_{1r} - s_{1r}^{k_q}) - (s_{1r}^{k_q} - t_{1r}^{k_q})(\xi_{2r} - 1/s_{1r}^{k_q}).$$

In the ξ_{1r}-ξ_{2r} plane, $h_{k_q}(\boldsymbol{\xi}) = 0$ represents the line connecting two points $\boldsymbol{s}_r^{k_q}$ and $\boldsymbol{t}_r^{k_q}$. It follows from (3.23) that $h_{k_{q+1}}(\boldsymbol{\xi}^{k_q}) > 0$ for each q while $h_{k_q}(\boldsymbol{\xi}) \le 0$ for $\boldsymbol{\xi} \in H(\boldsymbol{s}_r^{k_q}, \boldsymbol{t}_r^{k_q})$. Hence, we have

$$\lim_{q\to\infty} h_{k_{q+1}}(\boldsymbol{\xi}^{k_q}) = \lim_{q\to\infty} h_{k_q}(\boldsymbol{\xi}^{k_q}) = 0. \tag{3.25}$$

In addition to this, (3.23) implies that $s_{1r}^{k_q}$'s and $t_{1r}^{k_q}$'s have subsequences convergent to some $\bar{s}_{1r}$ and $\bar{t}_{1r}$, respectively, and that

$$s_{1r}^0 \le \bar{s}_{1r} \le \bar{t}_{1r} \le t_{1r}^0.$$

Since either $s_{1r}^{k_{q+1}}$ or $t_{1r}^{k_{q+1}}$ coincides with $v_{1r}^{k_q} = \sqrt{\xi_{1r}^{k_q}/\xi_{2r}^{k_q}}$, we have three possible cases to consider.

- $\bar{s}_{1r} < \bar{t}_{1r} = \bar{v}_{1r} \equiv \sqrt{\bar{\xi}_{1r}/\bar{\xi}_{2r}}$ and the limit of h_{k_q} is

$$\bar{h}(\boldsymbol{\xi}) = (1/\bar{s}_{1r} - 1/\bar{v}_{1r})(\xi_{1r} - \bar{s}_{1r}) - (\bar{s}_{1r} - \bar{v}_{1r})(\xi_{2r} - 1/\bar{s}_{1r}).$$

- $\bar{s}_{1r} = \bar{v}_{1r} < \bar{t}_{1r}$ and the limit of h_{k_q} is

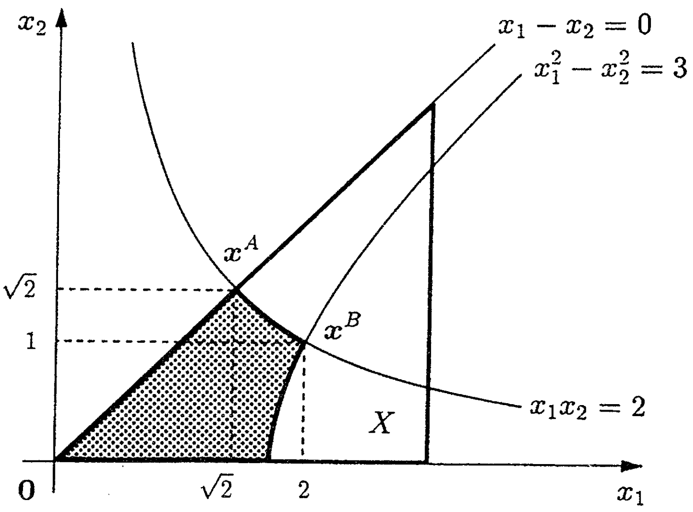

Figure 3.2. Example (3.27) of problem [P].

$$\bar{h}(\boldsymbol{\xi}) = (1/\bar{v}_{1r} - 1/\bar{t}_{1r})(\xi_{1r} - \bar{v}_{1r}) - (\bar{v}_{1r} - \bar{t}_{1r})(\xi_{2r} - 1/\bar{v}_{1r}).$$

- $\bar{s}_{1r} = \bar{t}_{1r} = \bar{v}_{1r}$ and the limit of h_{k_q} is

$$\bar{h}(\boldsymbol{\xi}) = -(\xi_{1r} - \bar{v}_{1r})/\bar{v}_{1r} + \bar{v}_{1r}(\xi_{2r} - 1/\bar{v}_{1r}).$$

