The Stability for Linear Combinations of Characteristic Polynomials for Discrete-time Systems

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Abstract

In this paper, stability for a linear combination of characteristic polynomials for discrete-time systems is studied. In order to investigate this problem, we study the transformation matrix derived by bilinear transformation and its properties. Under certain assumptions, necessary and sufficient conditions for a linear combination of k characteristic polynomials for discrete-time systems to be stable are obtained.

1 Introduction

The stability of convex combinations of two polynomials for continuous-time systems was studied by Białas and Garloff[2], Białas[3] and Bose[4]. And, the stability of polytope polynomials and linear combinations of k polynomials was studied by Bose[4]. On the other hand, for discrete-time systems, the stability of convex combinations of two polynomials was studied by Bose[4] and Ackermann and Barmish[5].

The objective of this paper is to study stability of a linear combination of k characteristic polynomials for discrete-time systems.

In Section 2, we will study a transformation matrix P_n derived by bilinear transformation and its properties. In Section 3, necessary and sufficient conditions for a linear combination of k characteristics polynomials to be stable will be studied under certain assumptions. Section 4 will give an illustrative example. Finally, Section 5 will make some concluding remarks.

2 Matrix Representation of Bilinear Transformation and Its Properties

In this section, some important relationships between the coefficient vector of n-th degree characteristic polynomials in the z-domain and the coefficient vector of the numerator of n-th degree characteristic polynomials in the s-domain by using bilinear transformations will be investigated.

We first introduce the following notations.

$$\begin{aligned}
\boldsymbol{C}^H &:= \{x \in \boldsymbol{C} \mid \mathrm{Re}[x] < 0\}. \\
\boldsymbol{C}^S &:= \{x \in \boldsymbol{C} \mid |x| < 1\}. \\
P_n^R(x) &:= \left\{ f(x) \;\middle|\; f(x) = \sum_{i=0}^{n} a_i x^i, a_i \in \boldsymbol{R} \text{ for } i = 0, 1, \ldots, n \right\}. \\
H_n^R(x) &:= \left\{ f(x) \in P_n^R(x) \;\middle|\; \{x \in \boldsymbol{C} | f(x) = 0\} \subset \boldsymbol{C}^H \right\}. \\
S_n^R(x) &:= \left\{ f(x) \in P_n^R(x) \;\middle|\; \{x \in \boldsymbol{C} | f(x) = 0\} \subset \boldsymbol{C}^S \right\}.
\end{aligned}$$

Here, we remark that $H_n^R(x)$ is the set of all n-th Hurwitz polynomials, and $S_n^R(x)$ is the set of all n-th Schur polynomials. Next, we consider the n-th characteristic polynomial of a discrete-time system given by

$$p(z) := a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 \in P_n^R(z), \qquad n \neq 0. \tag{1}$$

We apply the bilinear transformation $z = \dfrac{s+1}{s-1}$ $(s \neq 1)$ to polynomial (1). Then, we obtain the following relation:

$$\begin{aligned}
p(z) &= a_n \left(\frac{s+1}{s-1}\right)^n + a_{n-1} \left(\frac{s+1}{s-1}\right)^{n-1} + \cdots + a_0 \\
&= \frac{1}{(s-1)^n} \{a_n (s+1)^n + a_{n-1}(s+1)^{n-1}(s-1) + \cdots + a_0 (s-1)^n\} &(2) \\
&=: \frac{1}{(s-1)^n} \{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_0\}. &(3)
\end{aligned}$$

Further, we denote the numerator of equation (3) as

$$q(s) := b_n s^n + b_{n-1} s^{n-1} + \cdots + b_0. \tag{4}$$

From equations (2) and (3), a relationship between the coefficients a_j and b_i can be easily obtained as follows.

$$b_{n-i} = \sum_{j=0}^{n} \left\{ \sum_{k=0}^{i} \binom{n-j}{k} \binom{j}{i-k} (-1)^{i-k} \right\} a_{n-j} \tag{5}$$

where $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} := \dfrac{\alpha!}{(\alpha-\beta)!\beta!}$.

