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Abstract.

We consider the multiclass M/G/1 queues with random feedback. Each customer belongs to one of the several priority classes, and the customers of each class arrive at a station in a Poisson process from outside the system. The service time distribution for each class is arbitrary. After receiving a service, a customer at station i either departs from the system with probability p_{i0}, or feeds back to the system and proceeds to station k with probability p_{ik} $(i, k = 1, \ldots, J)$. We consider preemptive scheduling algorithms where every service discipline at every station is either FCFS and preemptive LCFS. We first formulate system performance measures as the cost functions of the system state. They include the mean sojourn times of customers arriving at every station. We consider the system at arbitrary states to obtain explicit formulae of these cost functions and then derive their steady state values.

Key words. Multiclass M/G/1 queues, feedbacks of customers, mean sojourn times, scheduling algorithms and Little's formula.

1. Introduction.

Single server queueing systems with feedback are useful to modeling multiprograming systems and manufacturing systems. In computer systems customers (tasks) that are scheduled for resources may be have to come back several times for additional service [1].

Disney [3] and Disney et al. [4] have been concerned with sojourn times in M/G/1 queues with instantaneous, Bernoulli feedback. Van den Berg et al. [1] considered the system in which each customer requires N services. Fed back customers return instantaneously, joining the end of the queue. They derived the set of linear equations in order that the mean sojourn times per visit can be explicitly solved. Simon [18] considered the system with c types of customers and m levels of priority. Class j customers may require service $N(j)$ times. The k^{th} time a class j customer enters the queue it is assigned priority level $f(j, k)$. He obtained the set of linear equations for the mean waiting times. Doshi and Kaufman [5] studied the sojourn time of a tagged customer who has just completed his m^{th} pass in an M/G/1 queue with Bernoulli feedback. They also considered the model with multiple customer classes. Recently, Epema [6] has investigated the general single server (M/G/1) time-sharing model with multiple queues and customer classes, priorities and feedback. Customers are served in passes, receiving a complete quantum of service on every pass, or their remaining service demand, whichever is the lesser. If a customer completes his service demand during the pass, it leaves the system. He derived a set of linear equations in the mean waiting times of the customer passes for all classes and queues. The priority queues are well investigated in [8, 10, 19, 23]. For further related topics in the field, see [11, 15].

We formulate system performance measures as the cost functions of system states. we first consider the system in arbitrary states in order to derive explicit formulae of the cost functions.

Then we consider the system under steady states. In Section 2. we describe our model in detail, and introduce notation and system states for given system parameters. Then we define a set of cost functions that represent the system performances such as the mean *sojourn times* and the expected values of *cumulative works*. Sets of equations satisfied by these cost functions are derived. These quantities are closely related to busy periods of the system. Hence we analyze busy periods in Section 3.. Section 4. is devoted to derive explicit formulae of two cost functions. We derive quantities with regard to transitions of states in Sections 5.. In Section 6. the sets of the equations are solved to derive explicit formulae of the two objective cost functions at arbitrary states. Uniqueness of the cost functions is established in Section 7.. Finally, in Section 8. we evaluate steady state values of the cost functions by the generalized Little's formula ($H = \lambda G$) and PASTA (Poisson arrivals see time averages) property.

2. Model and Notation.

We consider a multiclass priority queueing system with random feedbak. Let there be J classes of customers indexed as $1, 2, \ldots, J$. Customers arrive at station i from outside the system according to a Poisson process $\{A_i(t) : t \geq 0\}$ with rate λ_i $(i = 1, \ldots, J)$. Let $\lambda_j^+ = \sum_{i=1}^{j} \lambda_i$ and $\lambda = \lambda_J^+$. A customer at station i is called a *class i customer*. Service times S_i of class i customers are arbitrarily distributed. Customers in the system are served by a single server according to a predetermined *scheduling algorithm*. After a class i customer completes a service, he either departs from the system with probability p_{i0}, or feeds back to the system and proceeds to station j with probability p_{ij} $(i, j = 1, \ldots, J)$. For convenience, let $p_{0j} = 0$ $(j = 1, \ldots, J)$. Let $\mathbf{P}_l \equiv (p_{ij} : i, j = 1, \ldots, l)$ be the feedback probability matrix $(l = 1, \ldots, J)$. The arrival processes, the service times and the feedback processes are assumed to be independent of each other.