In any case, simple arithmetic shows that there is a positive constant α such that

$$\lim_{q\to\infty} h_{k_q}(\boldsymbol{\xi}^{k_q}) = \bar{h}(\bar{\boldsymbol{\xi}}) = \alpha\left(\sqrt{\bar{\xi}_{1r}\bar{\xi}_{2r}} - 1\right). \tag{3.26}$$

We see from (3.25) and (3.26) that $\bar{\xi}_{1r}\bar{\xi}_{2r} = 1$, which contradicts the assumption (3.24). Therefore, Algorithm DC terminates after finitely many iterations if $\epsilon > 0$. Since f is continuous and $\Phi_{k_q} \supset \Xi \cap \Phi_0$ for each q, we have

$$f(\bar{\boldsymbol{\xi}}) = \liminf_{q\to\infty} f(\boldsymbol{\xi}^{k_q}) \le f(\boldsymbol{\xi}), \quad \forall \boldsymbol{\xi} \in \Xi \cap \Phi_0,$$

which implies that $\bar{\boldsymbol{\xi}}$ is a globally optimal solution to [MP]. □

3.3. Numerical Example

Let us illustrate Algorithm DC with the following simple example:

$$\left|\begin{array}{ll} \text{minimize} & -4x_1 - 5x_2 \\ \text{subject to} & x_1 - x_2 \ge 0, \quad 0 \le x_1 \le 3, \quad x_2 \ge 0 \\ & x_1^2 - x_2^2 \le 3, \quad x_1 x_2 \le 2. \end{array}\right. \tag{3.27}$$

This problem has two locally optimal solutions $\boldsymbol{x}^A = (\sqrt{2}, \sqrt{2})^{\mathrm{T}}$ and $\boldsymbol{x}^B = (2,1)^{\mathrm{T}}$, among which $\boldsymbol{x}^B$ is globally optimal (see Figure 3.2).

Let

$$X = \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_1 - x_2 \geq 0,\ 0 \leq x_1 \leq 3,\ x_2 \geq 0\},$$

and let

$$\begin{aligned} &\boldsymbol{d}_{11} = (0.333, 0.333)^{\mathrm{T}}, \quad \delta_{11} = 0; \quad \boldsymbol{d}_{21} = (1.000, -1.000)^{\mathrm{T}}, \quad \delta_{21} = 0 \\ &\boldsymbol{d}_{12} = (0.500, 0.000)^{\mathrm{T}}, \quad \delta_{12} = 0; \quad \boldsymbol{d}_{22} = (0.000, 1.000)^{\mathrm{T}}, \quad \delta_{22} = 0. \end{aligned}$$

Then we see that problem (3.27) satisfies condition (2.2). Also, both conditions (2.4) and (2.5) are fulfilled by

$$\left.\begin{aligned} &\ell_{11} = 0.000, \quad u_{11} = 2.000; \quad \ell_{21} = 0.000, \quad u_{21} = 3.000 \\ &\ell_{12} = 0.000, \quad u_{12} = 1.500; \quad \ell_{22} = 0.000, \quad u_{22} = 3.000 \end{aligned}\right\}. \tag{3.28}$$

The objective function value $f(\boldsymbol{\xi})$ of the master problem [MP] is provided by

$$\left|\begin{array}{lll} \text{minimize} & -4x_1 - 5x_2 & \\ \text{subject to} & (x_1, x_2)^{\mathrm{T}} \in X & \\ & 0.333x_1 + 0.333x_2 \leq \xi_{11}, & 1.000x_1 - 1.000x_2 \leq \xi_{21} \\ & 0.500x_1 \qquad\qquad \leq \xi_{12}, & \qquad\quad 1.000x_2 \leq \xi_{22}. \end{array}\right. \tag{3.29}$$

Initialization (Step 0). First of all, we construct the initial relaxed problem $[\overline{\mathrm{MP}}_0]$ of [MP]. By substituting (3.28) into (3.2) and (3.5), we have

$$\begin{aligned} &\boldsymbol{s}_1^0 = (0.333, 3.000)^{\mathrm{T}}, \quad \boldsymbol{t}_1^0 = (2.000, 0.500)^{\mathrm{T}} \\ &\boldsymbol{s}_2^0 = (0.333, 3.000)^{\mathrm{T}}, \quad \boldsymbol{t}_2^0 = (1.500, 0.666)^{\mathrm{T}}. \end{aligned}$$