Now, we define the following coefficient vectors $\boldsymbol{a}$,$\boldsymbol{b}$ and matrix P_n:

$$\boldsymbol{a} := \begin{bmatrix} a_n \\ a_{n-1} \\ \vdots \\ a_{n-j} \\ \vdots \\ a_0 \end{bmatrix}, \boldsymbol{b} := \begin{bmatrix} b_n \\ b_{n-1} \\ \vdots \\ b_{n-i} \\ \vdots \\ b_0 \end{bmatrix}, P_n := \begin{bmatrix} p^n_{0,0} & p^n_{0,1} & \cdots & p^n_{0,j} & \cdots & p^n_{0,n} \\ p^n_{1,0} & p^n_{1,1} & \cdots & p^n_{1,j} & \cdots & p^n_{1,n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ p^n_{i,0} & p^n_{i,1} & \cdots & p^n_{i,j} & \cdots & p^n_{i,n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ p^n_{n,0} & p^n_{n,1} & \cdots & p^n_{n,j} & \cdots & p^n_{n,n} \end{bmatrix}, \tag{6}$$

where

$$p^n_{i,j} := \sum_{k=0}^{i} \begin{pmatrix} n-j \\ k \end{pmatrix} \begin{pmatrix} j \\ i-k \end{pmatrix} (-1)^{i-k} \tag{7}$$

for $i, j = 0, 1, \ldots, n$. From the equations (5) and (6), we have the following equality:

$$P_n \boldsymbol{a} = \boldsymbol{b}. \tag{8}$$

Here, the matrix P_n has the following well-known properties.

Lemma 2.1 Let P_n be the matrix defined by equations (6) and (7). Further, let $f(z) \in P_n^R(z)$ and $g(s) \in P_n^R(s)$. If the coefficient vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of $f(z)$ and $g(s)$ satisfy

$$P_n \boldsymbol{\alpha} = \boldsymbol{\beta},$$

then, the following two assertions are equivalent.

(i) $f(z) \in S_n^R(z)$.

(ii) $g(s) \in H_n^R(s)$. $\square$

Lemma 2.2 [1] Let P_n be the matrix defined by equations (6) and (7). Then, the following property holds.

$$P_n^2 = 2^n E_n \qquad (n \geq 1)$$

where E_n is a $(n+1) \times (n+1)$ identity matrix. Therefore, this implies the matrix P_n has the following inverse matrix:

$$P_n^{-1} = \frac{1}{2^n} P_n \qquad (n \geq 1). \quad \square$$

The next lemma can be easily obtained.

Lemma 2.3 Let $P_n = \{p^n_{i,j}\}$ be the matrix given by equations (6) and (7). Then, the elements $p^n_{i,j}$ $(i,j = 0,1,\dots,n)$ of P_n satisfy the following relations.

$$p^n_{i,j} = (-1)^i p^n_{i,n-j} \qquad (i,j = 0,1,\dots,n).$$

(proof) The proof follows from the following equations.

$$\begin{aligned} (-1)^i p^n_{i,n-j} &= \sum_{k=0}^{i} \binom{j}{k} \binom{n-j}{i-k} (-1)^{-k} \qquad (k' = i-k) \\ &= \sum_{k'=0}^{i} \binom{j}{i-k'} \binom{n-j}{k'} ((-1)^{-1})^{i-k'} \\ &= p^n_{i,j}. \quad \square \end{aligned}$$

The next lemma can be easily obtained from Lemma 2.3 and are used to prove our main results.

Lemma 2.4 Let $\boldsymbol{a}$,$\boldsymbol{b}$ and P_n be the coefficient vectors and the matrix given by equations (6) and (7), then

$$P_n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_i \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} (-1)^0 b_n \\ (-1)^1 b_{n-1} \\ \vdots \\ (-1)^i b_{n-i} \\ \vdots \\ (-1)^n b_0 \end{bmatrix}.$$

(proof) From the equation (5) and (7),

$$b_{n-i} = \sum_{j=0}^{n} p^n_{i,j} a_{n-j}.$$

Here, we define a vector $\boldsymbol{c}$ as follows:

$$\boldsymbol{c} = \begin{bmatrix} c_n \\ c_{n-1} \\ \vdots \\ c_0 \end{bmatrix} := P_n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

