Robust Disturbance-Rejection Problems for Linear ω-Periodic Discrete-Time Systems

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Abstract

In this paper, two robust disturbance-rejection problems with state feedback and with incomplete-state feedback for linear ω-periodic discrete-time systems are studied in the framework of the so-called geometric approach. And some necessary conditions and/or sufficient conditions for the problems to be solvable are presented. Finally an illustrative example is given.

key words : robustness, disturbance-rejection, ω-periodic discrete-time systems, simultaneous invariance, geometric approach

1 Introduction

In 1986 Ghosh[1] first studied the robust disturbance-rejection problem with state feedback for linear systems in the framework of the so-called geometric approach, and its solvability conditions were given. In 1995 Otsuka and Inaba *et al.*[3] studied the corresponding robust disturbance-rejection problem with incomplete-state feedback. On the other hand, in 1986 Grasselli and Longhi[2] studied the disturbance-rejection problem with state feedback for linear ω-periodic discrete-time systems.

The objective of this paper is to investigate two robust disturbance-rejection problems with state feedback and with incomplete-state feedback for linear ω-periodic discrete-time systems, and to study their solvability conditions.

This paper is organized as follows. Section 2 will give various notions of invariances and their important properties in order to formulate the problems. In Section 3, the two robust disturbance-rejection problems with state feedback and with incomplete-state feedback for linear ω-periodic discrete-time systems will be formulated, and some necessary conditions and/or sufficient conditions for their solvability will be presented. In Section 4, an illustrative example will be given. Finally, Section 5 will make some concluding remarks.

2 Preliminaries

In this section, some definitions of invariances are given, and their important properties are investigated.

The following notations will be used throughout this investigation. $\boldsymbol{Z}$:= the set of all integers, $\boldsymbol{Z}_{k_0}^{\omega} := \{k_0+1, k_0+2, \ldots, k_0+\omega \mid k_0 \in \boldsymbol{Z}\}$, $\boldsymbol{N}$:= the set of all natural numbers, $\boldsymbol{r} := \{1, 2, \ldots, r \mid r \in \boldsymbol{N}\}$, $\boldsymbol{R}^s$:= s-dimensional Euclidean space and $\boldsymbol{R}^{p\times q}$:= the set of all $p \times q$ real matrices. For a matrix-valued function $A(\cdot)$ $(A(k) \in \boldsymbol{R}^{p\times q}, k \in \boldsymbol{Z})$, Im$A(k)$:= the image of $A(k)$, Ker$A(k)$:= the kernel of $A(k)$ and $A^{-1}(k)\Omega := \{x \in \boldsymbol{R}^q \mid A(k)x \in \Omega\}$ for a subspace Ω of $\boldsymbol{R}^q$. And $A(\cdot)$ is said to be ω-periodic for a given $\omega \in \boldsymbol{N}$ if $A(k) = A(k+\omega)$ for all $k \in \boldsymbol{Z}$. For a subspace-valued function $V(\cdot)$ $(V(k) \subset \boldsymbol{R}^s, k \in \boldsymbol{Z})$, $V(\cdot)$ is said to be ω-periodic for a given $\omega \in \boldsymbol{N}$ if $V(k) = V(k+\omega)$ for all $k \in \boldsymbol{Z}$.

Now, consider a family of linear ω-periodic discrete-time systems $\{S_i^{\omega}; i \in \boldsymbol{r}\}$ given by

$$\{S_i^{\omega}; i \in \boldsymbol{r}\} : \begin{cases} x(k+1) &= A_i(k)x(k) + B_i(k)u(k) \\ y(k) &= C_i(k)x(k), \ k \in \boldsymbol{Z} \end{cases}$$

where $x(k) \in X := \boldsymbol{R}^n$ is the state, $u(k) \in U := \boldsymbol{R}^m$ is the input, $y(k) \in Y := \boldsymbol{R}^p$ is the incomplete-state (measurement output) and $A_i(\cdot)$ ($A_i(k) \in \boldsymbol{R}^{n\times n}$), $B_i(\cdot)$ ($B_i(k) \in \boldsymbol{R}^{n\times m}$) and $C_i(\cdot)$ ($C_i(k) \in \boldsymbol{R}^{p\times n}$) are ω-periodic for all $i \in \boldsymbol{r}$.

Now, the following definitions are given for the family of systems $\{S_i^\omega; i \in \boldsymbol{r}\}$.

Definition 2.1 Let $V(\cdot)$ ($V(k) \subset X$) be ω-periodic.

(i) $V(\cdot)$ is said to be $(A_i(\cdot), B_i(\cdot))$-invariant if there exists an ω-periodic feedback $F_i(\cdot)$ ($F_i(k) \in \boldsymbol{R}^{m\times n}$) such that

$$(A_i(k) + B_i(k)F_i(k))V(k) \subset V(k+1)$$

for all $k \in \boldsymbol{Z}$.

(ii) $V(\cdot)$ is said to be $\{(A_i(\cdot), B_i(\cdot)) \mid i \in \boldsymbol{r}\}$-invariant if there exists an ω-periodic feedback $F(\cdot)$ ($F(k) \in \boldsymbol{R}^{m\times n}$) such that

$$(A_i(k) + B_i(k)F(k))V(k) \subset V(k+1)$$

for all $i \in \boldsymbol{r}$ and $k \in \boldsymbol{Z}$. □

Definition 2.2 Let $V(\cdot)$ ($V(k) \subset X$) be ω periodic.

(i) $V(\cdot)$ is said to be $(A_i(\cdot), B_i(\cdot), C_i(\cdot))$-invariant if there exists an ω-periodic feedback $H_i(\cdot)$ ($H_i(k) \in \boldsymbol{R}^{m\times p}$) such that

$$(A_i(k) + B_i(k)H_i(k)C_i(k))V(k) \subset V(k+1)$$

for all $k \in \boldsymbol{Z}$.