Then $H(\boldsymbol{s}_1^0, \boldsymbol{t}_1^0)$ and $H(\boldsymbol{s}_2^0, \boldsymbol{t}_2^0)$ have the shapes shown by Figure 3.3, and their orthogonal product $\Phi_0 = H(\boldsymbol{s}_1^0, \boldsymbol{t}_1^0) \times H(\boldsymbol{s}_2^0, \boldsymbol{t}_2^0)$ is the feasible set of $[\overline{\mathrm{MP}}_0]$. The optimal solution to $[\overline{\mathrm{MP}}_0]$ is as follows:

$$\boldsymbol{\xi}^0 = (1.222, 0.306, 0.917, 1.833)^{\mathrm{T}}; \quad f(\boldsymbol{\xi}^0) = -16.500.$$

We then start iterating the algorithm with $\mathcal{P} = \{(\Phi_0, \boldsymbol{\xi}^0)\}$.

Iteration $k = 0$. We take $(\Phi_0, \boldsymbol{\xi}^0)$ from $\mathcal{P}$ and let

$[\overline{\mathrm{P}}_0]$ $\quad \text{minimize}\{f(\boldsymbol{\xi}) \mid \boldsymbol{\xi} \in \Phi_0 = H(\boldsymbol{s}_1^0, \boldsymbol{t}_1^0) \times H(\boldsymbol{s}_2^0, \boldsymbol{t}_2^0)\}.$

Since $\xi_{11}^0 \xi_{21}^0 = 0.373 < \xi_{12}^0 \xi_{22}^0 = 1.681$, we compute

$$\boldsymbol{v}_2^0 = \left(\sqrt{\xi_{12}^0/\xi_{22}^0}, \sqrt{\xi_{22}^0/\xi_{12}^0}\right)^{\mathrm{T}} = (0.707, 1.414)^{\mathrm{T}},$$

and define

$$\Psi_{10} = H(\boldsymbol{s}_1^0, \boldsymbol{t}_1^0) \times H(\boldsymbol{s}_2^0, \boldsymbol{v}_2^0), \quad \Psi_{20} = H(\boldsymbol{s}_1^0, \boldsymbol{t}_1^0) \times H(\boldsymbol{v}_2^0, \boldsymbol{t}_2)$$

(see Figure 3.3). The minima of f over Ψ_{10} and Φ_{20} are respectively

$$\begin{aligned} &\boldsymbol{\xi}^{10} = (0.943, 0.236, 0.707, 1.414)^{\mathrm{T}}; \quad f(\boldsymbol{\xi}^{10}) = -12.728 \\ &\boldsymbol{\xi}^{20} = (1.155, 1.768, 1.308, 0.848)^{\mathrm{T}}; \quad f(\boldsymbol{\xi}^{20}) = -14.702. \end{aligned}$$

We set $\mathcal{P} = \{(\boldsymbol{\xi}^{10}, \Psi_{10}), (\boldsymbol{\xi}^{20}, \Psi_{20})\}$.