Then,

$$c_{n-i} = \sum_{j=0}^{n} p^n_{i,n-j} a_{n-j}.$$

It follows from Lemma 2.3 that

$$\begin{aligned} c_{n-i} &= (-1)^i \sum_{j=0}^{n} p^n_{i,j} a_{n-j} \\ &= (-1)^i b_{n-i}. \end{aligned}$$

Therefore,

$$P_n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_i \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} (-1)^0 b_n \\ (-1)^1 b_{n-1} \\ \vdots \\ (-1)^i b_{n-i} \\ \vdots \\ (-1)^n b_0 \end{bmatrix}.$$

This completes the proof. □

Now, we give the following definitions.

Definition 2.1 Let $p(z) \in P_n^R(z)$ and $q(s) \in P_n^R(s)$.

(i) Define

$$\begin{aligned} p_s(z) &:= \frac{1}{2}\{p(z) + z^n p(z^{-1})\}, \\ p_a(z) &:= \frac{1}{2}\{p(z) - z^n p(z^{-1})\}. \end{aligned}$$

Then, $p_s(z)$ and $p_a(z)$ are said to be the symmetric part of $p(z)$ and the anti-symmetric part of $p(z)$, respectively.

(ii) Define

$$\begin{aligned} q_e(s) &:= \frac{1}{2}\{q(s) + q(-s)\}, \\ q_o(s) &:= \frac{1}{2}\{q(s) - q(-s)\}. \end{aligned}$$

Then, $q_e(s)$ and $q_o(s)$ are said to be the even part of $q(s)$ and the odd part of $q(s)$, respectively. □

Lemma 2.5 In Definition 2.1, all decomposition is unique.

(proof) First, we prove that $p(z)$ can be divided into $p_s(z)$ and $p_a(z)$ uniquely. We assume that $p(z)$ can be written by two way:

$$p(z) = p_{s1}(z) + p_{a1}(z) = p_{s2}(z) + p_{a2}(z) \tag{9}$$

where $p_{s1}(z)$ and $p_{s2}(z)$ are even parts of $p(z)$, and $p_{a1}(z)$ and $p_{a2}(z)$ are odd parts of $p(z)$. And, we remark that

$$\begin{aligned} p_s(z^{-1}) &= z^{-n}p_s(z), \\ p_a(z^{-1}) &= -z^{-n}p_a(z). \end{aligned}$$

From the equation (9),

$$f(z) := p_{s1}(z) - p_{s2}(z) = p_{a2}(z) - p_{a1}(z). \tag{10}$$

Then,

$$f(z^{-1}) = z^{-n}\{p_{s1}(z) - p_{s2}(z)\} = -z^{-n}\{p_{a2}(z) + p_{a1}(z)\}. \tag{11}$$

From the equation (10) and (11),

$$p_{a2}(z) - p_{a1}(z) = -p_{a2}(z) + p_{a1}(z).$$

Hence, we obtain

$$p_{a1}(z) = p_{a2}(z), \qquad p_{s1}(z) = p_{s2}(z).$$

Next, we prove the uniqueness of decomposition of $q(s)$ in the similarly way.

We assume that $q(s)$ can be written by two way:

$$q(s) = q_{e1}(s) + q_{o1}(s) = q_{e2}(s) + q_{o2}(s) \tag{12}$$

where $q_{e1}(s)$ and $q_{e2}(s)$ are even parts of $q(s)$, and $q_{o1}(s)$ and $q_{o2}(s)$ are odd parts of $q(s)$.

