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Abstract: A multiset which is also called a bag is a collection of elements in which a symbol can be repeated, and hence different from an ordinary set. Fuzzy bags have been proposed by Yager, but the relations and operations therein are incompatible with those for ordinary fuzzy sets. New multiset relations and basic operations including the α-cut are defined which are consistent with those for ordinary fuzzy sets, A number of theoretical properties are shown. Application to new rough approximations for fuzzy sets, multisets, and fuzzy multisets is considered. For example, fuzzy multisets are induced from a rough approximation of an ordinary fuzzy set.

1 Introduction

Crisp multisets [3] which are also called bags [8] have been generalized into fuzzy bags [12, 5, 6, 11]. They are defining a fuzzy bag as the crisp bag of the product space $\{(x_i, \mu_i)\}$ where μ_i is the membership for x_i. This definition has, however, an inconsistency, namely, (finite) crisp sets are regarded as a particular kind of bags with the basic set relations and operations which are essentially the same for the both, whereas an ordinary (finite) fuzzy set cannot be regarded as a special type of fuzzy bags with keeping the consisitency for the set relations and operations.

In this paper we consider new relations and operations for fuzzy multisets. Although the definition of fuzzy multisets itself is the same as Yager's definition, we distinguish fuzzy multisets herein and fuzzy bags by Yager, since the relations and operations are different between the both. Using the definitions in this paper, we easily see the consistency above mentioned: a mapping $\Gamma(\cdot)$ that imbeds an ordinary fuzzy set into a fuzzy multiset is used for this purpose. A notable property of the fuzzy multiset is that an α-cut is naturally defined which corresponds a fuzzy multiset to a one-parameter family of crisp multisets. Using the α-cut, it is proved that the collection of fuzzy multisets in a finite universal set forms a distributive lattice [7].

As an application, we consider a rough approximation of an ordinary fuzzy set which is different from rough fuzzy set by Dubois and Prade [2].

2 Multisets and fuzzy multisets

Let us briefly review the concept of crisp multisets [3, 8]. We hereafter prefer the term of multisets to bags. Assume that $X = \{x_1, x_2, ..., x_n\}$ is a finite set and all multisets are considered on this universal set. Let $\mathbf{N} = \{0, 1, 2, ...\}$ be the set of natural numbers. A multiset, say E, is characterized by a function $Count_E: X \to \mathbf{N}$ which means that the number of copies of x_i is given by $Count_E(x_i)$. We sometimes write $E = \{Count_E(x)/x : x \in X\}$.

Example 1. Assume $X = \{a, b, c, d\}$ and E is characterized by $Count_E(a) = 1$, $Count_E(b) = 2$, and $Count_E(c) = Count_E(d) = 0$. which means that there are one copy of a and two copies of b in E, but no copies of c and d. We may write $E = \{a, b, b\}$ or $E = \{1/a, 2/b\}$, ignoring elements of zero copies.

For two multisets E and F, the inclusion is defined by

$$E \subseteq F \iff Count_E(x) \leq Count_F(x), \quad \forall x \in X.$$

The union and intersection are defined by

$$\begin{aligned} Count_{E \cup F}(x) &= \max[Count_E(x), Count_F(x)], \quad \forall x \in X, \\ Count_{E \cap F}(x) &= \min[Count_E(x), Count_F(x)], \quad \forall x \in X. \end{aligned}$$

Readers may notice that there are resemblances between multiset operations and those for fuzzy sets.

An ordinary (finite) crisp set can be regarded as a particular type of multisets for which the function $Count$ takes values of zero or unity. The above definitions are consistent with the standard inclusions and union/intersection operations when crisp sets are regarded as particular multisets.

Although Yager [12] has proposed fuzzy bags and their operations, a part of the proposed relations and operations are inconsistent with those for ordinary fuzzy sets, when the class of ordinary fuzzy sets are regarded as a particular subclass of all fuzzy bags. (For more details concerning his definitions, see the appendix.)

This means that we should newly consider basic relations and operations for fuzzy multisets. In the following basic defintions and propositions concerning their theoretical properties are listed. The propositions need to be proved. For the ease of reference the proofs are given in the appendix.