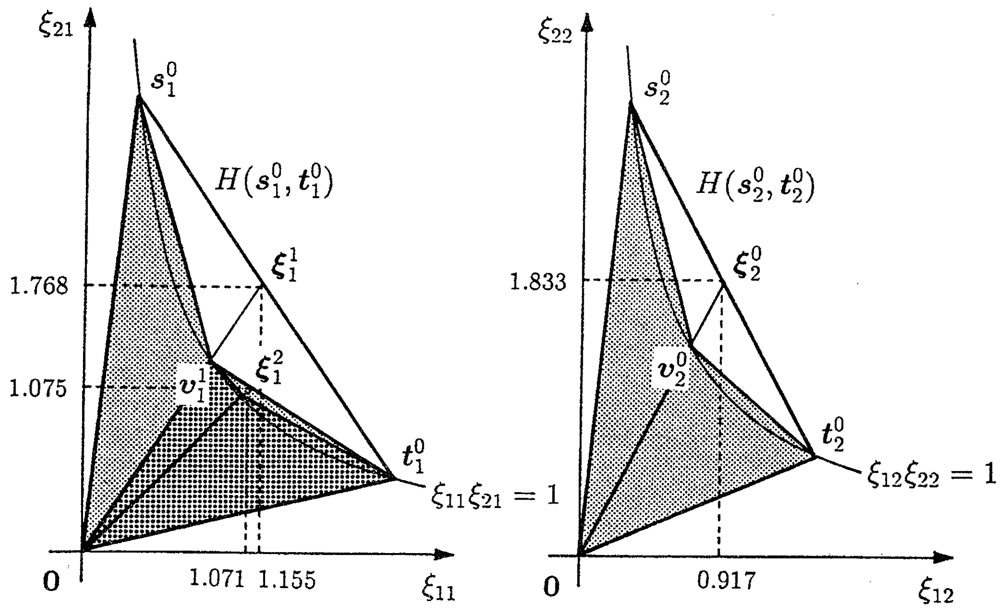

Figure 3.3. Feasible sets of the relaxed problems.

Iteration $k = 1$. Since $f(\boldsymbol{\xi}^{10}) > f(\boldsymbol{\xi}^{20})$, we take $(\boldsymbol{\xi}^{20}, \Psi_{20})$ as $(\boldsymbol{\xi}^1, \Psi_1)$ from $\mathcal{P}$ and let

$$[\overline{\mathrm{P}}_1] \quad \text{minimize}\{f(\boldsymbol{\xi}) \mid \boldsymbol{\xi} \in \Psi_1 = H(\boldsymbol{s}_1^1, \boldsymbol{t}_1^1) \times H(\boldsymbol{s}_2^1, \boldsymbol{t}_2^1)\},$$

where

$$\boldsymbol{s}_1^1 = \boldsymbol{s}_1^0 = (0.333, 3.000)^{\mathrm{T}}, \quad \boldsymbol{t}_1^1 = \boldsymbol{t}_1^0 = (2.000, 0.500)^{\mathrm{T}}$$
$$\boldsymbol{s}_2^1 = \boldsymbol{v}_2^0 = (0.707, 1.414)^{\mathrm{T}}, \quad \boldsymbol{t}_2^1 = \boldsymbol{t}_2^0 = (1.500, 0.666)^{\mathrm{T}}.$$

The optimal solution to the second relaxed problem $[\overline{\mathrm{MP}}_1]$ is given below by $[\overline{\mathrm{P}}_1]$:

$$\boldsymbol{\xi}^1 = (1.155, 1.768, 1.308, 0.848)^{\mathrm{T}}; \quad f(\boldsymbol{\xi}^1) = -14.702.$$

Since $\xi_{11}^1 \xi_{21}^1 = 2.041 > \xi_{12}^1 \xi_{22}^1 = 1.109$, we set

$$\boldsymbol{v}_1^1 = \left(\sqrt{\xi_{11}^1/\xi_{21}^1}, \sqrt{\xi_{21}^1/\xi_{11}^1}\right) = (0.808, 1.238)^{\mathrm{T}},$$
$$\Psi_{11} = H(\boldsymbol{s}_1^1, \boldsymbol{v}_1^1) \times H(\boldsymbol{s}_2^1, \boldsymbol{t}_2^1), \quad \Psi_{21} = H(\boldsymbol{v}_1^1, \boldsymbol{t}_1^1) \times H(\boldsymbol{s}_2^1, \boldsymbol{t}_2^1).$$

The minima of f over Ψ_{11} and Ψ_{21} are respectively

$$\boldsymbol{\xi}^{11} = (0.808, 1.238, 0.606, 1.212)^{\mathrm{T}}; \quad f(\boldsymbol{\xi}^{11}) = -10.909$$
$$\boldsymbol{\xi}^{21} = (1.071, 1.075, 1.072, 1.070)^{\mathrm{T}}; \quad f(\boldsymbol{\xi}^{21}) = -13.928.$$

We set $\mathcal{P} = \{(\boldsymbol{\xi}^{10}, \Psi_{10}), (\boldsymbol{\xi}^{11}, \Psi_{11}), (\boldsymbol{\xi}^{21}, \Psi_{21})\}$.

