An Automatic Design Procedure of IIR Digital Filters from an Analog Low-pass Filter

Koichi ICHIGE†, Naohisa OTSUKA††
and Rokuya ISHII†††

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† Master's Program in Science and Engineering, University of Tsukuba, Tsukuba, Ibaraki 305, Japan.
E-mail : ichi@wslab.is.tsukuba.ac.jp

†† Institute of Information Sciences and Electronics, University of Tsukuba, Tsukuba, Ibaraki 305, Japan.
E-mail : otsuka@wslab.is.tsukuba.ac.jp

††† Division of Electronics and Computer Engineering, Yokohama National University, Yokohama 240, Japan
E-mail : ishii@ishiilab.dnj.ynu.ac.jp

Abstract

IIR digital filters are generally designed from an analog low-pass filter by any combination of the bilinear transformation, frequency transformation for analog filters and that for IIR digital filters. But the direct application of these transformations yields a complicated formula of the target transfer function, which has to be reduced by hand computation into the form of a rational polynomial. This hand computation will be replaced by an automatic procedure if the relation between the coefficients of the transfer functions is formulated. This paper aims at deriving the explicit formulae which connect those coefficients by matrices. By using the proposed matrices, all the process to design IIR digital filters can be computed automatically.

key words : frequency transformation, bilinear transformation, analog filters, IIR digital filters.

1 Introduction

Frequency transformation derives filters of various types from a filter of low-pass type. The transformation formulae have been established for analog filters in the s-domain [1], and for IIR digital filters in the z-domain [2]. Bilinear transformation [3] derives IIR digital filters from analog filters, and also enables the inverse derivation. The transformation formulae is well-known and widely used. But the direct application of these transformations yields a complicated formula of the target transfer function, which has to be reduced by hand computation into the form of a rational polynomial. The yielded formula becomes more complicated for a transfer function with a higher order that can be used in practice to realize minute characteristics.

To dissolve this complexity, a transfer matrix has been formulated [4] which transforms coefficients of a polynomial in the s-domain to those in the z-domain. This matrix can replace the hand computation of the bilinear transformation for polynomials by an automatic procedure. Similarly, the hand computations of those transformations will be replaced by an automatic procedure if the relation between the coefficients of the transfer functions is formulated. This paper aims at deriving the explicit formulae which connect those coefficients by matrices. Sections 2, 3 and 4 describes formulae of the transfer matrices which represent the frequency transformation for analog filters, those for IIR digital filters, and the bilinear transformation for rational polynomials, respectively.

Generally, IIR digital filters are designed from an original analog low-pass filter by any combination of the bilinear transformation, frequency transformation for analog filters and that for IIR digital filters. The proposed matrices enable an automatic design of IIR digital filters from an analog low-pass filter. Section 5 investigates conditions to design IIR digital filters directly from an analog low-pass filter.

Section 6 shows some design examples of digital filters which are designed directly from an analog low-pass filter. The effectiveness of the proposed matrices can be evaluated how the process to design IIR digital filters is simplified by the proposed automatic procedure. Finally, Section 7 will make some concluding remarks.

2 Matrices Representation of the Frequency Transformation for Analog Filters

The explicit formulae of matrices are derived which represent the frequency transformation for analog filters. In the following discussion, frequency means angular frequency, and $\begin{pmatrix} a \\ b \end{pmatrix}$ denotes

$$\begin{pmatrix} a \\ b \end{pmatrix} := \begin{cases} 0, & b > a \text{ or } b < 0 \text{ or } \\ & b \text{ is not an integer,} \\ \dfrac{a!}{b!(a-b)!}, & \text{otherwise.} \end{cases}$$

Let the cut-off frequency of the original analog low-pass filter be $\Omega_0 = 1$, which is fixed throughout this investigation. The transfer function of this filter can be written as

$$H_1(s) := \frac{a_0 + a_1 s + \cdots + a_m s^m}{b_0 + b_1 s + \cdots + b_n s^n}, \qquad (m \leq n).$$

The transformation formulae [1] of the frequency transformation for analog filters are shown in Table 1. The transfer functions of the derived analog low-pass and high-pass filters can be written as

$$H_2(s) = \frac{c_0 + c_1 s + \cdots + c_n s^n}{d_0 + d_1 s + \cdots + d_n s^n}.$$

Those of the derived analog band-pass and band-stop filters can be written as

$$\tilde{H}_2(s) = \frac{c_0 + c_1 s + \cdots + c_n s^n + \cdots + c_{2n} s^{2n}}{d_0 + d_1 s + \cdots + d_n s^n + \cdots + d_{2n} s^{2n}}.$$

Then, matrices which transform the transfer function of the original low-pass filter to transfer functions of the derived filters are formulated as follows.