(ii) $V(\cdot)$ is said to be $\{(A_i(\cdot), B_i(\cdot), C_i(\cdot)) \mid i \in \boldsymbol{r}\}$-invariant if there exists an ω-periodic feedback $H(\cdot)$ ($H(k) \in \boldsymbol{R}^{m\times p}$) such that

$$(A_i(k) + B_i(k)H(k)C_i(k))V(k) \subset V(k+1)$$

for all $i \in \boldsymbol{r}$ and $k \in \boldsymbol{Z}$. □

Remark 2.1 We note that, for each system S_i^ω, an $(A_i(\cdot), B_i(\cdot))$-invariant (or an $(A_i(\cdot), B_i(\cdot), C_i(\cdot))$-invariant) $V(\cdot)$ has the property that if an arbitrary initial state $x(0) \in V(0)$ then there exists a state feedback input $u(k) = F_i(k)x(k)$ (or an incomplete-state feedback input $u(k) = H_i(k)y(k)$) such that $x(k) \in V(k)$ for all $k \geq 0$. On the other hand, for the family of systems $\{S_i^\omega\ ;\ i \in \boldsymbol{r}\}$, an $\{(A_i(\cdot), B_i(\cdot)) \mid i \in \boldsymbol{r}\}$-invariant (or an $\{(A_i(\cdot), B_i(\cdot), C_i(\cdot)) \mid i \in \boldsymbol{r}\}$-invariant) $V(\cdot)$ has the property that if an arbitrary initial state $x(0) \in V(0)$ then there exists a state feedback input $u(k) = F(k)x(k)$ (or an incomplete-state feedback input $u(k) = H(k)y(k)$) which is independent on i such that $x(k) \in V(k)$ for all $k \geq 0$. □

Now, the following lemma holds.

Lemma 2.1 Let $V(\cdot)$ ($V(k) \subset X$) be ω-periodic. Then, $V(\cdot)$ is $\{(A_i(\cdot), B_i(\cdot)) \mid i \in \boldsymbol{r}\}$-invariant if and only if $V(\cdot)$ satisfies

$$\begin{pmatrix} A_1(k) \\ A_2(k) \\ \vdots \\ A_r(k) \end{pmatrix} V(k) \subset \begin{pmatrix} V(k+1) \\ V(k+1) \\ \vdots \\ V(k+1) \end{pmatrix} + \mathrm{Im} \begin{pmatrix} B_1(k) \\ B_2(k) \\ \vdots \\ B_r(k) \end{pmatrix} \tag{1}$$

for all $k \in \boldsymbol{Z}$.

(Proof) Necessity is obvious. To prove sufficiency, suppose that $V(\cdot)$ satisfies (1). And now, we fix $k_0 \in \boldsymbol{Z}$. Then, there exists a basis $\{v_1^k, v_2^k, \cdots, v_{\mu(k)}^k\}$ of $V(k)$ for all $k \in \boldsymbol{Z}_{k_0}^{\omega}$ where $\mu(k)$ is a dimension of $V(k)$. It follows from (1) that there exists vectors $\begin{pmatrix} w_{1,j}^{k+1} \\ w_{2,j}^{k+1} \\ \vdots \\ w_{r,j}^{k+1} \end{pmatrix} \in \begin{pmatrix} V(k+1) \\ V(k+1) \\ \vdots \\ V(k+1) \end{pmatrix}$ and $u_j^k \in U$ such that

$$\begin{pmatrix} A_1(k) \\ A_2(k) \\ \vdots \\ A_r(k) \end{pmatrix} v_j^k = \begin{pmatrix} w_{1,j}^{k+1} \\ w_{2,j}^{k+1} \\ \vdots \\ w_{r,j}^{k+1} \end{pmatrix} + \begin{pmatrix} B_1(k) \\ B_2(k) \\ \vdots \\ B_r(k) \end{pmatrix} u_j^k \tag{2}$$

for all $j \in \{1, 2, \ldots, \mu(k)\}$ and $k \in \boldsymbol{Z}_{k_0}^{\omega}$.

Since $V(\cdot)$ is ω-periodic, the basis $\{v_1^k, v_2^k, \ldots, v_{\mu(k)}^k\}$ is also a basis of $V(k + l\omega)$ and $w_{i,j}^{k+1}$ is in $V(k + l\omega + 1)$ for all $l \in \boldsymbol{Z}$. Then, we define

$$v_j^{k+l\omega} := v_j^k \in V(k + l\omega) \tag{3}$$

$$w_{i,j}^{k+l\omega+1} := w_{i,j}^{k+1} \in V(k + l\omega + 1)$$

$$u_j^{k+l\omega} := u_j^k \in \boldsymbol{U} \tag{4}$$

for all $k \in \boldsymbol{Z}_{k_0}^{\omega}$, $l \in \boldsymbol{Z}$ and $j \in \{1, 2, \ldots, \mu(k)\}$. Then, (2) is satisfied for all $k \in \boldsymbol{Z}$. Now, define a state feedback $F_0(k) : V(k) \to U$ by

$$u_j^k = -F_0(k) v_j^k \tag{5}$$

for all $j \in \{1, 2, \dots, \mu(k)\}$ and $k \in \boldsymbol{Z}$. And let $F(k) : X \to U$ be any extension of $F_0(k)$ to X for all $k \in \boldsymbol{Z}$. Then, it follows from (3), (4) and (5) that $F(\cdot)$ is ω-periodic. And from (2) and definition of $F(\cdot)$,

$$\begin{pmatrix} A_1(k) \\ A_2(k) \\ \vdots \\ A_r(k) \end{pmatrix} v_j^k = \begin{pmatrix} w_{1,j}^{k+1} \\ w_{2,j}^{k+1} \\ \vdots \\ w_{r,j}^{k+1} \end{pmatrix} - \begin{pmatrix} B_1(k)F(k) \\ B_2(k)F(k) \\ \vdots \\ B_r(k)F(k) \end{pmatrix} v_j^k$$

for all $k \in \boldsymbol{Z}$. It follows that

$$(A_i(k) + B_i(k)F(k))v_j^k = w_{i,j}^{k+1} \in V(k+1)$$

for all $j \in \{1, 2, \dots, \mu(k)\}$, $i \in \boldsymbol{r}$ and $k \in \boldsymbol{Z}$ which imply

$$(A_i(k) + B_i(k)F(k))V(k) \subset V(k+1)$$

for all $i \in \boldsymbol{r}$ and $k \in \boldsymbol{Z}$. This completes the proof. □

For a given ω-periodic subspace-valued function $\Lambda(\cdot)$ ($\Lambda(k) \subset X$), define the following class of ω-periodic subspace-valued functions.

$$\boldsymbol{V}_s(A_i(\cdot), B_i(\cdot); \Lambda(\cdot) \mid i \in \boldsymbol{r}) := \Big\{ V(\cdot) \mid V(k) \subset \Lambda(k), V(\cdot) \text{ is } \{(A_i(\cdot), B_i(\cdot)) \mid i \in \boldsymbol{r}\}\text{-invariant} \Big\}.$$

For simplicity, $\boldsymbol{V}_s(A_i(\cdot), B_i(\cdot); \Lambda(\cdot) \mid i \in \boldsymbol{r})$ is denoted by $\boldsymbol{V}_s(\Lambda(\cdot))$.