Iteration $k = 2$. We take $(\boldsymbol{\xi}^{21}, \Psi_{21})$ as $(\boldsymbol{\xi}^2, \Psi_2)$ from $\mathcal{P}$ and let

$$[\overline{\mathrm{P}}_2] \quad \text{minimize}\{f(\boldsymbol{\xi}) \mid \boldsymbol{\xi} \in \Psi_2 = H(\boldsymbol{s}_1^2, \boldsymbol{t}_1^2) \times H(\boldsymbol{s}_2^2, \boldsymbol{t}_2^2)\},$$

where

$$\begin{aligned} \boldsymbol{s}_1^2 &= \boldsymbol{v}_1^1 = (0.808, 1.238)^{\mathrm{T}}, \quad \boldsymbol{t}_1^2 = \boldsymbol{t}_1^1 = (2.000, 0.500)^{\mathrm{T}} \\ \boldsymbol{s}_2^2 &= \boldsymbol{s}_2^1 = (0.707, 1.414)^{\mathrm{T}}, \quad \boldsymbol{t}_2^2 = \boldsymbol{t}_2^1 = (1.500, 0.666)^{\mathrm{T}}. \end{aligned}$$

The optimal solution to $[\overline{\mathrm{MP}}_2]$ is

$$\boldsymbol{\xi}^2 = (1.071, 1.075, 1.072, 1.070)^{\mathrm{T}}; \quad f(\boldsymbol{\xi}^2) = -13.928.$$

Since $\xi_{11}^2 \xi_{21}^2 = 1.151 > \xi_{12}^2 \xi_{22}^2 = 1.147$, we set

$$\begin{aligned} &\boldsymbol{v}_1^2 = \left(\sqrt{\xi_{11}^2/\xi_{21}^2}, \sqrt{\xi_{21}^2/\xi_{22}^2}\right) = (0.999, 1.001)^{\mathrm{T}}, \\ &\Psi_{12} = H(\boldsymbol{s}_1^2, \boldsymbol{v}_1^2) \times H(\boldsymbol{s}_2^2, \boldsymbol{t}_2^2), \quad \Psi_{22} = H(\boldsymbol{v}_1^2, \boldsymbol{t}_1^2) \times H(\boldsymbol{s}_2^2, \boldsymbol{t}_2^2). \end{aligned}$$

The minima of f over Ψ_{12} and Ψ_{22} are respectively

$$\begin{aligned} \boldsymbol{\xi}^{12} &= (0.999, 1.001, 0.865, 1.265)^{\mathrm{T}}; \quad f(\boldsymbol{\xi}^{12}) = -13.248 \\ \boldsymbol{\xi}^{22} &= (1.059, 0.971, 1.037, 1.103)^{\mathrm{T}}; \quad f(\boldsymbol{\xi}^{22}) = -13.812. \end{aligned}$$

We set $\mathcal{P} = \{(\boldsymbol{\xi}^{10}, \Psi_{10}), (\boldsymbol{\xi}^{11}, \Psi_{11}), (\boldsymbol{\xi}^{12}, \Psi_{12}), (\boldsymbol{\xi}^{22}, \Psi_{22})\}$.

In the next iteration, we take $(\boldsymbol{\xi}^{22}, \Psi_{22})$ as $(\boldsymbol{\xi}^3, \Psi_3)$ from $\mathcal{P}$. Then the optimal solution to $[\overline{\mathrm{MP}}_3]$ is as follows:

$$\boldsymbol{\xi}^3 = (1.220, 0.891, 1.037, 1.103)^{\mathrm{T}}; \quad f(\boldsymbol{\xi}^3) = -13.812.$$

In this way, Algorithm DC will generate a sequence $\{\boldsymbol{\xi}^k \mid k = 0, 1, \ldots\}$, whose accumulation point $\boldsymbol{\xi}^* = (1.000, 1.000, 1.000, 1.000)^{\mathrm{T}}$ is a globally optimal solution to [MP]. We obtain an optimal solution $\boldsymbol{x}^B$ to (3.27) and the optimal value -13.000 by solving (3.29) with $\boldsymbol{\xi} = (1.000, 1.000, 1.000, 1.000)^{\mathrm{T}}$.

4. Computational Experiment

We will report computational results of testing Algorithm DC on randomly generated problems of the form:

$$\left|\begin{array}{ll} \text{minimize} & -\tilde{\boldsymbol{c}}^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} & \tilde{A}\boldsymbol{x} \leq \tilde{\boldsymbol{b}}, \quad \boldsymbol{x} \geq \boldsymbol{0} \\ & (\boldsymbol{d}_{1j}^{\mathrm{T}}\boldsymbol{x} + \delta_{1j})(\boldsymbol{d}_{2j}^{\mathrm{T}}\boldsymbol{x} + \delta_{2j}) \leq 1, \quad j = 1, \ldots, p, \end{array}\right. \tag{4.1}$$