(a) Design of analog low-pass filter

The analog low-pass filter with the cut-off frequency Ω_1 is designed from the original analog low-pass filter. From the transformation formula in Table 1, the relation of coefficients from $\{a_i\}_{i=0}^m$ and $\{b_i\}_{i=0}^n$ into $\{c_i\}_{i=0}^n$ and $\{d_i\}_{i=0}^n$ can be easily written as

$$c_i = \begin{cases} \Omega_1^{-i} a_i, & i = 0, 1, \ldots, m, \\ 0, & i = m+1, m+2, \ldots, n, \end{cases}$$
$$d_i = \Omega_1^{-i} b_i.$$

Table 1: Transformation formulae of the frequency transformation of analog filters (from the original analog low-pass filter with the cut-off frequency $\Omega_0 = 1$)

Filter Type (Cut-off Frequency)	Transformation Formulae	Parameters
Low-pass (Ω_1)	$s = \dfrac{s}{\Omega_1}$	————
High-pass (Ω_1)	$s = \dfrac{\Omega_1}{s}$	————
Band-pass ($\Omega_1, \Omega_2(\Omega_1 < \Omega_2)$)	$s = \dfrac{s^2 + \Omega_a^2}{\Omega_b s}$	$\Omega_a = \sqrt{\Omega_1 \Omega_2}$, $\Omega_b = \Omega_2 - \Omega_1$
Band-stop ($\Omega_1, \Omega_2(\Omega_1 < \Omega_2)$)	$s = \dfrac{\Omega_b s}{s^2 + \Omega_a^2}$	$\Omega_a = \sqrt{\Omega_1 \Omega_2}$, $\Omega_b = \Omega_2 - \Omega_1$

On the other hand, we define the following two matrices

$$\boldsymbol{X}_{nm}^{LP} := [x_{ij}^{LP}] \in \boldsymbol{R}^{(n+1)\times(m+1)},$$

where

$$x_{ij}^{LP} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

and

$$\boldsymbol{S}_n^{LP} := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \Omega_1^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \Omega_1^{-n} \end{bmatrix} \in \boldsymbol{R}^{(n+1)\times(n+1)}.$$

Then, we can write the relation of coefficients as

$$\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = \boldsymbol{X}_{nm}^{LP} \boldsymbol{S}_m^{LP} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_m \end{bmatrix},$$

$$\begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_n \end{bmatrix} = \boldsymbol{S}_n^{LP} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

(b) Analog high-pass filter

The analog high-pass filter with the cut-off frequency Ω_1 is designed from the original analog low-pass filter. From the transformation formula in Table 1, the relation of coefficients from $\{a_i\}_{i=0}^m$ and $\{b_i\}_{i=0}^n$ into $\{c_i\}_{i=0}^n$ and $\{d_i\}_{i=0}^n$ can be easily written as

$$\begin{aligned} c_i &= \begin{cases} 0, & i = 0, 1, \ldots, n-m-1, \\ \Omega_1^{n-i} a_{n-i}, & i = n-m, n-m+1, \ldots, n, \end{cases} \\ d_i &= \Omega_1^{n-i} b_{n-i}. \end{aligned}$$

Then, the matrices which transform those coefficients can be easily obtained in the same manner as (a), *i.e.*,

$$\boldsymbol{X}_{nm}^{HP} := [x_{ij}^{HP}] \in \boldsymbol{R}^{(n+1)\times(m+1)},$$

where

$$x_{ij}^{HP} = \begin{cases} 1, & i = j+n-m, \\ 0, & i \neq j+n-m, \end{cases}$$

and

$$\boldsymbol{S}_n^{HP} := \begin{bmatrix} 0 & \cdots & 0 & \Omega_1^n \\ \vdots & \iddots & \iddots & 0 \\ 0 & \Omega_1 & \iddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix} \in \boldsymbol{R}^{(n+1)\times(n+1)}.$$