Definition 2.3 $V^*(\cdot)$ is said to be a supremal element of $\boldsymbol{V}_s(\Lambda(\cdot))$ if the following two conditions hold.

(i) $V^*(\cdot) \in \boldsymbol{V}_s(\Lambda(\cdot))$.

(ii) If $V(\cdot)$ is an arbitrary element of $\boldsymbol{V}_s(\Lambda(\cdot))$, then $V(k) \subset V^*(k)$ for all $k \in \boldsymbol{Z}$. □

Then, the following lemma holds.

Lemma 2.2 $\boldsymbol{V}_s(\Lambda(\cdot))$ has the supremal element $V^*(\cdot)$ in the sense of Definition 2.3.

(Proof) The proof can be easily shown by using the results of Wonham [4]. □

The computational algorithm of $V^*(\cdot)$ can be obtained as follows.

Lemma 2.3 Let $\Lambda(\cdot)$ ($\Lambda(k) \subset X$) be ω-periodic. For each $k \in \boldsymbol{Z}$, define the sequence $V^\mu(k)$ for $k \in Z$ according to

$$V^0(k) := \Lambda(k)$$

$$V^\mu(k) := \Lambda(k) \cap \begin{pmatrix} A_1(k) \\ A_2(k) \\ \vdots \\ A_r(k) \end{pmatrix}^{-1} \left[\begin{pmatrix} V^{\mu-1}(k+1) \\ V^{\mu-1}(k+1) \\ \vdots \\ V^{\mu-1}(k+1) \end{pmatrix} + \mathrm{Im} \begin{pmatrix} B_1(k) \\ B_2(k) \\ \vdots \\ B_r(k) \end{pmatrix} \right]$$

for $\mu \geq 0$. Then, the following statements hold.

(i) $V^{\mu}(k) \subset V^{\mu-1}(k)$ for all $k \in \boldsymbol{Z}$ and $\mu \geq 1$.

(ii) For fixed $k_0 \in \boldsymbol{Z}$, there exists $j \leq \max\{\dim[\Lambda(k)] \mid k \in Z_{k_0}^{\omega}\}$ such that $V^j(\cdot)$ is the supremal element of $\boldsymbol{V}_s(\Lambda(\cdot))$.

(Proof) The proof of (i) can be easily shown by induction with μ. Therefore, we will prove (ii). In order to prove (ii), it suffices to show the following two claims.

Claim 1 There exists $j \leq \max\{\dim[\Lambda(k)|k \in Z_{k_0}^{\omega}]\}$ such that $V^j(\cdot) \in \boldsymbol{V}_s(\Lambda(\cdot))$.

Claim 2 For an arbitrary element $V(\cdot) \in \boldsymbol{V}_s(\Lambda(\cdot))$, $V(k) \subset V^{\mu}(k)$ for all $k \in \boldsymbol{Z}$ and $\mu \geq 0$.

(Proof of Claim 1) We will first prove that $V^{\mu}(\cdot)$ is ω-periodic for all $j \geq 0$ by induction with μ. Since $\Lambda(\cdot)$ is ω-periodic, it is obvious that $V^0(\cdot)$ is ω-periodic. Assume that $V^{\mu-1}(\cdot)$ is ω-periodic. Since $A(\cdot)$, $B(\cdot)$, $\Lambda(\cdot)$ and $V^{\mu-1}(\cdot)$ are all ω-periodic, the following equalities hold.

$$\begin{aligned}
V^{\mu}(k) &= \Lambda(k) \cap \begin{pmatrix} A_1(k) \\ A_2(k) \\ \vdots \\ A_r(k) \end{pmatrix}^{-1} \left[\begin{pmatrix} V^{\mu-1}(k+1) \\ V^{\mu-1}(k+1) \\ \vdots \\ V^{\mu-1}(k+1) \end{pmatrix} + \operatorname{Im} \begin{pmatrix} B_1(k) \\ B_2(k) \\ \vdots \\ B_r(k) \end{pmatrix} \right] \\
&= \Lambda(k+\omega) \cap \begin{pmatrix} A_1(k+\omega) \\ A_2(k+\omega) \\ \vdots \\ A_r(k+\omega) \end{pmatrix}^{-1} \left[\begin{pmatrix} V^{\mu-1}(k+\omega+1) \\ V^{\mu-1}(k+\omega+1) \\ \vdots \\ V^{\mu-1}(k+\omega+1) \end{pmatrix} + \operatorname{Im} \begin{pmatrix} B_1(k+\omega) \\ B_2(k+\omega) \\ \vdots \\ B_r(k+\omega) \end{pmatrix} \right] \\
&= V^{\mu}(k+\omega)
\end{aligned}$$

for all $k \in \boldsymbol{Z}$. Then, $V^{\mu}(\cdot)$ is ω-periodic for all $\mu \geq 0$.

Next, since $V^{\mu}(\cdot)$ ($\mu \geq 0$) and $\Lambda(\cdot)$ are ω-periodic, there exists $j \leq \max\{\dim[\Lambda(k)|k \in Z_{k_0}^{\omega}]\}$ such that

$$V^j(k) = \Lambda(k) \cap \begin{pmatrix} A_1(k) \\ A_2(k) \\ \vdots \\ A_r(k) \end{pmatrix}^{-1} \left[\begin{pmatrix} V^j(k+1) \\ V^j(k+1) \\ \vdots \\ V^j(k+1) \end{pmatrix} + \operatorname{Im} \begin{pmatrix} B_1(k) \\ B_2(k) \\ \vdots \\ B_r(k) \end{pmatrix} \right]$$

for all $k \in \boldsymbol{Z}$ which imply

$$V^j(k) \subset \Lambda(k) \tag{6}$$

$$\begin{pmatrix} A_1(k) \\ A_2(k) \\ \vdots \\ A_r(k) \end{pmatrix} V^j(k) \subset \begin{pmatrix} V^j(k+1) \\ V^j(k+1) \\ \vdots \\ V^j(k+1) \end{pmatrix} + \mathrm{Im} \begin{pmatrix} B_1(k) \\ B_2(k) \\ \vdots \\ B_r(k) \end{pmatrix} \tag{7}$$

for all $k \in \boldsymbol{Z}$. Therefore, it follows from ω-periodicity of $V^j(\cdot)$, (6), (7) and Lemma 2.1 that Claim 1 holds.