Let v denote a *current work*, that is, a customer will receive v seconds of service potentially at the currently entered station on the visit (or, his current remaining service time at the station). The customer departs from the system or feeds back to the system after receiving v seconds of his service. Let $T_{ij}(v)$ be the total amount of service times of a customer who is currently at station i with his current work v receives until he departs from the system or leaves for one of stations $j + 1, \ldots, J$ for the first time $(i, j = 1, \ldots, J)$. Note that, even if $i > j$, it is assumed that he receives at least service v. The expected values of $T_{ij}(v)$ is given by

$$E[T_{ij}(v)] = v + \sum_{l=1}^{j} p_{il} E[T_{lj}(S_l)], \qquad i, j = 1, \ldots, J. \tag{2.1}$$

Specifically, if we let $T_{ij} \equiv T_{ij}(S_i)$, then

$$E[T_{ij}] = E[S_i] + \sum_{l=1}^{j} p_{il} E[T_{lj}], \qquad i, j = 1, \ldots, J. \tag{2.2}$$

So we can obtain their solutions in a vector form if $(\mathbf{I} - \mathbf{P}_j)^{-1}$ exists. We define intensities ρ_j^+ in the following manner:

$$\begin{aligned} \rho_0^+ &\equiv 0, \\ \rho_j^+ &\equiv \sum_{i=1}^{j} \lambda_i E[T_{ij}], \qquad j = 1, \ldots, J. \end{aligned}$$

Then we make the following assumption:

Assumption 2..

1. $\mathbf{P}_J^n \to 0$ as $n \to \infty$.

2. $\rho_J^+ < 1$. □

The first assumption is a sufficient condition for existence of $(\mathbf{I} - \mathbf{P}_j)^{-1}$ for $j = 1, \dots, J$. Since ρ_J^+ is the traffic intensity of the system, the second assumption is the usual condition.

Let $\mathcal{R}, \mathcal{R}_+, \mathcal{I}_+$ be respectively a set of real numbers, a set of nonnegative real numbers, and a set of nonnegative integers. Number of class i customers in the system is denoted by n_i and their J-tuple is denoted by $\mathbf{n} = (n_1, \dots, n_J) \in \mathcal{I}_+^J$. Customers in each station are queued in the order of their last arrivals to the station. Let v_{ik} be a current work of a class i customer in k^{th} position of its queue ($i = 1, \dots, J$ and $k = 1, \dots n_i$). Number of class i customers in the system at time t is denoted by $n_i^r(t)$ and their J-tuple is denoted by $\mathbf{n}^r(t) = (n_1^r(t), \dots, n_J^r(t))$. Let $v_{ik}(t)$ be a current work of a class i customer in k^{th} position of its queue at time t ($i = 1, \dots, J$ and $k = 1, \dots, n_i^r(t)$). All processes $\{v_{ik}(t) : t \geq 0\}$ ($i = 1, \dots, J$ and $k = 1, \dots, n_i^r(t)$) and all processes $\{n_i^r(t) : t \geq 0\}$ ($i = 1, \dots, J$) are right continuous with left-hand limits.

We assume that the classes of customers are priority classes such that class i has priority over class j if $i < j$. Customers are served preferentially served in order of priority. If a customer of high priority arrives when a customer of lower priority is being served, the server interrupts the current service and immediately starts to serve the customer of high priority. The preempted service for the customer of lower priority is commenced again from the point where it was interrupted. The service discipline at each station is either FCFS or preemptive LCFS(PR-LCFS). Station i with FCFS discipline serves customers according to first come first served basis if no customers are in stations $1, \dots, i-1$. Station i with PR-LCFS discipline serves customers according to preemptive resume last come first served basis if no customers are in stations $1, \dots, i-1$. We consider scheduling algorithms that are *work-conserving*: sums of remaining service times of all customers at any time $t \geq 0$ ($\sum_{i=1}^{J} \sum_{k=1}^{n_i^r(t)} T_{iJ}(v_{ik}(t))$: work-in-system; Wolff [21]) are the same for all scheduling algorithms considered, and the server is not idle whenever there is a customer in the system (non-idling). (We distinguish a term *current work* from a term *work*, where we use work to denote total amount of remaining service times of customers in the system that will be received until they depart from the system.)