(c) Analog band-pass filter

The analog band-pass filter with the cut-off frequency $\Omega_1, \Omega_2 (\Omega_1 < \Omega_2)$ is designed from the original analog low-pass filter. The matrices

$$\begin{aligned} \boldsymbol{X}_{nm}^{BP} &:= [x_{ij}^{BP}] \in \boldsymbol{R}^{(2n+1)\times(2m+1)}, \\ \boldsymbol{S}_n^{BP} &:= [s_{ij}^{BP}] \in \boldsymbol{R}^{(2n+1)\times(n+1)}, \end{aligned}$$

where

$$
\begin{aligned}
x_{ij}^{BP} &= \begin{cases} \Omega_b^{n-m}, & i = j+n-m, \\ 0, & i \neq j+n-m, \end{cases} \\
s_{ij}^{BP} &= \binom{i}{(i+j-n)/2} \Omega_a^{j-i+n} \Omega_b^{n-j},
\end{aligned}
$$

transform $\{a_i\}_{i=0}^{m}$ and $\{b_i\}_{i=0}^{n}$ into $\{c_i\}_{i=0}^{2n}$ and $\{d_i\}_{i=0}^{2n}$ as

$$
\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \\ \vdots \\ c_{2n} \end{bmatrix} = \boldsymbol{X}_{nm}^{BP} \boldsymbol{S}_m^{BP} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_m \end{bmatrix},
$$

$$
\begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_n \\ \vdots \\ d_{2n} \end{bmatrix} = \boldsymbol{S}_n^{BP} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{bmatrix}.
$$

(*d*) *Analog band-stop filter*

The analog band-stop filter with the cut-off frequency $\Omega_1, \Omega_2 (\Omega_1 < \Omega_2)$ is designed from the original analog low-pass filter. The matrices which transform $\{a_i\}_{i=0}^{m}$ and $\{b_i\}_{i=0}^{n}$ into $\{c_i\}_{i=0}^{2n}$ and $\{d_i\}_{i=0}^{2n}$ can be obtained in the same manner as (*c*), *i.e.*,

$$
\begin{aligned}
\boldsymbol{X}_{nm}^{BS} &:= [x_{ij}^{BS}] \in \boldsymbol{R}^{(2n+1)\times(2m+1)}, \\
\boldsymbol{S}_n^{BS} &:= [s_{ij}^{BS}] \in \boldsymbol{R}^{(2n+1)\times(n+1)},
\end{aligned}
$$

where

$$
\begin{aligned}
x_{ij}^{BS} &= \binom{n-m}{(i-j)/2} \Omega_a^{2n-2m+j-i}, \\
s_{ij}^{BS} &= s_{i,n-j}^{BP} = \binom{n-j}{(i-j)/2} \Omega_a^{2n-i+j} \Omega_b^{j},
\end{aligned}
$$

3 Matrices Representation of the Frequency Transformation for IIR Digital Filters

The explicit formulae of matrices are derived which represent the frequency transformation for IIR digital filters.

Let T denotes the sampling interval of digital filters, and the cut-off frequency of the original digital low-pass filter be ω_0. Assume that IIR digital filters are derived from analog filters by the bilinear transformation. Then the transfer functions of the original digital low-pass and high-pass filter can be easily obtained from [6] as

$$H_3(z) := \frac{e_0 + e_1 z^{-1} + \cdots + e_n z^{-n}}{f_0 + f_1 z^{-1} + \cdots + f_n z^{-n}}.$$

The transformation formulae [2] of the frequency transformation for IIR digital filters are shown in Table 2. The transfer functions of the derived digital low-pass and high-pass filters can be reduced to

$$H_4(z) = \frac{g_0 + g_1 z^{-1} + \cdots + g_n z^{-n}}{h_0 + h_1 z^{-1} + \cdots + h_n z^{-n}}.$$

Those of the derived digital band-pass and band-stop filters can be reduced to

$$\tilde{H}_4(z) = \frac{g_0 + g_1 z^{-1} + \cdots + g_n z^{-n} + \cdots + g_{2n} z^{-2n}}{h_0 + h_1 z^{-1} + \cdots + h_n z^{-n} + \cdots + h_{2n} z^{-2n}}.$$

Then, matrices which transform the transfer function of the original low-pass filter to transfer functions of the derived filters are formulated.

(a) Design of digital low-pass filter

The digital low-pass filter with the cut-off frequency ω_1 is designed from the original digital low-pass filter. The matrix

$$\boldsymbol{T}_n^{LP} := [t_{ij}^{LP}] \in \boldsymbol{R}^{(n+1)\times(n+1)},$$

where

$$t_{ij}^{LP} = \sum_{k=0}^{i} (-1)^{j-i} \begin{pmatrix} j \\ i-k \end{pmatrix} \begin{pmatrix} n-j \\ k \end{pmatrix} \alpha^{j-i+2k},$$

transforms $\{e_i\}_{i=0}^n$ and $\{f_i\}_{i=0}^n$ into $\{g_i\}_{i=0}^n$ and $\{h_i\}_{i=0}^n$ as

$$\begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ g_n \end{bmatrix} = \boldsymbol{T}_n^{LP} \begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ e_n \end{bmatrix}, \quad \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_n \end{bmatrix} = \boldsymbol{T}_n^{LP} \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