Let $I = [0, 1]$ be the unit interval. As introduced by Yager [12], a fuzzy multiset of X is characterized by a crisp multiset of $X \times I$. Namely, for a fuzzy multiset $A = \{(x_i, \mu_i)\}_{i=1,\ldots,p}$ which means that x_i has the degree of membership μ_i, (x_i, μ_i) and (x_j, μ_j) $(i \neq j)$ can have the same symbols: $x_i = x_j$ and/or $\mu_i = \mu_j$.

We can assemble, for each x_k $(1 \leq k \leq n)$, the collection of the corresponding membership μ''s: (μ'_{k1}, x_k),...,$(\mu'_{k\ell_k}, x_k)$, which means that there are copies of x_k with possibly different degrees of relevance represented by $\mu'_{k1}, \ldots, \mu'_{k\ell_k}$. Thus, the same set can be expressed as

$$A = \{\,\{\mu'_{11}, \ldots, \mu'_{1\ell_1}\}/x_1, \ldots, \{\mu'_{n1}, \ldots, \mu'_{n\ell_n}\}/x_n\,\}. \tag{1}$$

Notice that $\{\mu'_{k1}, \ldots, \mu'_{k\ell_k}\}$ is a crisp multiset of I. The latter representation is more convenient for the definitions below.

Example 2. Let $X = \{a, b, c, d\}$ and

$$\begin{aligned} A &= \{(a, 0.2), (b, 0.5), (b, 0.1), (a, 0.2), (a, 0.3), (d, 0.7)\} \\ &= \{\{0.2, 0.2, 0.3\}/a, \{0.5, 0.1\}/b, \{0.7\}/d\}. \end{aligned}$$

Thus, A has three copies of a with the degrees 0.2, 0.2, and 0.3; it has two copies of b with the degrees 0.5 and 0.1, and so on.

An operation for crisp multisets that is inapplicable to ordinary sets is the direct sum $\oplus$. Let E and F be two crisp multisets. Then

$$Count_{E\oplus F}(x) = Count_E(x) + Count_F(x), \quad \forall x \in X.$$

The sum of two fuzzy multisets of X is defined by the direct sum of the corresponding crisp multisets of $X \times I$. Namely, let $A = \{(x_i, \mu_i)\}_{i=1,...,p}$ and $B = \{(x'_j, \mu'_j)\}_{j=1,...,q}$. Then

$$A \oplus B = \{(x_1, \mu_1), ..., (x_p, \mu_p), (x'_1, \mu'_1), ..., (x'_q, \mu'_q)\}.$$

This definition is due to Yager [12].

Since an ordinary fuzzy set $A = \sum_i \mu_i/x_i$ can be regarded as a particular fuzzy multiset $\{(x_i, \mu_i)\}_{i=1,..,n}$ for which $x_i \neq x_j$, $i \neq j$. To distinguish these two representations, we define a mapping $\Gamma(\cdot)$ which is defined on the set of all fuzzy set of X into the set of fuzzy multisets of X: For $A = \sum_i \mu_i/x_i$,

$$\Gamma(A) = \{(x_i, \mu_i)\}_{i=1,..,n}.$$

The basic relations and operations for fuzzy multisets should be consistent with those for ordinary fuzzy sets. Namely, for arbitrary fuzzy sets A and B, the inclusion, union, and intersection should de defined so that they satisfy

$$A \subseteq B \iff \Gamma(A) \subseteq \Gamma(B), \quad (2)$$
$$\Gamma(A \cup B) = \Gamma(A) \cup \Gamma(B), \quad (3)$$
$$\Gamma(A \cap B) = \Gamma(A) \cap \Gamma(B). \quad (4)$$

To obtain the relation and the operations which satisfy (2), (3), and (4), third representation for the fuzzy multiset using a *grade sequence* should be introduced. Namely, the set $\{\mu'_{k1}, ..., \mu'_{k\ell_k}\}$ corresponding to x_k is rearranged into a sequence $(\mu''_{k1}, ..., \mu''_{k\ell_k})$ of decreasing order: $\mu''_{k1} \geq \mu''_{k2} \geq ... \geq \mu''_{k\ell_k}$. This sequence is called the grade sequence for x_k. Thus, a fuzzy multiset is represented by the third form:

$$A = \{\, (\mu''_{11}, ..., \mu''_{1\ell_1})/x_1, ..., (\mu''_{n1}, ..., \mu''_{n\ell_n})/x_n \,\}. \quad (5)$$

Notice also that when we consider a finite number of fuzzy multisets, the length of the grade sequences ℓ_k can be taken to be a constant, say $\ell_k = p$ for all x_k and for all the fuzzy multisets under consideration, by appending zero grades: $(\mu''_{k1}, ..., \mu''_{k\ell_k}, 0, ..., 0)$.