(Proof of Claim 2) The proof also can be shown by the induction with μ. Let $V(\cdot)$ be an arbitrary element of $\boldsymbol{V}_s(\Lambda(\cdot))$. It is obvious that

$$V(k) \subset \Lambda(k) = V^0(k)$$

for all $k \in \boldsymbol{Z}$. Assume that $V(k) \subset V^{\mu-1}(k)$ for all $k \in \boldsymbol{Z}$. Then by using Lemma 2.1, the following inclusions hold.

$$\begin{aligned} V(k) &\subset \Lambda(k) \cap \begin{pmatrix} A_1(k) \\ A_2(k) \\ \vdots \\ A_r(k) \end{pmatrix}^{-1} \left[\begin{pmatrix} V(k+1) \\ V(k+1) \\ \vdots \\ V(k+1) \end{pmatrix} + \mathrm{Im} \begin{pmatrix} B_1(k) \\ B_2(k) \\ \vdots \\ B_r(k) \end{pmatrix} \right] \\ &\subset \Lambda(k) \cap \begin{pmatrix} A_1(k) \\ A_2(k) \\ \vdots \\ A_r(k) \end{pmatrix}^{-1} \left[\begin{pmatrix} V^{\mu-1}(k+1) \\ V^{\mu-1}(k+1) \\ \vdots \\ V^{\mu-1}(k+1) \end{pmatrix} + \mathrm{Im} \begin{pmatrix} B_1(k) \\ B_2(k) \\ \vdots \\ B_r(k) \end{pmatrix} \right] \\ &= V^{\mu}(k) \end{aligned}$$

for all $k \in \boldsymbol{Z}$. Hence, the proof of Claim 2 was completed.

This completes the proof of this lemma. □

3 Problems Formulation and Main Results

In this section, we first formulate our robust disturbance-rejection problems and then give some solvability conditions for the problems.

Consider the following linear ω-periodic discrete-time system $S^{\omega}(\alpha, \beta, \gamma, \delta, \sigma)$.

$$S^{\omega}(\alpha, \beta, \gamma, \delta, \sigma) : \begin{cases} x(k+1) &= A(k)x(k) + B(k)u(k) + M(k)\xi(k) \\ y(k) &= C(k)x(k) \\ z(k) &= D(k)x(k), \ k \in \boldsymbol{Z} \end{cases}$$

where $x(k) \in X := \boldsymbol{R}^{n \times n}$, $u(k) \in U := \boldsymbol{R}^{n \times m}$, $\xi(k) \in Q := \boldsymbol{R}^s$, $y(k) \in Y := \boldsymbol{R}^p$ and $z(k) \in Z := \boldsymbol{R}^q$ are the state, the input, the disturbance, the incomplete state and the controlled output, respectively, and $A(k), B(k), M(k), C(k)$ and $D(k)$ have uncertainties which are assumed to have the following convex combinations of two given matrices, respectively.

$$\begin{aligned} A(k) &= \alpha A_1(k) + (1-\alpha)A_2(k) \\ B(k) &= \beta B_1(k) + (1-\beta)B_2(k) \\ M(k) &= \gamma M_1(k) + (1-\gamma)M_2(k) \\ C(k) &= \delta C_1(k) + (1-\delta)C_2(k) \\ D(k) &= \sigma D_1(k) + (1-\sigma)D_2(k) \end{aligned} \tag{8}$$

where $A_i(\cdot)$ $(A_i(k) \in \boldsymbol{R}^{n\times n})$, $B_i(\cdot)$ $(B_i(k) \in \boldsymbol{R}^{n\times m})$, $M_i(\cdot)$ $(M_i(k) \in \boldsymbol{R}^{n\times s})$, $C_i(\cdot)$ $(C_i(k) \in \boldsymbol{R}^{p\times n})$ and $D_i(\cdot)$ $(D_i(k) \in \boldsymbol{R}^{q\times n})$ are all ω-periodic for $i \in \mathbf{2}$, and $\alpha, \beta, \gamma, \delta, \sigma \in [0,1]$.

Now, we consider an incomplete-state feedback of the form

$$u(k) = H(k)y(k) = H(k)C(k)x(k)$$

where $H(\cdot)$ $(H(k) \in \boldsymbol{R}^{m\times p})$ is the ω-periodic incomplete-state feedback. Then, we obtain the following closed loop system $S_{cl}^{\omega}(\alpha,\beta,\gamma,\delta,\sigma)$ (see Figure 1).

$$S_{cl}^{\omega}(\alpha,\beta,\gamma,\delta,\sigma) : \begin{cases} x(k+1) &= (A(k) + B(k)H(k)C(k))x(k) + M(k)\xi(k) \\ z(k) &= D(k)x(k), \ k \in \boldsymbol{Z} \end{cases}$$

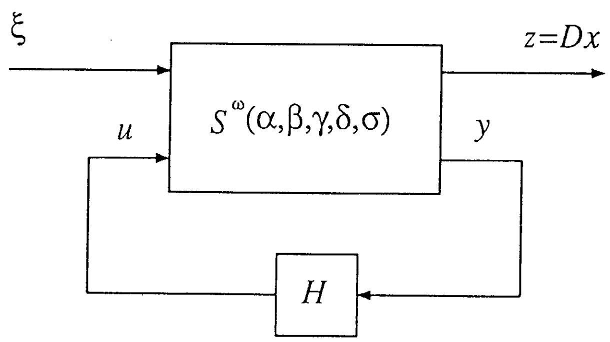

Figure 1: Block diagram

For the system $S_{cl}^{\omega}(\alpha,\beta,\gamma,\delta,\sigma)$, we use the following notations.

$$\begin{aligned} A^H(k) &:= A(k) + B(k)H(k)C(k) \\ \Phi^H(k,k_0) &:= A^H(k-1)A^H(k-2)\cdots A^H(k_0) \qquad (k > k_0) \\ \Phi^H(k,k) &:= I_n \end{aligned}$$

where I_n is an $(n \times n)$-th identify matrix. Then, the following lemma holds.