Table 2: Transformation formulae of frequency transformation of digital filters (from the original digital low-pass filter with the cut-off frequency ω_0)

Filter Type (Cut-off Frequency)	Transformation Formulae / Parameters
Low-pass (ω_1)	$z^{-1} = \dfrac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$ $\alpha = \dfrac{\sin\left(\dfrac{\omega_0 - \omega_1}{2}\right)T}{\sin\left(\dfrac{\omega_0 + \omega_1}{2}\right)T}$
High-pass (ω_1)	$z^{-1} = -\dfrac{z^{-1} + \tilde{\alpha}}{1 + \tilde{\alpha} z^{-1}},$ $\tilde{\alpha} = -\dfrac{\cos\left(\dfrac{\omega_0 + \omega_1}{2}\right)T}{\cos\left(\dfrac{\omega_0 - \omega_1}{2}\right)T}$
Band-pass ($\omega_1, \omega_2(\omega_1 < \omega_2)$)	$z^{-1} = -\dfrac{z^{-2} + vz^{-1} + u}{uz^{-2} + vz^{-1} + 1}$ $u = \dfrac{k-1}{k+1}, \quad v = -\dfrac{2\alpha k}{k+1},$ $k = \cot\left(\dfrac{\omega_2 - \omega_1}{2}\right)T \tan\dfrac{\omega_0}{2}T,$ $\alpha = \dfrac{\cos\left(\dfrac{\omega_2 + \omega_1}{2}\right)T}{\cos\left(\dfrac{\omega_2 - \omega_1}{2}\right)T}$
Band-stop ($\omega_1, \omega_2(\omega_1 < \omega_2)$)	$z^{-1} = \dfrac{z^{-2} + \tilde{v}z^{-1} + \tilde{u}}{\tilde{u}z^{-2} + \tilde{v}z^{-1} + 1}$ $\tilde{u} = -\dfrac{\tilde{k}-1}{\tilde{k}+1}, \quad \tilde{v} = -\dfrac{2\tilde{\alpha}}{\tilde{k}+1},$ $\tilde{k} = \tan\left(\dfrac{\omega_2 - \omega_1}{2}\right)T \tan\dfrac{\omega_0}{2}T,$ $\tilde{\alpha} = \dfrac{\cos\left(\dfrac{\omega_2 + \omega_1}{2}\right)T}{\cos\left(\dfrac{\omega_2 - \omega_1}{2}\right)T}$

(*b*) *Digital high-pass filter*

The digital high-pass filter with the cut-off frequency ω_1 is designed from the original digital low-pass filter. The matrix which transforms $\{e_i\}_{i=0}^n$ and $\{f_i\}_{i=0}^n$ into $\{g_i\}_{i=0}^n$ and $\{h_i\}_{i=0}^n$ can be obtained in the same manner as (*a*), *i.e.*,

$$\begin{aligned} \boldsymbol{T}_n^{HP} &:= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & (-1)^n \end{bmatrix} \boldsymbol{T}_n^{LP}\Big|_{\alpha=\tilde{\alpha}} \\ &= [t_{ij}^{HP}] \in \boldsymbol{R}^{(n+1)\times(n+1)}, \end{aligned}$$

where

$$\begin{aligned} t_{ij}^{HP} &= (-1)^i t_{ij}^{LP}\Big|_{\alpha=\tilde{\alpha}} \\ &= \sum_{k=0}^{i} (-1)^j \binom{j}{i-k} \binom{n-j}{k} \tilde{\alpha}^{j-i+2k}. \end{aligned}$$

(*c*) *Digital band-pass filter*

The digital band-pass filter with the cut-off frequency $\omega_1, \omega_2 (\omega_1 < \omega_2)$ is designed from the original digital low-pass filter. The matrix

$$\boldsymbol{T}_n^{BP} := [t_{ij}^{BP}] \in \boldsymbol{R}^{(2n+1)\times(n+1)},$$

where

$$\begin{aligned} t_{ij}^{BP} &= \sum_{\ell=0}^{i} (-1)^j \beta_{i-\ell,j} \gamma_{\ell,j}, \\ \beta_{\ell,j} &= \sum_{k=0}^{\lceil \ell/2 \rceil} \binom{j}{\ell-k} \binom{\ell-k}{k} u^{j-\ell+k} v^{\ell-2k}, \\ \gamma_{\ell,j} &= \sum_{k=0}^{\lceil \ell/2 \rceil} \binom{n-j}{\ell-k} \binom{\ell-k}{k} u^k v^{\ell-2k}, \end{aligned}$$

transforms $\{e_i\}_{i=0}^n$ and $\{f_i\}_{i=0}^n$ into $\{g_i\}_{i=0}^{2n}$ and $\{h_i\}_{i=0}^{2n}$ as

$$\begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ g_n \\ \vdots \\ g_{2n} \end{bmatrix} = \boldsymbol{T}_n^{BP} \begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ e_n \end{bmatrix}, \quad \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_n \\ \vdots \\ h_{2n} \end{bmatrix} = \boldsymbol{T}_n^{BP} \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

In the above formulation of $\beta_{\ell,j}$ and $\gamma_{\ell,j}$, $\lceil \ell/2 \rceil$ means the maximum integer not exceeding $\ell/2$.