Let us assume that customers are numbered in the order of their arrivals from outside the system and that the system is operated under a fixed scheduling algorithm. Let us consider an e^{th} customer arrives from outside the system at one of the stations at some epoch σ_0^e ($e = 1, 2, \dots$). Let M^e be the number of his visits to the stations from his arrival at time σ_0^e until his departure from the system. Then let σ_k^e be a time epoch just when he would arrive at one of the stations after completing his k^{th} service ($k = 1, 2, \dots, M^e$). For convenience, $\sigma_k^e = \sigma_{M^e}^e$ for $k > M^e$. We must specify informations of the system in order to operate it according to a predetermined scheduling algorithm. Let $\mathbf{l}_{im}(t) = (v_{im}(t), \sigma_0^e, \dots, \sigma_{N_F^e(t)}^e, \infty, \dots) \in \mathcal{R}_+ \times (\mathcal{R}_+ \cup \{\infty\})^\infty$ be an information vector of a class i customer in m^{th} position of its queue where $N_F^e(t)$ is the number of his feedbacks up to time t ($\sigma_k^e \equiv \infty$ for $k > N_F^e(t)$ in the information vector). If there is not a class i customer in m^{th} position of its queue at time t, let $\mathbf{l}_{im}(t) = (0, \infty, \dots)$. The *customer list* is a set of these information vectors such that $L(t) = (\mathbf{l}_{im}(t) : i = 1, \dots, J \text{ and } m = 1, 2, \dots)$. Let us consider transition epochs of these processes consist of customer arrival epochs and service completion epochs. Then let $X(t)$ denote a station where a customer arrives at the last transition epoch before t ($t \geq 0$). $X(t)$ is right continuous with left-hand limits and $X(t) = 0$ if a customer departs from the system at the last transition epoch before t. Let $n_i(t) \equiv n_i^r(t)$ for all $t \geq 0$ ($i = 1, \dots, J$) except for transition epochs τ of customers at which $n_i(\tau) \equiv n_i^r(\tau)$ ($i \neq X(\tau)$) and $n_{X(\tau)}(\tau) \equiv n_{X(\tau)}^r(\tau) - 1$. Hence $\{n_i(t) : t \geq 0\}$ denotes the number of customers in station i just prior to arrivals of customers. Let $\mathbf{n}(t) \equiv (n_1(t), \dots, n_J(t))$. Further let $\mathbf{v}(t) = (v_1(t), \dots, v_J(t))$ denotes the vector of *total amount of current works* $v_i(t) \equiv \sum_{m=1}^{n_i(t)} v_{im}(t)$ at time t ($i = 1, \dots, J$). Then we define the stochastic process $\mathcal{Q} = \{\mathbf{Y}(t) = (X(t), \mathbf{v}(t), \mathbf{n}(t), L(t)) : t \geq 0\}$ that represents an evolution of the system. For any scheduling algorithm defined above, $\mathcal{Q}$ embeds a Markov process with a stationary transition probability whose transition epochs consist of customer arrival epochs and service completion epochs. Possible values of $\mathbf{Y}(t)$ $(t \geq 0)$ are called *states*. The state space of $\mathcal{Q}$ is denoted by $\mathcal{E}$.