From the equation (12),

$$g(s) := q_{e1}(s) - q_{e2}(s) = q_{o2}(s) - q_{o1}(s). \tag{13}$$

Then,

$$g(-s) = q_{e1}(s) - q_{e2}(s) = -q_{o2}(s) + q_{o1}(s). \tag{14}$$

From the equation (13) and (14),

$$q_{o2}(s) - q_{o1}(s) = -q_{o2}(s) + q_{o1}(s).$$

Hence,

$$q_{o1}(s) = q_{o2}(s), \qquad q_{e1}(s) = q_{e2}(s).$$

This completes the proof. □

Let $p_s(z)$ and $p_a(z)$ be the symmetric part of polynomial (1) and the anti-symmetric part of polynomial (1), respectively. Further, Let $q_e(s)$ and $q_o(s)$ be the even part of polynomial (4) and the odd part of polynomial (4), respectively. Then, if n is even, the coefficient vectors $\boldsymbol{a}_s$ of $p_s(z)$, $\boldsymbol{a}_a$ of $p_a(z)$, $\boldsymbol{b}_e$ of $q_e(s)$ and $\boldsymbol{b}_o$ of $q_o(s)$ can be written as follows.

$$\boldsymbol{a}_s = \frac{1}{2}\begin{bmatrix} a_n + a_0 \\ a_{n-1} + a_1 \\ a_{n-2} + a_2 \\ \vdots \\ a_1 + a_{n-1} \\ a_0 + a_n \end{bmatrix}, \boldsymbol{a}_a = \frac{1}{2}\begin{bmatrix} a_n - a_0 \\ a_{n-1} - a_1 \\ a_{n-2} - a_2 \\ \vdots \\ a_1 - a_{n-1} \\ a_0 - a_n \end{bmatrix}, \boldsymbol{b}_e = \begin{bmatrix} b_n \\ 0 \\ b_{n-2} \\ \vdots \\ 0 \\ b_0 \end{bmatrix}, \boldsymbol{b}_o = \begin{bmatrix} 0 \\ b_{n-1} \\ 0 \\ \vdots \\ b_1 \\ 0 \end{bmatrix}. \tag{15}$$

If n is odd, they can be written as follows.

$$\boldsymbol{a}_s = \frac{1}{2}\begin{bmatrix} a_n + a_0 \\ a_{n-1} + a_1 \\ a_{n-2} + a_2 \\ \vdots \\ a_1 + a_{n-1} \\ a_0 + a_n \end{bmatrix}, \boldsymbol{a}_a = \frac{1}{2}\begin{bmatrix} a_n - a_0 \\ a_{n-1} - a_1 \\ a_{n-2} - a_2 \\ \vdots \\ a_1 - a_{n-1} \\ a_0 - a_n \end{bmatrix}, \boldsymbol{b}_e = \begin{bmatrix} 0 \\ b_{n-1} \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix}, \boldsymbol{b}_o = \begin{bmatrix} b_n \\ 0 \\ b_{n-2} \\ \vdots \\ b_1 \\ 0 \end{bmatrix}. \tag{16}$$

The next Theorem can be proved by using Lemma 2.4 and plays an important role to prove our main results.

Theorem 2.1 Suppose that $p(z) \in P_n^R(z)$ and $q(s) \in P_n^R(s)$ are defined by polynomials (1) and (4), respectively. And, let $\boldsymbol{a}$, $\boldsymbol{b}$ and P_n be the vectors and the matrix given by equations (6) and (7). Moreover, let $\boldsymbol{a}_s$, $\boldsymbol{a}_a$, $\boldsymbol{b}_e$ and $\boldsymbol{b}_o$ be the coefficient vectors given by (15) or (16). Then, if n is odd,

$$P_n\boldsymbol{a}_s = \boldsymbol{b}_o, \qquad P_n\boldsymbol{a}_a = \boldsymbol{b}_e,$$

and if n is even,

$$P_n\boldsymbol{a}_s = \boldsymbol{b}_e, \qquad P_n\boldsymbol{a}_a = \boldsymbol{b}_o.$$

(proof) The proof is given only for the case that n is even. In the case that n is odd, we can prove it, similarly.