Example 3. Let

$$\begin{aligned} A &= \{(a, 0.2), (b, 0.5), (b, 0.1), (a, 0.2), (a, 0.3), (d, 0.7)\} \\ &= \{\{0.2, 0.2, 0.3\}/a, \{0.5, 0.1\}/b, \{0.7\}/d\}, \\ B &= \{(c, 0.4), (b, 0.7), (b, 0.1), (a, 0.2)\} \\ &= \{\{0.2\}/a, \{0.7, 0.1\}/b, \{0.4\}/c\}. \end{aligned}$$

Then, the grade sequence forms are given by $p = 3$ and

$$\begin{aligned} A &= \{(0.3, 0.2, 0.2)/a, (0.5, 0.1, 0)/b, (0, 0, 0)/c, (0.7, 0, 0)/d\}, \\ B &= \{(0.2, 0, 0)/a, (0.7, 0.1, 0)/b, (0.4, 0, 0)/c, (0, 0, 0)/d\}. \end{aligned}$$

We hereafter take this form of the grade sequence as the standard representation for the fuzzy multiset and the element μ''_{kj} in $(\mu''_{k1}, ..., \mu''_{kp})/x_k$ for A is also referred to as $\mu^j_A(x_k)$: $(\mu^j_A(x_k) =_{def} \mu''_{kj})$ for the ease of reference.

Now, we can define basic relations and operations for fuzzy multisets using the grade sequence form. Namely, for two fuzzy multisets A and B of X,

(I) [inclusion]

$$A \subseteq B \iff \mu^j_A(x) \le \mu^j_B(x), \quad j = 1, 2, ..., p, \quad \forall x \in X.$$

(II) [equality]

$$A = B \iff \mu^j_A(x) = \mu^j_B(x), \quad j = 1, 2, ..., p, \quad \forall x \in X.$$

(III) [union]

$$\mu^j_{A\cup B}(x) = \max[\mu^j_A(x), \mu^j_B(x)], \quad j = 1, 2, ..., p, \quad \forall x \in X.$$

(IV) [intersection]

$$\mu^j_{A\cap B}(x) = \min[\mu^j_A(x), \mu^j_B(x)], \quad j = 1, 2, ..., p, \quad \forall x \in X.$$

Next, we define α-cut for fuzzy multisets.

(V) [α-cut] For arbitrarily given $\alpha \in (0, 1]$, the number of copies of x in A_α, the α - cut of A is defined by

$$\begin{array}{lll} Count_{A_\alpha}(x) = 0 & \iff & \mu^1_A(x) < \alpha \\ Count_{A_\alpha}(x) = k & \iff & \mu^k_A(x) \ge \alpha \text{ and } \mu^k_A(x) < \alpha, \quad (k < p) \\ Count_{A_\alpha}(x) = p & \iff & \mu^p_A(x) \ge \alpha. \end{array}$$

Example 4. For A and B in Example 3,

$$\begin{aligned} A \cup B &= \{(0.3, 0.2, 0.2)/a, (0.7, 0.1, 0)/b, (0.4, 0, 0)/c, (0.7, 0, 0)/d\}, \\ A \cap B &= \{(0.2, 0, 0)/a, (0.5, 0.1, 0)/b, (0, 0, 0)/c, (0, 0, 0)/d\}, \\ A_{0.2} &= \{3/a, 1/b, 1/c, 1/d\}. \end{aligned}$$

Then we have

Proposition 1. The grade sequences in (III) and (IV) are well-defined, i.e.,

$$\begin{aligned} \mu^1_{A\cup B}(x) \ge \mu^2_{A\cup B}(x) \ge ... \ge \mu^p_{A\cup B}(x), \\ \mu^1_{A\cap B}(x) \ge \mu^2_{A\cap B}(x) \ge ... \ge \mu^p_{A\cap B}(x), \end{aligned}$$

for all $x \in X$.