We would like to derive two types of cost functions defined below. First type of the cost functions represents the mean sojourn times of customers. We define

$$C^e_{Wj}(t) \equiv \begin{cases} 1, & \text{if an } e^{th} \text{ customer stays at station } j \text{ at time } t, \\ 0, & \text{if an } e^{th} \text{ customer does not stay at station } j \text{ at time } t, \end{cases} \tag{2.3}$$

for $t \geq 0$, $j = 1, \ldots, J$ and $e = 1, 2, \ldots$. Then we define

$$W_j(\mathbf{Y}, e, l) \equiv E\left[\int_{\sigma^e_l}^{\infty} C^e_{Wj}(t)dt | \mathbf{Y}(\sigma^e_l) = \mathbf{Y}\right], \quad j = 1, \ldots, J, \tag{2.4}$$

where $\mathbf{Y} = (i, \mathbf{v}, \mathbf{n}, L) \in \mathcal{E}$, $l = 0, 1, \ldots$ and $e = 1, 2, \ldots$. $W_j(\mathbf{Y}, e, l)$ denotes the mean *sojourn time* of an e^{th} customer spent at station j after time σ^e_l given that the system is in state $\mathbf{Y}$ at that time. Trivially, $W_j(\mathbf{Y}, e, l) \equiv 0$ for $\mathbf{Y} = (0, \mathbf{v}, \mathbf{n}, L) \in \mathcal{E}$. Let us consider an e^{th} customer is in station i at time σ^e_l. His *initial stay* denotes a period from time σ^e_l until he completes his first service at station i. The length of his initial stay is called the initial sojourn time. Then we define

$$W^I_j(\mathbf{Y}, e, l) \equiv E\left[\int_{\sigma^e_l}^{\sigma^e_{l+1}} C^e_{Wj}(t)dt | \mathbf{Y}(\sigma^e_l) = \mathbf{Y}\right], \quad j = 1, \ldots, J, \tag{2.5}$$

where $\mathbf{Y} = (i, \mathbf{v}, \mathbf{n}, L) \in \mathcal{E}$, $l = 0, 1, \ldots$ and $e = 1, 2, \ldots$. $W^I_j(\mathbf{Y}, e, l)$ denotes the mean *initial sojourn time* of an e^{th} customer spent at station j during his initial stay after time σ^e_l given that the system is in state $\mathbf{Y}$ at that time. Trivially, $W^I_j(\mathbf{Y}, e, l) \equiv 0$ for $\mathbf{Y} = (i, \mathbf{v}, \mathbf{n}, L) \in \mathcal{E}$ and $i \neq j$. Then the mean sojourn time $W_j(\cdot)$ of an e^{th} customer spent at station j $(j = 1, \ldots, J)$ is decomposed into two parts: the mean initial sojourn time and the mean sojourn time after his initial stay at station i. We mathematically express the fact as follows:

$$W_j(\mathbf{Y}, e, l) = W^I_j(\mathbf{Y}, e, l) + E[W_j(\mathbf{Y}(\sigma^e_{l+1}), e, l+1) | \mathbf{Y}(\sigma^e_l) = \mathbf{Y}], \tag{2.6}$$

for $\mathbf{Y} = (i, \mathbf{v}, \mathbf{n}, L) \in \mathcal{E}$, $l = 0, 1, \ldots$ and $e = 1, 2, \ldots$ $(j = 1, \ldots, J)$. Of course, every scheduling algorithm has its own cost functions.

Second type of the cost functions represents the expected *cumulative current works* of customers. Let $C^e_{Gj}(t)$ be a *current work* at time t of an e^{th} customer at station j. For example, if he enters station j at epoch σ^e_l, then $C^e_{Gj}(\sigma^e_l) = S_j$. The value of $C^e_{Gj}(t)$ gradually decreases as the server serves him. If he completes his current work at epoch σ^e_{l+1}, then $C^e_{Gj}(\sigma^e_{l+1}-) = 0$. If he again enters station j at $\sigma^e_{l'} \geq \sigma^e_{l+1}$, then $C^e_{Gj}(\sigma^e_{l'}) = S_j$. The value of $C^e_{Gj}(t)$ gradually decreases until the server completes his current work at epoch $\sigma^e_{l'+1}$. Then, $C^e_{Gj}(\sigma^e_{l'+1}-) = 0$ and so forth. Then we define