From the equation (15), the coefficient vector $\boldsymbol{a}_s$ can be written by

$$\boldsymbol{a}_s = \frac{1}{2}\begin{bmatrix} a_n + a_0 \\ a_{n-1} + a_1 \\ a_{n-2} + a_2 \\ \vdots \\ a_1 + a_{n-1} \\ a_0 + a_n \end{bmatrix},$$

It follows from equations (8), (15) and Lemma 2.4 that

$$P_n\boldsymbol{a}_s = \frac{1}{2}\left(P_n\begin{bmatrix} a_n \\ a_{n-1} \\ a_{n-2} \\ \vdots \\ a_1 \\ a_0 \end{bmatrix} + P_n\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix}\right) = \frac{1}{2}\left(\begin{bmatrix} b_n \\ b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \\ b_0 \end{bmatrix} + \begin{bmatrix} b_n \\ -b_{n-1} \\ b_{n-2} \\ \vdots \\ -b_1 \\ b_0 \end{bmatrix}\right) = \begin{bmatrix} b_n \\ 0 \\ b_{n-2} \\ \vdots \\ 0 \\ b_0 \end{bmatrix} = \boldsymbol{b}_e.$$

Similarly, we can easily obtain

$$P_n\boldsymbol{a}_a = \boldsymbol{b}_o.$$

Hence, The proof can be completed. □

3 The Stability for A Linear Combination of Characteristic Polynomials

In this section, we give necessary and sufficient conditions for a linear combination of characteristic polynomials for discrete-time systems to be stable under certain assumptions.

The next lemma concerning continuous-time systems can be easily obtained from the results of Bose[4].

Lemma 3.1 Let $q_i(s) \in P_n^R(s)$ $(i = 1,2,\ldots,k)$, and let $q_{ei}(s)$ and $q_{oi}(s)$ $(i = 1,2,\ldots,k)$ be the even part of $q_i(s)$ and the odd part of $q_i(s)$ $(i = 1,2,\ldots,k)$, respectively.

Suppose that the following condition (a) or (b) is satisfied.

(a) $q_{e1}(s) = q_{e2}(s) = \cdots = q_{ek}(s)$.

(b) $q_{o1}(s) = q_{o2}(s) = \cdots = q_{ok}(s)$.

Then, the following two assertions are equivalent.

(i) $\quad q_i(s) \in H_n^R(s)$ $(i = 1, 2, \ldots, k)$.

(ii) $\quad \sum_{i=1}^{k} \lambda_i q_i(s) \in H_n^R(s)$ for all $\lambda_i \in [0, 1]$ $(i = 1, 2, \ldots, k)$. $\square$

The next theorem is one of our main results. This theorem is a discrete-time systems version of Lemma 3.1.

Theorem 3.1 Let $p_i(z) \in P_n^R(z)$ $(i = 1, 2, \ldots, k)$, and let $p_{si}(z)$ and $p_{ai}(z)$ $(i = 1, 2, \ldots, k)$ be the symmetric parts of $p_i(z)$ and the anti-symmetric parts of $p_i(z)$ $(i = 1, 2, \ldots, k)$, respectively.

Suppose that the following condition (a) or (b) is satisfied.

(a) $\quad p_{s1}(z) = p_{s2}(z) = \cdots = p_{sk}(z)$.

(b) $\quad p_{a1}(z) = p_{a2}(z) = \cdots = p_{ak}(z)$.

Then, the following two assertions are equivalent.

(i) $\quad p_i(z) \in S_n^R(z)$ $(i = 1, 2, \ldots, k)$.

(ii) $\quad \sum_{i=1}^{k} \lambda_i p_i(z) \in S_n^R(z)$ for all $\lambda_i \in [0, 1]$ $(i = 1, 2, \ldots, k)$.

(proof) Proof is given only for the case that n is even. In the case that n is odd, we can easily prove it, similarly.

First, let P_n be the matrix given by equations (6) and (7), and let

$$p_i(z) =: a_{in} z^n + a_{in-1} z^{n-1} + \cdots + a_{i0} \qquad (i = 1, 2, \ldots, k).$$

Further, let $\boldsymbol{a}_i$ denote the coefficient vector of $p_i(z)$ $(i = 1, 2, \ldots, k)$, and define vectors $\boldsymbol{b}_i$ $(i = 1, 2, \ldots, k)$ as follows:

$$\boldsymbol{b}_i = \begin{bmatrix} b_{in} \\ b_{in-1} \\ \vdots \\ b_{i0} \end{bmatrix} := P_n \boldsymbol{a}_i \qquad (i = 1, 2, \ldots, k). \tag{17}$$