Proposition 2. The relations (i – ii) and the operations (iii – v) are consistent with those for ordinary fuzzy sets. Namely, an ordinary fuzzy set can be regarded as a fuzzy multiset by putting $p = 1$ in defining the grade sequence. Then, the relations (2), (3), and (4) are satisfied. Moreover, for arbitrary fuzzy set A and $\alpha \in (0, 1]$,

$$\Gamma(A_\alpha) = [\Gamma(A)]_\alpha. \tag{6}$$

Proposition 3. For arbitrary fuzzy multisets A and B,

$$A \subseteq B \iff A_\alpha \subseteq B_\alpha, \quad \forall \alpha \in (0,1], \tag{7}$$
$$A = B \iff A_\alpha = B_\alpha, \quad \forall \alpha \in (0,1], \tag{8}$$
$$(A \cup B)_\alpha = A_\alpha \cup B_\alpha, \quad \forall \alpha \in (0,1], \tag{9}$$
$$(A \cap B)_\alpha = A_\alpha \cap B_\alpha, \quad \forall \alpha \in (0,1]. \tag{10}$$

Proposition 4. The union and intersection for arbitrary fuzzy multisets A, B, and C satisfy the following laws:
(a) [the commutative law]

$$A \cup B = B \cup A, \qquad A \cap B = B \cap A.$$

(b) [the associative law]

$$A \cup (B \cup C) = (A \cup B) \cup C, \qquad A \cap (B \cap C) = (A \cap B) \cap C.$$

(c) [the distributive law]

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Thus, the class of all fuzzy multisets of X forms a distributive lattice [7].

The intersection and union are frequently generalized by using t-norms and t-conorms. Let t and s be arbitrary t-norm and t-conorm for ordinary fuzzy sets. Furthermore, $\cap_t$ and $\cup_s$ are defined by

$$\mu_{A\cap_t B}(x) = t(\mu_A(x), \mu_B(x)),$$
$$\mu_{A\cup_s B}(x) = s(\mu_A(x), \mu_B(x)),$$

for two ordinary fuzzy sets A and B.

Now, it is straightforward to generalize these to fuzzy multisets. Namely, if A and B are two fuzzy multisets, we can define $\cap_t$ and $\cup_s$ by

$$\mu^j_{A\cap_t B}(x) = t(\mu^j_A(x), \mu^j_B(x)), \quad j = 1, 2, ..., p$$
$$\mu^j_{A\cup_s B}(x) = s(\mu^j_A(x), \mu^j_B(x)), \quad j = 1, 2, ..., p.$$

Proposition 5. The operations $\cup_t$ and $\cup_s$ are well-defined, namely,

$$\mu^1_{A\cap_t B}(x) \geq \mu^2_{A\cap_t B}(x) \geq ... \geq \mu^p_{A\cap_t B}(x),$$
$$\mu^1_{A\cup_s B}(x) \geq \mu^2_{A\cup_s B}(x) \geq ... \geq \mu^p_{A\cup_s B}(x),$$

for all $x \in X$.

Proposition 6. Let A and B be two ordinary fuzzy sets. Then,

$$\Gamma(A \cap_t B) = \Gamma(A) \cap_t \Gamma(B),$$
$$\Gamma(A \cup_s B) = \Gamma(A) \cup_s \Gamma(B).$$

3 Application to rough approximations

Let $X = \{x_1, x_2, ..., x_n\}$ be a finite universal set and R be an equivalence relation. Rough sets introduced by Pawlak [10] concern the lower and upper approximations for a given subset. Given a fuzzy set $A = \{\mu_1/x_1, ..., \mu_n/x_n\}$, its lower and upper approximations have been considered by Dubois and Prade [2]. We consider here another type of an approximation using the fuzzy multisets. Let the equivalence classes induced from R be $X/R = \{Y_1, ..., Y_m\}$. Assume that

$$A \cap Y_k = \{\mu_{ki}/x_{ki}, ..., \mu_{k\ell}/x_{k\ell}\}.$$

Now, a fuzzy multiset approximation for A is defined by

$$\hat{A} = \{\{\mu_{ki}, ..., \mu_{k\ell}\}/Y_k\}_{1 \le k \le m}.$$

Namely, the information on the elements are lost but the membership values are kept. It is clear that the rough fuzzy sets [2] use the minimum and maximum values of $\{\mu_{ki}, ..., \mu_{k\ell}\}$ for each Y_k.