Lemma 3.1 The following relations hold.

$$\Phi^H(k, k-h\omega) = \left\{\Phi^H(k, k-\omega)\right\}^h \tag{9}$$

$$\Phi^H(i,j)\Phi^H(j,k) = \Phi^H(i,k) \tag{10}$$

$$\Phi^H(k+\omega, k+\omega-h) = \Phi^H(k, k-h) \tag{11}$$

(Proof) The proof is easily shown and is omitted. □

Our robust disturbance-rejection problem can be stated as follows. Given ω-periodic matrix-valued functions $A_1(\cdot)$, $A_2(\cdot)$, $B_1(\cdot)$, $B_2(\cdot)$, $M_1(\cdot)$, $M_2(\cdot)$, $C_1(\cdot)$, $C_2(\cdot)$, $D_1(\cdot)$ and $D_2(\cdot)$ of the system $S^\omega(\alpha,\beta,\gamma,\delta,\sigma)$, find (if possible) an ω-periodic incomplete-state feedback $H(\cdot)$ ($H(k) \in \boldsymbol{R}^{m\times p}$) such that

$$D(k)\sum_{h=k_0}^{k-1}\Phi^H(k,h+1)M(h)\xi(h) = 0$$

for all $\alpha,\beta,\gamma,\delta,\sigma \in [0,1]$, $\xi(\cdot)$ and $k \in \boldsymbol{Z}$ where k_0 is an initial time.

It remarks that a subspace generated by the disturbance $\xi(\cdot)$ is the following subspace.

$$\sum_{h=k_0}^{k-1}\Phi^H(k,h+1)\mathrm{Im}M(h) \tag{12}$$

Before formulating our problem, some properties related to subspace (12) are shown.

The following lemma will be used to prove Lemmas 3.3 and 3.4.

Lemma 3.2 The following inclusion holds.

$$\Phi^H(k, k-(n+h)\omega-l)\mathrm{Im}M(k-(n+h)\omega-l-1) \subset \sum_{i=0}^{n-1}\left\{\Phi^H(k,k-i\omega-l)\mathrm{Im}M(k-i\omega-l-1)\right\}$$

for all $l \in \{0,1,\ldots,\omega-1\}$, $h \geq 0$ and $k \in \boldsymbol{Z}$.

(Proof) See Appendix 1. □

Lemma 3.3 The following assertions hold.

(i) If $k - n\omega < k_0$, then

$$\sum_{h=k_0}^{k-1}\Phi^H(k,h+1)\mathrm{Im}M(h) \subset \sum_{h=k-n\omega}^{k-1}\Phi^H(k,h+1)\mathrm{Im}M(h)$$

(ii) If $k_0 \leq k - n\omega$, then

$$\sum_{h=k_0}^{k-1}\Phi^H(k,h+1)\mathrm{Im}M(h) = \sum_{h=k-n\omega}^{k-1}\Phi^H(k,h+1)\mathrm{Im}M(h)$$

(Proof) See Appendix 2. □

By virtue of Lemma 3.3, our robust disturbance-rejection problems can be formulated as follows.

Robust Disturbance-Rejection Problem with Incomplete-State Feedback (RDRP-ISF) Given ω-periodic matrix-valued functions $A_1(\cdot)$, $A_2(\cdot)$, $B_1(\cdot)$, $B_2(\cdot)$, $M_1(\cdot)$, $M_2(\cdot)$, $C_1(\cdot)$, $C_2(\cdot)$, $D_1(\cdot)$ and $D_2(\cdot)$ of the system $S^\omega(\alpha, \beta, \gamma, \delta, \sigma)$, find (if possible) an ω-periodic incomplete-state feedback gain $H(\cdot)$ ($H(k) \in \boldsymbol{R}^{m \times p}$) such that

$$\sum_{h=k-n\omega}^{k-1} \Phi^H(k, h+1)\mathrm{Im}M(h) \subset \mathrm{Ker}D(k)$$

for all $\alpha, \beta, \gamma, \delta, \sigma \in [0,1]$ and $k \in \boldsymbol{Z}$. □

Remark 3.1 If $C(\cdot)$ satisfies $C(k) = I_n$ for all $k \in \boldsymbol{Z}$, the above problem (RDRPISF) reduced to the corresponding problem with state feedback, and is called Robust Disturbance-Rejection Problem with State Feedback (RDRPSF). □

Next theorem is our main result.

Theorem 3.1 If there exists an $\{(A_i(\cdot), B_j(\cdot), C_l(\cdot)) | i, j, l \in \mathbf{2}\}$-invariant $V(\cdot)$ such that

$$(\mathrm{Im}M(k-1) + \mathrm{Im}M(k-1)) \subset V(k) \subset (\mathrm{Ker}D_1(k) \cap \mathrm{Ker}D_2(k)) \tag{13}$$

for all $k \in \boldsymbol{Z}$, then the RDRPISF is solvable.

(Proof) Suppose that $\{(A_i(\cdot), B_j(\cdot), C_l(\cdot)) | i, j, l \in \mathbf{2}\}$-invariant $V(\cdot)$ satisfies (13) for all $k \in \boldsymbol{Z}$. Then, there exists an ω-periodic feedback $H(\cdot)$ ($H(k) \in \boldsymbol{R}^{m \times q}$) such that

$$(A_i(k) + B_j(k)H(k)C_l(k))V(k) \subset V(k+1) \tag{14}$$

for $i, j, l \in \mathbf{2}$ and $k \in \boldsymbol{Z}$. Choose an arbitrary element $x(k) \in V(k)$. Then, using (8) and (14), the following relations hold.

$$\begin{aligned}
&(A(k) + B(k)H(k)C(k))x(k) \\
&\quad = \alpha(A_1(k) + B_1(k)H(k)C_1(k))x(k) + (\beta\delta - \alpha)(A_2(k) + B_1(k)H(k)C_1(k))x(k) \\
&\qquad + \beta(1-\delta)(A_2(k) + B_1(k)H(k)C_2(k))x(k) + (1-\beta)\delta(A_2(k) + B_2(k)H(k)C_1(k))x(k) \\
&\qquad + (1-\beta)(1-\delta)(A_2(k) + B_2(k)H(k)C_2(k))x(k) \\
&\quad \in V(k+1)
\end{aligned}$$

for all $\alpha, \beta, \gamma, \delta, \sigma \in [0,1]$ and $k \in \boldsymbol{Z}$ which imply

$$A^H(k)V(k) = (A(k) + B(k)H(k)C(k))V(k) \subset V(k+1)$$

for all $\alpha, \beta, \delta \in [0,1]$ and $k \in \boldsymbol{Z}$. Further, it follows that

$$\begin{aligned}
\Phi^H(k,h+1)V(h+1) &= A^H(k-1)A^H(k-2)\cdots A^H(h+2)A^H(h+1)V(h+1) \\
&\subset A^H(k-1)A^H(k-2)\cdots A^H(h+2)V(h+2) \\
&\vdots \\
&\subset V(k)
\end{aligned} \tag{15}$$

for all $k, h+1 \in \boldsymbol{Z}$ $(k \geq h+1)$. Moreover, since $V(\cdot)$ satisfies (13),

$$\mathrm{Im}M(k-1) \subset V(k) \subset \mathrm{Ker}D(k) \tag{16}$$

for $\gamma, \sigma \in [0,1]$ and $k \in \boldsymbol{Z}$.