Defining $q_i(s)$ by

$$q_i(s) := b_{in} s^n + b_{in-1} s^{n-1} + \cdots + b_{i0} \qquad (i = 1, 2, \ldots, k),$$

and denoting $q_{ei}(s)$ and $q_{oi}(s)$ by the even part of $q_i(s)$ and the odd part of $q_i(s)$. Further, we define

$$\begin{aligned} q_\lambda(s) &:= \sum_{i=1}^{k} \lambda_i q_i(s),\ \lambda_i \in [0,1],\ i = 1,2,\ldots,k \\ p_\lambda(z) &:= \sum_{i=1}^{k} \lambda_i p_i(z),\ \lambda_i \in [0,1],\ i = 1,2,\ldots,k. \end{aligned}$$

Then, it follows from Lemma 2.1 and equation (17) that the following claim holds.

Claim 1 $\quad p_i(z) \in S_n^R(z),\ (i = 1,2,\ldots,k) \iff q_i(s) \in H_n^R(s),\ (i = 1,2,\ldots,k).$

Further, from Theorem 2.1 and the hypothesis (a) or (b), the following condition (1) or (2) is satisfies:

(1) $\quad q_{e1}(s) = q_{e2}(s) = \cdots = q_{ek}(s).$

(2) $\quad q_{o1}(s) = q_{o2}(s) = \cdots = q_{ok}(s).$

Then, it follows from Lemma 3.1 that

Claim 2 $\quad q_i(s) \in H_n^R(s),\ (i = 1,2,\ldots,n) \iff q_\lambda(s) \in H_n^R(s).$

If we show the following claim, the proof of this theorem follows from Claim 1-3.

Claim 3 $\quad q_\lambda(s) \in H_n^R(s) \iff p_\lambda(z) \in S_n^R(z).$

Therefore, we will prove Claim 3. First, we can write $q_\lambda(s)$ in the following form:

$$q_\lambda(s) = \sum_{i=0}^{n} (\lambda_1 b_{1n-i} + \lambda_2 b_{2n-i} + \cdots + \lambda_k b_{kn-i}) s^{n-i}.$$

Then, the coefficient vector $\boldsymbol{b}_\lambda$ of $q_\lambda(s)$ can be written as

$$\begin{aligned} \boldsymbol{b}_\lambda &:= \begin{bmatrix} \lambda_1 b_{1n} + \lambda_2 b_{2n} + \cdots + \lambda_k b_{kn} \\ \lambda_1 b_{1n-1} + \lambda_2 b_{2n-1} + \cdots + \lambda_k b_{kn-1} \\ \vdots \\ \lambda_1 b_{10} + \lambda_2 b_{20} + \cdots + \lambda_k b_{k0} \end{bmatrix} \\ &= \lambda_1 \boldsymbol{b}_1 + \lambda_2 \boldsymbol{b}_2 + \cdots + \lambda_k \boldsymbol{b}_k \\ &= \lambda_1 P_n \boldsymbol{a}_1 + \lambda_2 P_n \boldsymbol{a}_2 + \cdots + \lambda_k P_n \boldsymbol{a}_k. \end{aligned} \tag{18}$$

Since the inverse matrix P_n^{-1} exists from Lemma 2.2, it follows from (18) that

$$\begin{aligned} P_n^{-1}\boldsymbol{b}_\lambda &= \lambda_1\boldsymbol{a}_1 + \lambda_2\boldsymbol{a}_2 + \cdots + \lambda_k\boldsymbol{a}_k \\ &= \begin{bmatrix} \lambda_1 a_{1n} + \lambda_2 a_{2n} + \cdots + \lambda_k a_{kn} \\ \lambda_1 a_{1n-1} + \lambda_2 a_{2n-1} + \cdots + \lambda_k a_{kn-1} \\ \vdots \\ \lambda_1 a_{10} + \lambda_2 a_{20} + \cdots + \lambda_k a_{k0} \end{bmatrix} \\ &=: \boldsymbol{a}_\lambda \end{aligned}$$