$$G_j(\mathbf{Y}, e, l) \equiv E\left[\int_{\sigma^e_l}^{\infty} C^e_{Gj}(t)dt | \mathbf{Y}(\sigma^e_l) = \mathbf{Y}\right], \quad j = 1, \ldots, J, \tag{2.7}$$

where $\mathbf{Y} = (i, \mathbf{v}, \mathbf{n}, L) \in \mathcal{E}$, $l = 0, 1, \ldots$ and $e = 1, 2, \ldots$. $G_j(\mathbf{Y}, e, l)$ denotes the expected value of the *cumulative current work* of an e^{th} customer, who arrives at station i at σ^e_l just when the system is in state $\mathbf{Y} = (i, \mathbf{v}, \mathbf{n}, L) \in \mathcal{E}$, accumulated during he stays at station j until he departs from the system. Trivially, $G_j(\mathbf{Y}, e, l) \equiv 0$ for $\mathbf{Y} = (0, \mathbf{v}, \mathbf{n}, L) \in \mathcal{E}$. Further, we define

$$G^I_j(\mathbf{Y}, e, l) \equiv E\left[\int_{\sigma^e_l}^{\sigma^e_{l+1}} C^e_{Gj}(t)dt | \mathbf{Y}(\sigma^e_l) = \mathbf{Y}\right], \quad j = 1, \ldots, J, \tag{2.8}$$

where $\mathbf{Y} = (i, \mathbf{v}, \mathbf{n}, L) \in \mathcal{E}$, $l = 0, 1, \dots$ and $e = 1, 2, \dots$. $G_j^I(\mathbf{Y}, e, l)$ denotes the expected value of the *initial cumulative current work* of an e^{th} customer, who arrives at station i at time σ_l^e just when the system is in state $\mathbf{Y} = (i, \mathbf{v}, \mathbf{n}, L)$, accumulated during his initial stay at the station. Trivially, $G_j^I(\mathbf{Y}, e, l) \equiv 0$ for $\mathbf{Y} = (i, \mathbf{v}, \mathbf{n}, L) \in \mathcal{E}$ and $i \neq j$. Then the expected cumulative current work $G_j(\cdot)$ of an e^{th} customer accumulated at station j $(j = 1, \dots, J)$ is decomposed into two parts: the expected initial cumulative currnet work and the expected cumulative current work after his initial stay at station i. We mathematically express the fact as follows:

$$G_j(\mathbf{Y}, e, l) \;=\; G_j^I(\mathbf{Y}, e, l) + E[G_j(\mathbf{Y}(\sigma_{l+1}^e), e, l+1) | \mathbf{Y}(\sigma_l^e) = \mathbf{Y}], \tag{2.9}$$

for $\mathbf{Y} = (i, \mathbf{v}, \mathbf{n}, L) \in \mathcal{E}$, $l = 0, 1, \dots$ and $e = 1, 2, \dots$ $(j = 1, \dots, J)$. Of course, every scheduling algorithm has its own cost functions.

After stating some assumptions, we will explicitly solve equations (2.6) and (2.9) in Section 6..

3. Busy periods of the system.

The quantities defined in the last section are shown to be closely related to busy periods. Let B^j be the first time until the system is cleared of customers from classes 1 through j $(j = 1, \dots, J)$. Let

$$B^j(v) \equiv \begin{cases} \text{the first time until the system is cleared of customers from classes} \\ \text{1 through } j \text{ with an `exceptional' service time } v;\ j = 1, \dots, J. \end{cases} \tag{3.1}$$

For notational convenience, let $B^0(v) \equiv v$. According to usual queueing parlances, B^j is called a *busy period* composed of customers from classes 1 through j, and $B^j(v)$ is called an *Exceptional First Service Busy Period* (EFSBP) composed of customers from classes 1 through j. We will call B^j and $B^j(v)$ simply a *class j busy period* and a *class j busy period initiated with exceptional service v*, respectively. Their expected values are given by

$$E[B^j] = \frac{\rho_j^+}{\lambda_j^+(1 - \rho_j^+)}, \quad j = 1, \dots, J, \tag{3.2}$$