(d) Digital band-stop filter

The digital band-stop filter with the cut-off frequency $\omega_1, \omega_2 (\omega_1 < \omega_2)$ is designed from the original digital low-pass filter. The matrix which transforms $\{e_i\}_{i=0}^n$ and $\{f_i\}_{i=0}^n$ into $\{g_i\}_{i=0}^{2n}$ and $\{h_i\}_{i=0}^{2n}$ can be obtained in the same manner as (c), *i.e.*,

$$\begin{aligned} \boldsymbol{T}_n^{BS} &:= \boldsymbol{T}_n^{BP}\Big|_{u=\tilde{u},v=\tilde{v}} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & (-1)^n \end{bmatrix} \\ &= [t_{ij}^{BS}] \in \boldsymbol{R}^{(2n+1)\times(n+1)}, \end{aligned}$$

where

$$\begin{aligned} t_{ij}^{BS} &= (-1)^j t_{ij}^{BP}\Big|_{u=\tilde{u},v=\tilde{v}} \\ &= \sum_{\ell=0,}^{i} \beta_{i-\ell,j}\gamma_{\ell,j}\Bigg|_{u=\tilde{u},v=\tilde{v}}. \end{aligned}$$

4 Matrices Representation of the Bilinear Transformation for Rational Polynomials

The transfer matrices are derived which represent the bilinear transformation for rational polynomials.

The transfer function of an analog filter was written as

$$H_1(s) := \frac{a_0 + a_1 s + \cdots + a_m s^m}{b_0 + b_1 s + \cdots + b_n s^n}, \qquad (m \le n).$$

The transformation formula [3] of the bilinear transformation is given as

$$s = \frac{1 - z^{-1}}{1 + z^{-1}}.$$

Then the transfer function of the derived digital filter can be reduced [6] to

$$H_3(z) = \frac{e_0 + e_1 z^{-1} + \cdots + e_n z^{-n}}{f_0 + f_1 z^{-1} + \cdots + f_n z^{-n}}.$$

Then the matrices $\boldsymbol{Y}_{nm}$ and $\boldsymbol{Q}_n$ which transform $\{a_i\}_{i=0}^m$ and $\{b_i\}_{i=0}^n$ into $\{e_i\}_{i=0}^n$ and $\{f_i\}_{i=0}^n$ as

$$\begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ e_n \end{bmatrix} = \boldsymbol{Y}_{nm}\boldsymbol{Q}_m \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_m \end{bmatrix},$$

$$\begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix} = \boldsymbol{Q}_n \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{bmatrix},$$

can be written as follows. The recursive formula to obtain $\boldsymbol{Q}_n$ is shown in [4].

$$\boldsymbol{Y}_{nm} := [y_{ij}] \in \boldsymbol{R}^{(n+1)\times(m+1)},$$

$$\boldsymbol{Q}_n := [q_{ij}] \in \boldsymbol{R}^{(n+1)\times(n+1)},$$

where

$$y_{ij} = \begin{pmatrix} n-m \\ i-j \end{pmatrix},$$

$$q_{ij} = \sum_{k=0}^{i}(-1)^{i-k}\begin{pmatrix} j \\ i-k \end{pmatrix}\begin{pmatrix} n-j \\ k \end{pmatrix}.$$

5 Conditions for the direct design of IIR digital filters

Conditions to design IIR digital filters directly from an analog Low-pass filter are investigated.

In designing an IIR digital filter, two different ways can be considered as shown in Fig.1., *i.e.*,

(i) Frequency Transformation for Analog Filters first, and then Bilinear Transformation.

(ii) Bilinear Transformation first, and then Frequency Transformation for IIR Digital Filters.

They look quite different from each other at a glance. However, it was proven that the transformation formulae of them had been totally same so that the same digital filter is designed [5]. Hence, we can choose more simple one.