According to the definitions in the previous section, we can consider rough relations and operations.

Proposition 7. Given two fuzzy sets A and B of X, the followings hold.

(i) $A \subseteq B \Rightarrow \hat{A} \subseteq \hat{B}$.

(ii) $A = B \Rightarrow \hat{A} = \hat{B}$.

(iii) $|A \cap B| \le |\hat{A} \cap \hat{B}|, \quad |A \cup B| \ge |\hat{A} \cup \hat{B}|$.

where $|\cdot|$ is the cardinality for fuzzy sets and fuzzy multisets. For a fuzzy multiset A, $|A|$ is defined by the sum of all of its membership values for all $x \in X$: $|A| = \sum_{x \in X} \sum_{j=1}^{p} \mu_A^j(x)$.

It is straightforward to generalize the above definitions to the cases of approximations of multisets and fuzzy multisets of X. Namely, let $E = \{\nu_1/x_1, ..., \nu_n/x_n\}$ be a crisp multiset and $E \cap Y_k = \{\nu_{ki}/x_{ki}, ..., \nu_{k\ell}/x_{k\ell}\}$. We define $\hat{E} = \{\{\nu_{ki}, ..., \nu_{k\ell}\}/Y_k\}_{1 \le k \le m}$. Given two multisets E and F, (i') $E \subseteq F \Rightarrow \hat{E} \subseteq \hat{F}$, (ii') $E = F \Rightarrow \hat{E} = \hat{F}$, and (iii') $|E \cap F| \le |\hat{E} \cap \hat{F}|$, $|E \cup F| \ge |\hat{E} \cup \hat{F}|$. similar to (i)–(iii) in Proposition 7 hold. Moreover for a fuzzy multiset $C = \{\mu_1/x'_1, ..., \mu_q/x'_q\}$ in which x'_i and x'_j may be identical, the approximation $\hat{C}$ is defined in exactly the same way. The properties analogous to those in Proposition 7 are obtained. We omit the detail.

4 Conclusions

We have developed new relations and operations of fuzzy multisets. Thus, a fuzzy multiset is regarded as the collection of its α-cuts as crisp multisets. Application to a new rough approximation of fuzzy sets given an equivalence relation has been considered.

Although consideration for a fuzzy multirelation are omited herein, its definition and consideration for the max–min composition are straightforward.

Future studies include advanced operations for fuzzy multisets, and further development in the above defined rough approximations. Applications to other areas such as fuzzy databases and information systems are moreover possible.

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Appendix

A. Bag relations and operations proposed by Yager

Yager [12] has introduced a notation $C.M_A(x)$ for a fuzzy bag A:

$$C.M_A(x_k) = \{\mu'_{k1}, ..., \mu'_{k\ell_k}\}$$

corresponding to (1). Thus,

$$A = \{C.M_A(x_1)/x_1, ..., C.M_A(x_n)/x_n\}.$$

Notice that $C.M_A(x)$ is a crisp multiset of the unit interval. Then he defined the inclusion, the union, and the intersection by

$$A \subseteq B \iff C.M_A(x) \subseteq C.M_B(x), \quad \forall x \in X,$$

$$\begin{aligned} C.M_{A\cup B}(x) &= C.M_A(x) \cup C.M_B(x), \quad \forall x \in X, \\ C.M_{A\cap B}(x) &= C.M_A(x) \cap C.M_B(x), \quad \forall x \in X. \end{aligned}$$

These defitions do not satisfy (2), (3), and (4), respectively. To see this, Let $A = \{0.5/a\}$ and $B = \{1.0/a\}$ be two fuzzy sets. Then, from the ordinary fuzzy set relation and operations,

$$A \subseteq B, \qquad A \cup B = B, \qquad A \cap B = A.$$

In contrast, from $\Gamma(A) = \{\{0.5\}/a\}$ and $\Gamma(A) = \{\{1.0\}/a\}$, which means $\Gamma(A) \subseteq \Gamma(B)$ does not hold. Moreover,

$$\Gamma(A) \cup \Gamma(B) = \{\{0.5, 1.0\}/a\}, \qquad \Gamma(A) \cap \Gamma(B) = \emptyset.$$