Therefore, it follows from (15) and (16) that

$$\begin{aligned}
\sum_{h=k-n\omega}^{k-1} \Phi^H(k,h+1)\mathrm{Im}M(h) &\subset \sum_{h=k-n\omega}^{k-1} \Phi^H(k,h+1)V(h+1) \\
&\subset \sum_{h=k-n\omega}^{k-1} V(k) \\
&= \underbrace{V(k)+V(k)+\cdots+V(k)}_{n\omega \text{ times}} \\
&\subset \mathrm{Ker}D(k)
\end{aligned}$$

for all $k \in \boldsymbol{Z}$ and $\alpha, \beta, \gamma, \delta, \sigma \in [0,1]$ which prove that RDRPISF is solvable. □

Corollary 3.1 It is assumed that $C(\cdot)$ satisfies $C(k) = I_n$ for all $k \in \boldsymbol{Z}$. And, define $\Omega_{12}(k) := \mathrm{Ker}D_1(k) \cap \mathrm{Ker}D_2(k)$ for $k \in \boldsymbol{Z}$ and let $V^*(\cdot)$ be the supremal element of $\boldsymbol{V}_s(A_i(\cdot), B_j(\cdot); \Omega_{12}(\cdot) \mid i,j \in \mathbf{2})$. If

$$(\mathrm{Im}M_1(k-1) + \mathrm{Im}M_2(k-1)) \subset V^*(k)$$

for all $k \in \boldsymbol{Z}$, then RDRPSF is solvable.

(Proof) The proof follows from Theorem 3.1 □

Next lemma will be used to prove Theorem 3.2.

Lemma 3.4 Let $V(k) := \sum_{h=k-n\omega}^{k-1} \Phi^H(k,h+1)\mathrm{Im}M(h)$. Then, $V(\cdot)$ is ω-periodic and satisfies

$$A^H(k)V(k) \subset V(k+1)$$

for all $k \in \boldsymbol{Z}$, *i.e.* $V(\cdot)$ is $(A(\cdot), B(\cdot), C(\cdot))$-invariant.

(Proof) See Appendix 3. □

Theorem 3.2 If RDRPISF is solvable, then there exist $(A_i(\cdot), B_j(\cdot), C_l(\cdot))$-invariant $V_{ijl}(\cdot)$ for $i, j, l \in \mathbf{2}$ such that

(i) $(\mathrm{Im}M_1(k-1) + \mathrm{Im}M_2(k-1)) \subset \bigcap_{i,j,l \in \mathbf{2}} V_{ijl}(k)$ for all $k \in \boldsymbol{Z}$

(ii) $\sum_{i,j,l \in \mathbf{2}} V_{ijl}(k) \subset (\mathrm{Ker}D_1(k) \cap \mathrm{Ker}D_2(k))$ for all $k \in \boldsymbol{Z}$

(iii) $\bigcap_{i,j,l \in \mathbf{2}} \boldsymbol{H}(A_i(\cdot), B_j(\cdot), C_l(\cdot); V_{ijl}(\cdot)) \neq \phi$

where

$$\begin{aligned} &\boldsymbol{H}(A_i(\cdot), B_j(\cdot), C_l(\cdot); V_{ijl}(\cdot)) \\ &\quad := \left\{ H_{ijl}(\cdot) \mid (A_i(k) + B_j(k)H_{ijl}(k)C_l(k))V_{ijl}(k) \subset V_{ijk}(k+1) \text{ for all } k \in \boldsymbol{Z} \right\} \end{aligned}$$

(Proof) Suppose that RDRPISF is solvable. Then, there exists an ω-periodic incomplete-state feedback $H(\cdot)$ such that

$$\sum_{h=k-n\omega}^{k-1} \Phi^H(k, h+1)\mathrm{Im}M(h) \subset \mathrm{Ker}D(k)$$

for $\alpha, \beta, \gamma, \delta, \sigma \in [0,1]$. Thus, for $i, j, l, p, q \in \mathbf{2}$ we have

$$\sum_{h=k-n\omega}^{k-1} \Phi^H_{ijl}(k, h+1)\mathrm{Im}M_p(h) \subset \mathrm{Ker}D_q(k)$$

where

$$\begin{aligned} A^H_{ijl}(k) &:= A_i(k) + B_j(k)H(k)C_l(k) \\ \Phi^H_{ijl}(k, k_0) &:= A^H_{ijl}(k-1)A^H_{ijl}(k-2)\cdots A^H_{ijl}(k_0) \quad (k > k_0) \\ \Phi^H_{ijl}(k, k) &:= I_n. \end{aligned}$$

Then, the following inclusions hold.

$$V^p_{ijl}(k) := \sum_{h=k-n\omega}^{k-1} \Phi^H_{ijl}(k, h+1)\mathrm{Im}M_p(h) \subset (\mathrm{Ker}D_1(k) \cap \mathrm{Ker}D_2(k)) \tag{17}$$

for $i, j, l, p \in \mathbf{2}$. And, define

$$V_{ijl}(k) := V^1_{ijl}(k) + V^2_{ijl}(k)$$

for all $k \in \boldsymbol{Z}$. Then, since $V^1_{ijl}(\cdot)$ and $V^2_{ijl}(\cdot)$ are $(A_i(\cdot), B_j(\cdot), C_l(\cdot))$-invariant with incomplete-state feedback $H(\cdot)$ by (17) and Lemma 3.4, it follows that

$$(A_i(k) + B_j(k)H(k)C_l(k))V_{ijl}(k) \subset V_{ijl}(k+1)$$

for all $i, j, l \in \mathbf{2}$ and $k \in \boldsymbol{Z}$ which imply

$$H(\cdot) \in \bigcap_{i,j,l \in \mathbf{2}} \boldsymbol{H}(A_i(\cdot), B_j(\cdot), C_l(\cdot); V_{ijl}(\cdot))$$