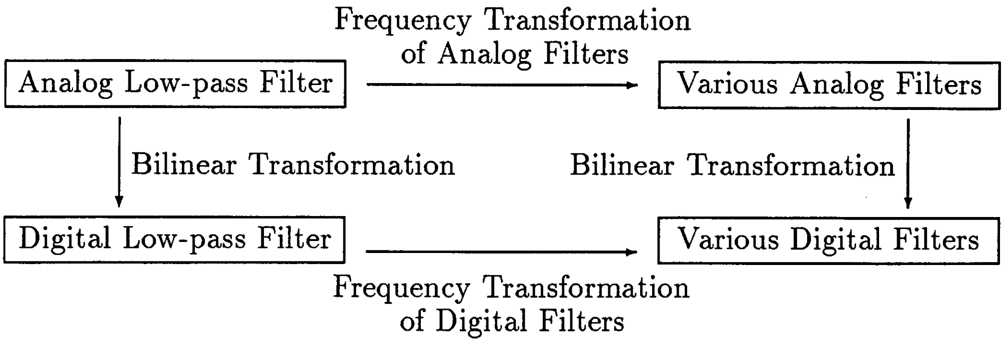

Figure 1: The Design process of IIR digital filters

Since the frequency transformation formulae to design analog low-pass and high-pass filters can be simply defined, it looks more simple to design digital low-pass and high-pass filters by the way **(i)** than by the way **(ii)**. It is unknown which way is more simple to design digital band-pass and band-stop filters.

To design any type of IIR digital filters by the way **(i)**, the cut-off frequency ω_0 of the original digital low-pass filter must be

$$\omega_0 = \frac{\pi}{2T}.$$

To design digital low-pass and high-pass filter with cut-off frequency ω_1 by the way **(ii)**, the cut-off frequency Ω_1 of analog filters must be

$$\Omega_1 = \tan\frac{\omega_1}{2}T.$$

To design band-pass and band-stop digital filter with cut-off frequency ω_1, ω_2 $(\omega_1 < \omega_2)$ by the way **(ii)**, the cut-off frequency Ω_1, Ω_2 of analog filters must be

$$\Omega_1 = \tan\frac{\omega_1}{2}T, \quad \Omega_2 = \tan\frac{\omega_2}{2}T.$$

6 Design Examples of IIR Digital Filters

Some simple digital filters are designed to evaluate the effectiveness of the proposed matrices. By using the matrices, transfer functions of IIR digital filters of various types are derived directly and automatically from a transfer function of an analog low-pass filter.

(*a*) *Original analog filter of low-pass type*

Let the original analog low-pass filter be the second-order normalized Butterworth low-pass filter. The transfer function is given as

$$H_1(s) \quad := \quad \frac{1}{1+\sqrt{2}s+s^2}.$$

Coefficient vectors are

$$[a_0] \quad = \quad [1], \quad \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}.$$

The amplitude characteristics of this filter is shown in Figure 2.

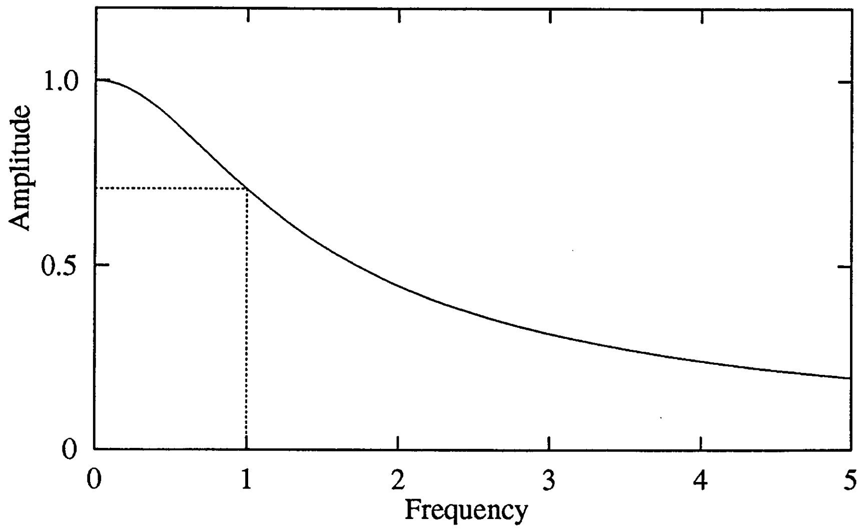

Figure 2: Amplitude characteristics of the original analog low-pass filter

(b) Design of IIR digital filter of high-pass type

The digital high-pass filter with the cut-off frequency $\omega_1 = 0.2\pi/T$ is designed from the original analog low-pass filter. This filter is designed by the way **(i)**, which is more simple one.