Table 4.1. Performance of Algorithm DC for (4.1) when $\epsilon = 10^{-5}$.

p	2				3			
m	30	30	50	50	30	30	50	50
n	20	50	30	70	20	50	30	70
CPU time taken at Step 0								
	0.65	2.27	3.20	12.12	0.84	3.29	4.89	18.18
	(0.14)	(0.35)	(0.55)	(1.80)	(0.10)	(0.36)	(0.59)	(2.56)
# of subproblems								
	15.0	27.2	19.2	30.0	46.8	73.2	62.8	72.8
	(8.29)	(11.57)	(6.29)	(10.71)	(24.40)	(28.77)	(21.84)	(18.43)
CPU time taken at Steps 1 – 4								
	2.91	16.73	11.12	77.66	15.39	61.35	74.86	297.78
	(1.74)	(5.60)	(7.43)	(45.85)	(12.34)	(29.83)	(28.76)	(146.81)
Total CPU time								
	3.56	19.00	14.32	89.78	16.22	64.63	79.74	315.96
	(1.77)	(5.61)	(7.46)	(46.07)	(12.31)	(29.97)	(28.62)	(146.64)
p	4				5			
m	30	30	50	50	30	30	50	50
n	20	50	30	70	20	50	30	70
CPU time taken at Step 0								
	1.19	4.76	7.04	26.09	1.54	5.59	8.39	28.67
	(0.14)	(0.31)	(0.96)	(3.68)	(0.13)	(0.72)	(1.10)	(2.42)
# of subproblems								
	77.8	156.4	129.8	122.4	134.8	390.8	215.0	400.2
	(64.14)	(68.42)	(49.82)	(76.28)	(93.60)	(152.30)	(161.01)	(189.07)
CPU time taken at Steps 1 – 4								
	31.92	175.23	246.12	509.82	72.17	694.47	540.51	2175.89
	(21.95)	(64.87)	(152.83)	(218.92)	(50.48)	(356.32)	(427.06)	(1067.39)
Total CPU time								
	33.11	179.99	253.16	535.91	73.71	700.06	548.90	2204.55
	(21.98)	(65.00)	(152.16)	(218.83)	(50.43)	(356.37)	(426.28)	(1067.78)

Table 4.2. Effect of varying ϵ on Algorithm DC.

(m,n,p)	(30, 20, 3)				(30, 20, 4)			
ϵ	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
# of subproblems								
	36.4	46.8	55.0	65.0	62.2	77.8	89.0	96.2
	(19.62)	(24.40)	(27.54)	(32.12)	(53.73)	(77.80)	(89.00)	(76.19)
CPU time taken at Steps 1 – 4								
	11.60	15.39	18.23	21.65	25.43	31.92	36.61	39.27
	(8.66)	(12.34)	(14.22)	(17.48)	(18.27)	(21.95)	(24.43)	(25.64)

where $\tilde{A} \in \mathbb{R}^{m\times n}$, $\tilde{\boldsymbol{b}} \in \mathbb{R}^m$, $\tilde{\boldsymbol{c}}$, $\boldsymbol{d}_{ij} \in \mathbb{R}^n$ and $\delta_{ij} \in \mathbb{R}$.

Algorithm DC was coded in double precision C language according to the description in Section 3.2. At Step 0 in DC, we have to solve $4p$ linear programs of size (m,n) to compute ℓ_{ij}'s and u_{ij}'s, and one linear program of size $(m+3p, n+2p)$ to solve the initial relaxed problem $[\overline{\text{MP}}]$. The procedure employed to solve them was the usual revised simplex method. Each relaxed subproblem $[\overline{\text{P}}_{ik}]$ to be solved at Step 3 is also equivalent to a linear program of size $(m+3p, n+2p)$. In our code, the dual simplex method solved it using a solution provided by the preceding relaxed subproblem as the starting dual feasible solution.

The test problems (4.1) had 16 different sizes; (m,n,p) ranged from $(30,20,2)$ to $(50,70,5)$. Components of $\tilde{A}$, $\tilde{\boldsymbol{b}}$ and $\tilde{\boldsymbol{c}}$ were drawn from a uniform distribution in the interval $[0.00, 1.00]$, and those of $\boldsymbol{d}_{ij}$'s and δ_{ij} were in the interval $[0.50, 1.00]$. Out of the resulting instances of each problem size, we selected the first ten instances which were feasible but had no trivial solutions, and solved them on a microSPARC II computer (85 MHz).