Further, $V_{ijl}(\cdot)$ satisfy the following inclusions.

$$(\mathrm{Im}M_1(k-1)+\mathrm{Im}M_2(k-1)) \subset V_{ijl}(k) \subset (\mathrm{Ker}D_1(k)\cap \mathrm{Ker}D_2(k))$$

for all $i,j,l \in \mathbf{2}$ and $k \in \boldsymbol{Z}$. Therefore,

$$(\mathrm{Im}M_1(k-1)+\mathrm{Im}M_2(k-1)) \subset \bigcap_{i,j,l\in\mathbf{2}} V_{ijl}(k)$$

$$\sum_{i,j,l\in\mathbf{2}} V_{ijl}(k) \subset (\mathrm{Ker}D_1(k)\cap \mathrm{Ker}D_2(k))$$

$$\bigcap_{i,j,l\in\mathbf{2}} H(A_i(\cdot),B_j(\cdot),C_l(\cdot);V_{ijl}(\cdot)) \neq \phi$$

for all $k \in \boldsymbol{Z}$. This completes the proof of this theorem. □

4 An Illustrative Example

Consider the following two-periodic uncertain system:

$$S^{\omega}(\alpha): \begin{cases} A(k) = \begin{bmatrix} 1 & 1 & \cos^2\frac{\pi k}{2} \\ 1+\alpha(\sin^2\frac{\pi k}{2}-1) & \alpha+(1-\alpha)\cos^2\frac{\pi k}{2} & \sin^2\frac{\pi k}{2} \\ \alpha\cos^2\frac{\pi k}{2} & (1-\alpha)\sin^2\frac{\pi k}{2} & \sin^2\frac{\pi k}{2} \end{bmatrix} \\ B(k) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad M(k) = \begin{bmatrix} \cos^2\frac{\pi k}{2} \\ 1 \\ 1 \end{bmatrix} \\ C(k) = I_3 \quad D(k) = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \end{cases}$$

where α is an uncertainty in $[0,1]$.

By using Lemma 2.3, the supremal element $V^*(\cdot)$ of $\boldsymbol{V}_s(A_i(\cdot),B_j(\cdot);\Omega_{12}(\cdot)|i,j\in\mathbf{2})$ can be computed as

$$V^*(k) = \mathrm{span}\left\{ \begin{bmatrix} \sin^2\frac{\pi k}{2} \\ \cos^2\frac{\pi k}{2} \\ \cos^2\frac{\pi k}{2} \end{bmatrix} \right\}.$$

Since it can be easily checked that $V^*(\cdot)$ satisfies the conditions of Corollary 3.1, RDRPSF is solvable. Indeed, the following state feedback gain solves the problem.

$$F(k) = \begin{bmatrix} \cos\pi k & -1 & 0 \end{bmatrix}$$

Next, consider the system $S^{\omega}(\alpha)$ with $C(k) = \begin{bmatrix} -1 & \cos\pi k & 0 \end{bmatrix}$. Then, $V^*(\cdot)$ is also $\{(A_i(\cdot),B_j(\cdot),C_l(\cdot)) \mid i,j,l\in\mathbf{2}\}$-invariant with an incomplete-state feedback gain

$$H(k) = -\cos\pi k$$

and, since $V^*(\cdot)$ satisfies conditions of Theorem 3.1, RDRPISF is solvable.

5 Concluding Remarks

In this paper, the two robust disturbance-rejection problems with state feedback and with incomplete-state feedback for linear ω-periodic discrete-time systems were formulated, and then some necessary conditions and/or sufficient conditions for these problems to be solvable were obtained. The results are extensions of the results of Ghosh[1] and Otsuka and Inaba *et al.*[3] to the ω-periodic discrete-time systems, and of the results of Grasselli and Longhi[2] to the robust problems.

Appendix 1 (Proof of Lemma 3.2)

It follows from (9), (10), Cayley-Hamilton's theorem and ω-periodicity of $M(\cdot)$ that the following inclusions hold.

$$\begin{aligned}
&\Phi^H(k, k-(n+h)\omega - l)\mathrm{Im}M(k-(n+h)\omega - l - 1)\\
&\quad = \Phi^H(k, k-l)\left\{\Phi^H(k-l, k-\omega-l)\right\}^{n+h}\mathrm{Im}M(k-(n+h)\omega-l-1) \qquad \text{(by (9), (10))}\\
&\quad = \Phi^H(k, k-l)\left\{\sum_{i=0}^{n-1}\left(\Phi^H(k-l, k-\omega-l)\right)^i\right\}\mathrm{Im}M(k-(n+h)\omega-l-1)\\
&\qquad\qquad \text{(by Cayley-Hamilton's theorem)}\\
&\quad = \left\{\sum_{i=0}^{n-1}\Phi^H(k, k-i\omega-l)\right\}\mathrm{Im}M(k-(n+h)\omega-l-1) \qquad \text{(by (9), (10))}\\
&\quad \subset \sum_{i=0}^{n-1}\left\{\Phi^H(k, k-i\omega-l)\mathrm{Im}M(k-i\omega-l-1)\right\} \qquad \text{(by } \omega\text{-periodicity of } M(\cdot))
\end{aligned}$$

This completes the proof of this lemma. □

Appendix 2 (Proof of Lemma 3.3)

(i) and (ii) in the case that $k_0 = k - n\omega$ are obvious. So, we prove (ii) in the case that $k_0 < k - n\omega$. Then, it is obvious that the following inclusion holds.

$$\sum_{h=k_0}^{k-1}\Phi^H(k, h+1)\mathrm{Im}M(h) \supset \sum_{h=k-n\omega}^{k-1}\Phi^H(k, h+1)\mathrm{Im}M(h) \tag{18}$$

Since $k_0 < k - n\omega$, there exists $j \in \{0, 1, 2, \ldots\}$ such that

$$\left\{(k-n\omega) - j\omega\right\} - (k_0+1) \in \{0, 1, \ldots, \omega-1\}$$

Then, the following relations are obtained from Lemma 3.2.