The cut-off frequency of the analog high-pass filter, which is derived by the frequency transformation for analog filters, is given as

$$\Omega_1 = \tan\frac{\omega_1}{2}T = \tan 0.1\pi \simeq 0.3249.$$

The transfer matrices $\boldsymbol{Q}_2, \boldsymbol{X}_{2,0}^{HP}, \boldsymbol{S}_0^{HP}$ and $\boldsymbol{S}_2^{HP}$ are given as

$$\boldsymbol{Q}_2 = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}, \quad \boldsymbol{X}_{2,0}^{HP} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\boldsymbol{S}_0^{HP} = [1], \quad \boldsymbol{S}_2^{HP} = \begin{bmatrix} 0 & 0 & 0.1056 \\ 0 & 0.3249 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then the coefficient vectors of the derived digital high-pass filter are obtained as

$$\begin{bmatrix} g_0 \\ g_1 \\ g_2 \end{bmatrix} = \boldsymbol{Q}_2 \boldsymbol{X}_{2,0}^{HP} \boldsymbol{S}_0^{HP} [a_0] = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} h_0 \\ h_1 \\ h_2 \end{bmatrix} = \boldsymbol{Q}_2 \boldsymbol{S}_2^{HP} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1.5651 \\ -1.7888 \\ 0.6461 \end{bmatrix}.$$

The derived transfer function of this filter is written as

$$H_4(z) = \frac{1 + 2z^{-1} + z^{-2}}{1.5651 - 1.7888z^{-1} + 0.6461z^{-2}}.$$

The amplitude characteristics of this filter is shown in Figure 3. The designed digital filter certainly realizes high-pass characteristics.

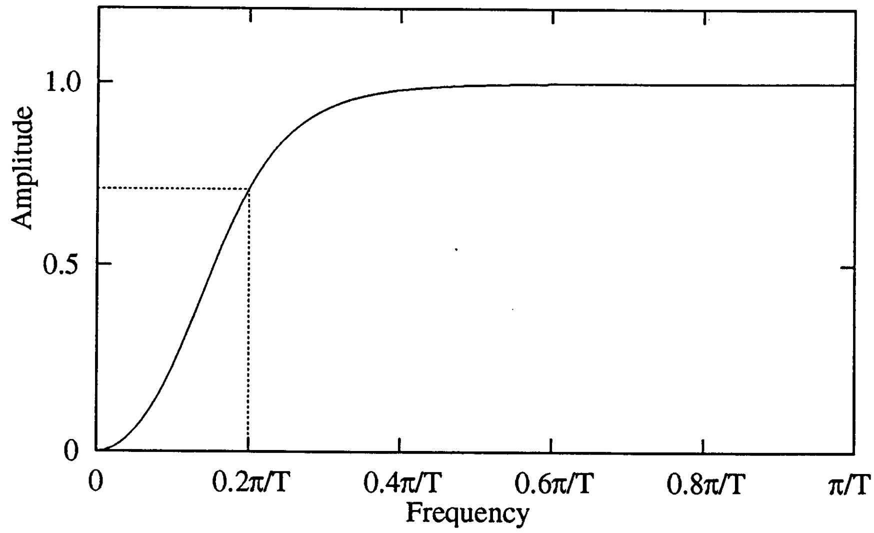

Figure 3: Amplitude characteristics of the derived digital high-pass filter

(c) IIR digital filter of band-pass type

The digital band-pass filter with the cut-off frequency $\omega_1 = 0.2\pi/T, \omega_2 = 0.6\pi/T$ is designed from the original analog low-pass filter. This filter is designed by both the ways (i) and (ii).

(c-i) Frequency Transformation of Analog filters first

The cut-off frequency of the analog band-pass filter, which is derived by the frequency transformation for analog filters, are given as

$$\Omega_1 = \tan\frac{\omega_1}{2}T = \tan 0.1\pi \simeq 0.3249,$$
$$\Omega_2 = \tan\frac{\omega_2}{2}T = \tan 0.3\pi \simeq 1.3764,$$

The transfer matrices $\boldsymbol{Q}_4, \boldsymbol{X}_{2,0}^{BP}, \boldsymbol{S}_0^{BP}$ and $\boldsymbol{S}_2^{BP}$ are given as

$$\boldsymbol{Q}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 2 & 0 & -2 & -4 \\ 6 & 0 & -2 & 0 & 6 \\ 4 & -2 & 0 & 2 & -4 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix},$$

$$\boldsymbol{X}_{2,0}^{BP} = \begin{bmatrix} 0 \\ 0 \\ 1.1056 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{S}_0^{BP} = [1],$$