Table 4.1 shows the average performance of Algorithm DC when the tolerance ϵ was fixed at 10^{-5}. For each (m,n,p), it gives the CPU time (in second) taken at Step 0, the number of relaxed subproblems $[\overline{\text{P}}_{ik}]$'s solved at Step 3, the CPU time taken at Steps 1 – 4, and the total computational time. The standard deviations of these numbers are also given in the brackets. Table 4.2 shows the effect of varying the tolerance ϵ on DC for $(m,n,p) = (30,20,3)$, $(30,20,4)$. For each value of ϵ, the number of $[\overline{\text{P}}_{ik}]$'s solved at Step 3 and the CPU time taken at Steps 1 – 4 are listed in it.

We see from Tables 4.1 and 4.2 that the main factor affecting the performance of Algorithm DC is the size of p. The number of subproblems generated through computation increases sharply as a function of p, and the computational time taken at Steps 1 – 4 does even more sharply. For each p, however, it is worth noting that the number of generated subproblems is rather insensitive to the variation of (m,n). Obviously, the total computational time is dominated by that needed to solve $[\overline{\mathrm{P}}_{ik}]$'s. Therefore, we may conclude that Algorithm DC has the potential to solve (4.1) with much larger (m, n) as long as p is a small number, say less than five. In that case, DC will require a more efficient procedure such as an interior-point algorithm or sophisticated implementation of the dual simplex method, to solve linear programs associated with $[\overline{\mathrm{P}}_{ik}]$'s.

5. Concluding Remark

Before closing the paper, let us devote a little space to discussing the general class of [P] defined below:

$$\left|\begin{array}{ll} \text{minimize} & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} & A'\boldsymbol{x} \geq \boldsymbol{b}', \quad \boldsymbol{x} \geq \boldsymbol{0} \\ & (\boldsymbol{d}_{1j}^{\mathrm{T}}\boldsymbol{x} + \delta_{1j})(\boldsymbol{d}_{2j}^{\mathrm{T}}\boldsymbol{x} + \delta_{2j}) \leq 1, \quad j = 1, \ldots, p', \end{array}\right. \tag{5.1}$$

where $A' \in \mathbb{R}^{m' \times n}$, $\boldsymbol{b}' \in \mathbb{R}^{m'}$, and the other notations are similar to [P] but $\boldsymbol{d}_{ij}^{\mathrm{T}}\boldsymbol{x} + \delta_{ij}$'s can take both positive and negative values on the polytope

$$X' = \{\boldsymbol{x} \in \mathbb{R}^n \mid A'\boldsymbol{x} \geq \boldsymbol{b}', \ \boldsymbol{x} \geq \boldsymbol{0}\}. \tag{5.2}$$

According to the signs of $\boldsymbol{d}_{ij}^{\mathrm{T}}\boldsymbol{x} + \delta_{ij}$'s, the problem (5.1) can be decomposed into $4^{p'}$ subproblems, each of which is of the form:

$$\left|\begin{array}{llll} \text{minimize} & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} & & \\ \text{subject to} & \boldsymbol{x} \in X' & & \\ & (\boldsymbol{d}_{1j}^{\mathrm{T}}\boldsymbol{x} + \delta_{1j})(\boldsymbol{d}_{2j}^{\mathrm{T}}\boldsymbol{x} + \delta_{2j}) \leq 1, & & j = 1, \ldots, p' \\ & \boldsymbol{d}_{ij}^{\mathrm{T}}\boldsymbol{x} + \delta_{ij} \leq 0, & i = 1, 2, & j \in J_1 \\ & \boldsymbol{d}_{ij}^{\mathrm{T}}\boldsymbol{x} + \delta_{ij} \geq 0, & i = 1, 2, & j \in J_2 \\ & \boldsymbol{d}_{1j}^{\mathrm{T}}\boldsymbol{x} + \delta_{2j} \leq 0, & \boldsymbol{d}_{2j}^{\mathrm{T}}\boldsymbol{x} + \delta_{2j} \geq 0, & j \in J_3, \end{array}\right. \tag{5.3}$$

where $J_1 \cup J_2 \cup J_3 = \{1, \ldots, p'\}$ and $J_1 \cap J_2 = J_2 \cap J_3 = J_3 \cap J_1 = \emptyset$. Since the last $2|J_3|$ constraints imply that