$$
\begin{aligned}
&\sum_{h=k_0}^{k-n\omega-1} \Phi^H(k,h+1)\mathrm{Im}M(h) \\
&= \underbrace{\begin{array}{l}\Phi^H(k,k-n\omega)\mathrm{Im}M(k-n\omega-1)+\Phi^H(k,k-n\omega-1)\mathrm{Im}M(k-n\omega-2) \\ \quad +\cdots+\Phi^H(k,k-n\omega-(\omega-1))\mathrm{Im}M(k-n\omega-(\omega-1)-1)\end{array}}_{\omega \text{ times}} \\
&\quad + \underbrace{\begin{array}{l}\Phi^H(k,k-(n+1)\omega)\mathrm{Im}M(k-(n+1)\omega-1) \\ \quad +\cdots+\Phi^H(k,k-(n+1)\omega-(\omega-1))\mathrm{Im}M(k-(n+1)\omega-(\omega-1)-1)\end{array}}_{\omega \text{ times}} \\
&\quad \vdots \\
&\quad + \underbrace{\begin{array}{l}\Phi^H(k,k-(n+j-1)\omega)\mathrm{Im}M(k-(n+j-1)\omega-1) \\ \quad +\cdots+\Phi^H(k,k-(n+j-1)\omega-(\omega-1))\mathrm{Im}M(k-(n+j-1)\omega-(\omega-1)-1)\end{array}}_{\omega \text{ times}} \\
&\quad + \underbrace{\begin{array}{l}\Phi^H(k,k-(n+j)\omega)\mathrm{Im}M(k-(n+j)\omega-1) \\ \quad +\cdots+ \Big\{\Phi^H(k,k-(n+j)\omega-\big[\{(k-n\omega)-j\omega\}-(k_0+1)\big]) \\ \qquad\qquad \mathrm{Im}M(k-(n+j)\omega-\big[\{(k-n\omega)-j\omega\}-(k_0+1)\big]-1\Big\}\end{array}}_{[\{(k-n\omega)-j\omega\}-(k_0+1)]+1\le\omega \text{ times}} \\
&\subset \sum_{i=0}^{n-1}\Phi^H(k,k-i\omega)\mathrm{Im}M(k-i\omega-1)+\sum_{i=0}^{n-1}\Phi^H(k,k-i\omega-1)\mathrm{Im}M(k-i\omega-2) \\
&\qquad +\cdots+\sum_{i=0}^{n-1}\Phi^H(k,k-i\omega-(\omega-1))\mathrm{Im}M(k-i\omega-(\omega-1)-1) \qquad \text{(by Lemma 3.2)} \\
&= \sum_{h=k-n\omega}^{k-1}\Phi(k,h+1)\mathrm{Im}M(h) \qquad (19)
\end{aligned}
$$

Thus, the following inclusions follows from (19).

$$
\begin{aligned}
\sum_{h=k_0}^{k-1}\Phi(k,h+1)\mathrm{Im}M(h) &= \sum_{h=k-n\omega}^{k-1}\Phi(k,h+1)\mathrm{Im}M(h)+\sum_{h=k_0}^{k-n\omega-1}\Phi(k,h+1)\mathrm{Im}M(h) \\
&\subset \sum_{h=k-n\omega}^{k-1}\Phi(k,h+1)\mathrm{Im}M(h) \qquad (20)
\end{aligned}
$$

Hence, (ii) in the case that $k_0 < k - n\omega$ follows from (18) and (20).

This completes the proof. □

Appendix 3 (Proof of Lemma 3.4)

The following equalities hold.

$$
V(k+\omega) = \sum_{h=k+\omega-n\omega}^{k+\omega-1}\Phi^H(k+\omega,h+1)\mathrm{Im}M(h)
$$

$$
\begin{aligned}
&= \Phi^H(k+\omega, k+\omega-n\omega+1)\mathrm{Im}M(k+\omega-n\omega) \\
&\qquad +\Phi^H(k+\omega, k+\omega-n\omega+2)\mathrm{Im}M(k+\omega-n\omega+1) \\
&\qquad +\cdots+\Phi^H(k+\omega, k+\omega)\mathrm{Im}M(k+\omega-1) \\
&= \Phi^H(k, k-n\omega+1)\mathrm{Im}M(k-n\omega)+\Phi^H(k, k-n\omega+2)\mathrm{Im}M(k-n\omega+1) \\
&\qquad +\cdots+\Phi^H(k,k)\mathrm{Im}M(k-1) \qquad\qquad \text{(by (11) and } \omega\text{-periodicity of } M(\cdot)) \\
&= \sum_{h=k-n\omega}^{k-1} \Phi^H(k, h+1)\mathrm{Im}M(h) \\
&= V(k)
\end{aligned}
$$

for all $k \in \boldsymbol{Z}$. Thus, $V(\cdot)$ is ω-periodic.

Further, it follows from Lemma 3.2 that the following relations are obtained.

$$
\begin{aligned}
&A^H(k)V(k) \\
&= A^H(k)\sum_{h=k-n\omega}^{k-1} \Phi^H(k, h+1)\mathrm{Im}M(h) \\
&\subset \sum_{h=k-n\omega}^{k-1} \Phi^H(k+1, h+1)\mathrm{Im}M(h)+\Phi^H(k+1, k+1)\mathrm{Im}M(k) \\
&= \Phi^H(k+1, k-n\omega+1)\mathrm{Im}M(k-n\omega)+\sum_{h=k-n\omega+1}^{k} \Phi^H(k+1, h+1)\mathrm{Im}M(h) \\
&= \Phi^H(k+1, (k+1)-n\omega)\mathrm{Im}M((k+1)-n\omega-1)+\sum_{h=k+1-n\omega}^{k} \Phi^H(k+1, h+1)\mathrm{Im}M(h) \\
&\subset \sum_{i=0}^{n-1}\left\{\Phi^H(k+1, k+1-i\omega)\mathrm{Im}M(k-i\omega)\right\}+\sum_{h=(k+1)-n\omega}^{(k+1)-1} \Phi^H(k+1, h+1)\mathrm{Im}M(h) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 3.2)} \\
&= \sum_{h=(k+1)-n\omega}^{(k+1)-1} \Phi^H(k+1, h+1)\mathrm{Im}M(h) \\
&= V(k+1)
\end{aligned}
$$

for all $k \in \boldsymbol{Z}$. This completes the proof. □

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