$$\boldsymbol{S}_2^{BP} = \begin{bmatrix} 0 & 0 & 0.2 \\ 0 & 0.4702 & 0 \\ 1.1056 & 0 & 0.8944 \\ 0 & 1.0515 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, the coefficient vectors of the derived digital high-pass filter are obtained as

$$\begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \\ g_4 \end{bmatrix} = \boldsymbol{Q}_4 \boldsymbol{X}_{2,0}^{BP} \boldsymbol{S}_0^{BP} [a_0] = \begin{bmatrix} 1.1056 \\ 0 \\ -2.2112 \\ 0 \\ 1.1056 \end{bmatrix},$$

$$\begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \end{bmatrix} = \boldsymbol{Q}_4 \boldsymbol{S}_2^{BP} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 5.352 \\ -4.844 \\ 3.2 \\ -1.556 \\ 1.048 \end{bmatrix}.$$

The derived transfer function of this filter is written as

$$\tilde{H}_4(z) = \frac{1.1056 - 2.2112z^{-2} + 1.1056z^{-4}}{5.352 - 4.844z^{-1} + 3.2z^{-2} - 1.556z^{-3} + 1.048z^{-4}}.$$

The amplitude characteristics of this filter is shown in Figure 4. The designed digital filter certainly realizes band-pass characteristics.

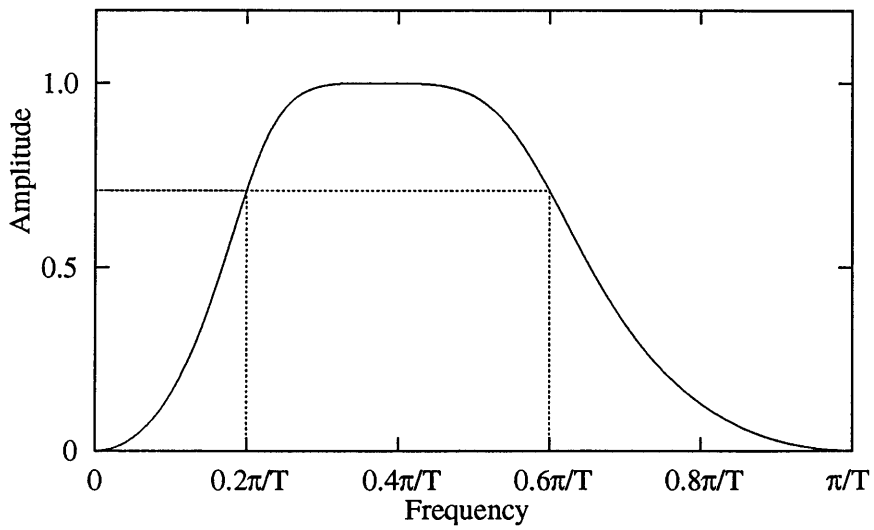

Figure 4: Amplitude characteristics of the derived digital band-pass filter

(*c-ii*) *Bilinear Transformation first*

The transfer matrices $\boldsymbol{T}_2^{BP}, \boldsymbol{Y}_{2,0}$ and $\boldsymbol{Q}_0$ are given as

$$\boldsymbol{T}_2^{BP} = \begin{bmatrix} 1 & -0.1584 & 0.0251 \\ -0.885 & 0.5126 & -0.1402 \\ 0.5126 & -1.2209 & 0.5126 \\ -0.1402 & 0.5126 & -0.885 \\ 0.0251 & -0.1584 & 1 \end{bmatrix},$$

$$\boldsymbol{Y}_{2,0} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \boldsymbol{Q}_0 = [1].$$

Then, the coefficient vectors of the derived digital high-pass filter are obtained as

$$\begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \\ g_4 \end{bmatrix} = \boldsymbol{T}_2^{BP}\boldsymbol{Y}_{2,0}\boldsymbol{Q}_0[a_0] = \begin{bmatrix} 0.7083 \\ 0 \\ -1.4166 \\ 0 \\ 0.7083 \end{bmatrix},$$

$$\begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \end{bmatrix} = \boldsymbol{T}_2^{BP} \boldsymbol{Q}_2 \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 3.4289 \\ -3.1037 \\ 2.0503 \\ -0.9970 \\ 0.6713 \end{bmatrix}.$$

These coefficients are different from those obtained by the way **(i)**. But the derived transfer functions become same. Therefore, the same digital filter was certainly designed.

7 Conclusion

This paper proposed the matrices which represent the relation of the coefficients of the frequency transformation for analog filters, that for IIR digital filters and the bilinear transformation for rational polynomials. The proposed matrices could replace hand computations in these transformations by automatic procedures. By using the proposed matrices, an automatic procedure was established to design various types of IIR digital filters from an analog low-pass filter.

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