$$(\boldsymbol{d}_{1j}^{\mathrm{T}}\boldsymbol{x} + \delta_{1j})(\boldsymbol{d}_{2j}^{\mathrm{T}} + \delta_{2j}) \leq 1, \quad j \in J_3, \tag{5.4}$$

we can remove (5.4) from (5.3). Hence, (5.3) is reduced to

$$\left|\begin{array}{ll} \text{minimize} & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} & \boldsymbol{x} \in X \\ & (\tilde{\boldsymbol{d}}_{1j}^{\mathrm{T}}\boldsymbol{x} + \tilde{\delta}_{1j})(\tilde{\boldsymbol{d}}_{2j}^{\mathrm{T}}\boldsymbol{x} + \tilde{\delta}_{2j}) \leq 1, \quad j \in J_1 \cup J_2, \end{array}\right. \tag{5.5}$$

where

$$(\tilde{\boldsymbol{d}}_{1j}^{\mathrm{T}}, \tilde{\delta}_{1j}, \tilde{\boldsymbol{d}}_{2j}^{\mathrm{T}}, \tilde{\delta}_{2j}) = \begin{cases} -(\boldsymbol{d}_{1j}^{\mathrm{T}}, \delta_{1j}, \boldsymbol{d}_{2j}^{\mathrm{T}}, \delta_{2j}), & \text{if } j \in J_1 \\ (\boldsymbol{d}_{1j}^{\mathrm{T}}, \delta_{1j}, \boldsymbol{d}_{2j}^{\mathrm{T}}, \delta_{2j}), & \text{if } j \in J_2, \end{cases} \tag{5.6}$$

and

$$X = X' \cap \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} \tilde{\boldsymbol{d}}_{ij}^{\mathrm{T}}\boldsymbol{x} + \tilde{\delta}_{ij} \geq 0, & i = 1,2,\ j \in J_1 \cup J_2 \\ \boldsymbol{d}_{1j}^{\mathrm{T}}\boldsymbol{x} + \delta_{1j} \leq 0, & \boldsymbol{d}_{2j}^{\mathrm{T}}\boldsymbol{x} + \delta_{2j} \geq 0,\ \ j \in J_3 \end{array} \right\}. \tag{5.7}$$

Then we immediately see that

$$\tilde{\boldsymbol{d}}_{ij}^{\mathrm{T}}\boldsymbol{x} + \tilde{\delta}_{ij} \geq 0,\ \ \forall \boldsymbol{x} \in X,\quad i = 1,2,\ j \in J_1 \cup J_2. \tag{5.8}$$

Since we have not assumed that X nor X' has a nonempty interior, both the bounds $\tilde{\ell}_{ij}$ and $\tilde{u}_{ij}$ of $(\tilde{\boldsymbol{d}}_{ij}^{\mathrm{T}}\boldsymbol{x} + \tilde{\delta}_{ij})$ on X might coincide for some $j \in J_1 \cup J_2$. In such a case, however, $\tilde{\boldsymbol{d}}_{ij}^{\mathrm{T}}\boldsymbol{x} + \tilde{\delta}_{ij}$ is constant valued on X, and hence the constraint $(\tilde{\boldsymbol{d}}_{1j}^{\mathrm{T}}\boldsymbol{x} + \tilde{\delta}_{1j})(\tilde{\boldsymbol{d}}_{2j}^{\mathrm{T}}\boldsymbol{x} + \tilde{\delta}_{2j}) \leq 1$ can be regarded as a linear inequality. We can therefore assume for (5.5) that

$$0 \leq \tilde{\ell}_{ij} < \tilde{u}_{ij},\quad i = 1,2,\ j \in J_1 \cup J_2. \tag{5.9}$$

Also, in the same way as in Section 2, the other essential condition needed in Algorithm DC can be assumed as follows:

$$\tilde{\ell}_{1j}\tilde{\ell}_{2j} \leq 1 < \tilde{u}_{1j}\tilde{u}_{2j},\quad j \in J_1 \cup J_2. \tag{5.10}$$

These two conditions (5.9) and (5.10) allow us to apply Algorithm DC to (5.5). In other words, we can solve the general class (5.1) by applying DC to $4^{p'}$ problems belonging to its subclass [P]. When p' is two or three, Algorithm DC will generate a reasonably good approximate solution of (5.1) in a practical amount of time, as shown in Section 4.

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