B. Proofs of the propositions

Proof of Proposition 1. Let t and s be arbitrary t - norm and t - conorm for ordinary fuzzy sets, respectively. Assume that $a, b, c,$ and d are arbitrary real numbers in the unit interval such that $a \geq c \geq 0$ and $b \geq d \geq 0$. Then, from the definitions of the t - norm and conorm ([4], p.23),

$$\begin{aligned} t(a,b) &\geq t(c,d), && (11) \\ s(a,b) &\geq s(c,d). && (12) \end{aligned}$$

It is well-known that the min and max operations are a t - norm and a t - conorm, respectively. Taking $a = \mu_A^k(x)$, $b = \mu_B^k(x)$, $c = \mu_A^{k+1}(x)$, and $d = \mu_B^{k+1}(x)$, we have the desired conclusion.

Proof of Proposition 2. The consistency expressed by (2), (3), and (4) is obvious, since for an ordinary fuzzy set A, its grade sequence consists of one member $\mu_A^1(x) = \mu_A(x)$ (the right hand side is the ordinary membership) and $p = 1$. Then, it is easy to see that the multiset inclusion, equality, union, and intersection coincide with those for the respective relations and operations for ordinary fuzzy sets.

The definition of the α -cut for fuzzy multisets in the case of $p = 1$ reduces to

$$Count_A(x) = \begin{cases} 1 & (\mu_A^1(x) \geq \alpha), \\ 0 & (\mu_A^1(x) < \alpha), \end{cases}$$

which coincides with the ordinary α-cut for fuzzy sets, since $\mu_A^1(x) = \mu_A(x)$, $\forall x \in X$. Thus, the consistency for the α-cut (6) also holds.

Proof of Proposition 3. For proving (7), we define a sequence of fuzzy sets $A^1, A^2, ..., A^p$ for a given fuzzy multiset A by

$$\mu_{A^k}(x) = \mu_A^k(x), \quad k = 1, 2, ..., p, \quad \forall x \in X.$$

Then it is easy to see that, for two fuzzy multisets A and B,

$$A \subseteq B \iff A^k \subseteq B^k, \quad k = 1, 2, ..., p. \tag{13}$$

Let $k = Count_{A_\alpha}(x)$. Then, from the definition of $Count_{A_\alpha}$,

$$\begin{aligned} \mu_{A^k}(x) &= \mu_A^k(x) \geq \alpha, \\ \mu_{A^{k+1}}(x) &= \mu_A^{k+1}(x) < \alpha \end{aligned}$$

In other words,

$$\begin{aligned} &x \in (A^k)_\alpha, && j = 1, .., k, \\ &x \notin (A^k)_\alpha, && j = k+1, .., p \end{aligned}$$

where $(A^k)_\alpha$ and $(B^k)_\alpha$ are ordinary α-cuts for A^k and B^k, respectively. Thus, we have

$$Count_{A_\alpha}(x) = \max\{k : x \in (A^k)_\alpha\}$$

This means that the following lemma holds.

Lemma. For every $\alpha \in (0, 1]$,

$$A_\alpha \subseteq B_\alpha \iff (A^k)_\alpha \subseteq (B^k)_\alpha, \quad k = 1, ..., p. \tag{14}$$

(Proof of the lemma) The above discussion implies that

$$\begin{aligned} A_\alpha \subseteq B_\alpha &\iff count_{A_\alpha}(x) \leq count_{B_\alpha}(x), \ \forall x \in X \\ &\iff \max\{k : x \in (A^k)_\alpha\} \leq \max\{k : x \in (B^k)_\alpha\}, \quad \forall x \in X \\ &\iff if\ x \in (A^k)_\alpha\ then\ x \in (B^k)_\alpha, \quad \forall x \in X, \quad k = 1, ..., p \\ &\iff (A^k)_\alpha \subseteq (B^k)_\alpha, \quad k = 1, ..., p. \end{aligned}$$

(End of the proof of the lemma)

Furthermore, it is well-known that

$$A^k \subseteq B^k \iff (A^k)_\alpha \subseteq (B^k)_\alpha, \quad \forall \alpha \in (0, 1], \quad k = 1, .., p. \tag{15}$$

From (13), (14), and (15), it is now obvious that (7) holds.