Now, construct the polynomial with coefficient vector $\boldsymbol{a}_\lambda$ as follows:

$$\begin{aligned} &(\lambda_1 a_{1n} + \cdots + \lambda_k a_{kn})z^n + (\lambda_1 a_{1n-1} + \cdots + \lambda_k a_{kn-1})z^{n-1} + \cdots + (\lambda_1 a_{10} + \cdots + \lambda_k a_{k0}) \\ &= \lambda_1(a_{1n}z^n + a_{1n-1}z^{n-1} + \cdots + a_{10}) + \cdots + \lambda_k(a_{kn}z^n + a_{kn-1}z^{n-1} + \cdots + a_{k0}) \\ &= \sum_{i=1}^{k} \lambda_i p_i(z) \\ &= p_\lambda(z) \end{aligned}$$

Therefore, the coefficient vectors $\boldsymbol{a}_\lambda$ and $\boldsymbol{b}_\lambda$ of $p_\lambda(z)$ and $q_\lambda(s)$ satisfy the following equality. are related by P_n as follows:

$$P_n\boldsymbol{a}_\lambda = \boldsymbol{b}_\lambda.$$

By Lemma 2.1, Claim 3 was proved.

This completes the proof of this theorem. □

4 An Example

Consider the following polynomial:

$$p_\lambda(z) = (12\lambda_1 + 13\lambda_2 + 11\lambda_3)z^3 + (8\lambda_1 + 9\lambda_2 + 7\lambda_3)z^2 - (\lambda_1 + 2\lambda_2)z - (\lambda_1 + 2\lambda_2)$$

where $\lambda_i \in [0,1]$ $(i = 1,2,3)$. Then, $p_\lambda(z)$ is represented as a linear combination of the following three polynomials.

$$p_\lambda(z) = \lambda_1 p_1(z) + \lambda_2 p_2(z) + \lambda_3 p_3(z)$$

where

$$\begin{aligned} p_1(z) &= 12z^3 + 8z^2 - z - 1 \\ p_2(z) &= 13z^3 + 9z^2 - 2z - 2 \\ p_3(z) &= 11z^3 + 7z^2 \end{aligned}$$

It can be easily checked that the hypothesis (a) of Theorem 3.1 is satisfied, *i.e.*,

$$p_{s1}(z) = p_{s2}(z) = p_{s3}(z) = \frac{11}{2}z^3 + \frac{7}{2}z^2 + \frac{7}{2}z + \frac{11}{2}$$

where $p_{si}(z)$ is the symmetric part of $p_i(z)$ for $i = 1, 2, 3$.

Then, the stability of $p_\lambda(z)$ can be determined by checking the stability of the polynomials $p_1(z)$, $p_2(z)$ and $p_3(z)$. In fact, since p_1, p_2 and p_3 are all schur stable, we can see p_λ is schur stable for all $\lambda \in [0, 1]$.

5 Conclusions

In this paper, necessary and sufficient conditions for a linear combination of characteristic polynomials for discrete-time systems to be stable were given under certain assumptions. This result is a discrete-time systems version of the result[4] of continuous-time systems.

References

[1] N. K. Bose, "Properties of the Q_n-Matrix in Bilinear Transformation," Proceedings of The IEEE, Vol.71, pp.1110-1111, 1983.

[2] S. Białas and J. Garloff, "Convex Combinations of Stable Polynomials," Journal of the Franklin Inst., Vol.319, No.3, pp.373-377, 1985.

[3] S. Białas, "A Necessary and Sufficient Condition for the Stability of Convex Combinations of Stable Polynomials or Matrices," Bulletin of the Polish Academy of Sci. Tech. Sci., Vol.33, No.9-10, pp.473-480, 1985.

[4] N. K. Bose, "A System-Theoretic Approach to Stability of Sets of Polynomials," Contemporary Mathematics, Vol.47, pp.25-34, 1985.

[5] J. E. Ackermann and B. R. Barmish, "Robust Schur Stability of a Polytope of Polynomials," IEEE Trans. on Automatic Control, Vol.33, No.10, pp.984-986, 1988.