It is now easily seen that (8) is valid, since

$$\begin{aligned} A = B &\iff A \subseteq B, \quad B \subseteq A \\ &\iff A_\alpha \subseteq B_\alpha, \quad B_\alpha \subseteq A_\alpha, \quad \forall \alpha \in (0, 1] \\ &\iff A_\alpha = B_\alpha \quad \forall \alpha \in (0, 1]. \end{aligned}$$

The sequnces $\{A^k\}$ and $\{B^k\}$ are also used for proving (9) and (10). Notice that these equations hold for ordinary fuzzy sets [1, 9].

Since A^k and B^k ($k = 1, 2, ..., p$) are ordinary fuzzy sets, $(A^k \cup B^k)_\alpha = (A^k)_\alpha \cup (B^k)_\alpha$ are valid. Moreover,

$$\mu_{A \cup B}^k(x) = \max[\mu_A^k(x), \mu_B^k(x)] = \max[\mu_{A^k}(x), \mu_{B^k}(x)] = \mu_{A^k \cup B^k}(x).$$

Thus, for all $x \in X$,

$$\begin{aligned} count_{(A\cup B)_\alpha}(x) &= \max\{k : \mu^k_{A\cup B}(x) \geq \alpha\} \\ &= \max\{k : \mu_{A^k\cup B^k}(x) \geq \alpha\} \\ &= \max\{k : x \in (A^k)_\alpha \ or \ x \in (B^k)_\alpha\} \\ &= \max[\max\{k : x \in (A^k)_\alpha\}, \max\{j : x \in (B^j)_\alpha\}] \\ &= \max[count_{A_\alpha}(x), count_{B_\alpha}(x)] \\ &= count_{A_\alpha \cup B_\alpha}(x). \end{aligned}$$

Hence we have $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$. The equation (10) is proved in the same way.

Proof of Proposition 4. Let us prove one equation for the distributive law. Notice that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ is equivalent to

$$[(A \cup B) \cap C]_\alpha = [(A \cap C) \cup (B \cap C)]_\alpha$$

for all $\alpha \in (0, 1]$. Noting that the distributive law holds for crisp multisets [12] and using (9), (10), we have

$$\begin{aligned} [(A \cup B) \cap C]_\alpha &= (A \cup B)_\alpha \cap C_\alpha \\ &= (A_\alpha \cup B_\alpha) \cap C_\alpha \\ &= (A_\alpha \cap C_\alpha) \cup (B_\alpha \cap C_\alpha) \\ &= (A \cap C)_\alpha \cup (B \cap C)_\alpha \\ &= [(A \cap C) \cup (B \cap C)]_\alpha. \end{aligned}$$

The other laws are proved in the same way.

Proof of Proposition 5. Notice the property described in the proof of Proposition 1 for arbitrary t-norm and t-conorm. By putting $a = \mu^k_A(x)$, $b = \mu^k_B(x)$, $c = \mu^{k+1}_A(x)$, and $d = \mu^{k+1}_B(x)$, we see that this proposition can be proved in the same way as Proposition 1.

Proof of Proposition 6. To see that the conclusion holds, it is sufficient to note that the remark given in the beginning of the proof of Proposition 2 can be applied in this case. We omit the detail.

Proof of Proposition 7. The properties (i) and (ii) are obvious. For the first inequality in (iii), it is sufficient to note that for four nonnegative numbers a_1, a_2, b_1, b_2 such that $a_1 \geq a_2$ and $b_1 \geq b_2$,

$$\min[a_1, b_1] + \min[a_2, b_2] \geq \min[a_1, b_2] + \min[a_2, b_1]$$

holds. This inequality is proved by a straightforward calculation. Now, given two grade sequences $\mu^j_A(x)$ and $\mu^j_B(x)$ and for an arbitrary permutation $(k_1, ..., k_p)$ of $(1, 2, ..., p)$, it is not difficult to see that

$$\sum_{j=1}^{p} \min[\mu^j_A(x), \mu^j_B(x)] \geq \sum_{j=1}^{p} \min[\mu^j_A(x), \mu^{k_j}_B(x)]$$

using the previous inequality. $|A \cap B|$ is attained at some permutation, say $(k'_1, ..., k'_p)$. Thus, the first inequality is valid. The second inequality in (iii) is proved likewise.