$$E[B^j(v)] = \frac{v}{1 - \rho_j^+}, \quad j = 1, \dots, J. \tag{3.3}$$

Let $\mathbf{Y} = (i, \mathbf{v}, \mathbf{n}, L) \in \mathcal{E}$ be a state of the system at some transition epoch of $\mathcal{Q}$. We now consider any set of customers currently in the system and denote it by $\mathcal{C} = \mathcal{C}(\mathbf{Y})$. For example, if a class j customer in k^{th} position of its queue belongs to the set, we express it as $(j, k) \in \mathcal{C}$. A set of customers who are initially in the system and are not in $\mathcal{C}$ when the system is in state $\mathbf{Y}$ is denoted by $\mathcal{C}^c = \mathcal{C}^c(\mathbf{Y}; \mathcal{C})$. We then define

$$B^0(\mathbf{Y}; \mathcal{C}) \;\equiv\; \begin{cases} \text{the first time at which current works of the customers in } \mathcal{C} \\ \text{have been completed, given that the system is initially in} \\ \text{state } \mathbf{Y} \in \mathcal{E} \text{ at some transition epoch; } j = 1, \dots, J. \end{cases} \tag{3.4}$$

$$B^j(\mathbf{Y}; \mathcal{C}) \;\equiv\; \begin{cases} \text{the first time at which all current works of the customers} \\ \text{in } \mathcal{C} \text{ have been completed and the system is cleared of} \\ \text{customers from classes 1 through } j \text{, except for the customers} \\ \text{in } \mathcal{C}^c \text{, given that the system is initially in state } \mathbf{Y} \in \mathcal{E} \\ \text{at some transition epoch; } j = 1, \dots, J. \end{cases} \tag{3.5}$$

We will call $B^j(\mathbf{Y}; \mathcal{C})$ a *class j busy period initiated with* $\{\mathbf{Y}; \mathcal{C}\}$. Then their expected values are given by the usual method [21]:

$$E[B^0(\mathbf{Y}; \mathcal{C})] \;=\; \sum_{(k,l) \in \mathcal{C}} v_{kl}, \tag{3.6}$$

$$E[B^j(\mathbf{Y};\mathcal{C})] = \frac{\sum_{(k,l)\in\mathcal{C}} E[T_{kj}(v_{kl})]}{1-\rho_j^+}, \quad j=1,\ldots,J. \tag{3.7}$$

Remark. In order to obtain the above expressions for any j ($j = 1,\ldots,J$), any system state $\mathbf{Y} \in \mathcal{E}$ and any set $\mathcal{C} = \mathcal{C}(\mathbf{Y})$ of customers initially in the system, we consider a scheduling algorithm where all customers from classes 1 through j and all customers in $\mathcal{C}$ are served *nonpreemptively* in an LCFS order until they depart from the system or leave for one of stations $j+1,\ldots,J$ and where customers from classes $j+1$ through J, except for the customers in $\mathcal{C}$, can be served only when there is not any customer from classes 1 through j in the system. Although this does not belong to the class of scheduling algorithms defined in this section, busy periods $B^j(\mathbf{Y};\mathcal{C})$ of the system with the scheduling algorithm are equivalent to those of the system with any scheduling algorithm in the class.

Let $\mathbf{Y} = (i,\mathbf{v},\mathbf{n},L) \in \mathcal{E}$ be any state of the system at some transition epoch of the process $\mathcal{Q}$ and let $\mathcal{C} = \mathcal{C}(\mathbf{Y})$ be any subset of customers initially in the system. Let $V_l^j(\mathbf{Y};\mathcal{C})$ and $N_l^j(\mathbf{Y};\mathcal{C})$ ($0 \le j < l \le J$) be, respectively, the total amount of current works and the number of customers at station l at a completion epoch of $B^j(\mathbf{Y};\mathcal{C})$. Hence, by considering the scheduling algorithm introduced in the above remark, it can be shown that these random variables are respectively sums of random variables:

$$V_{kl}^j(v_{km}) \equiv \begin{cases} \text{the total amount of current works of customers at station } l \\ \text{at a completion epoch of } B^j \text{ initiated with an exceptional} \\ \text{service } T_{kj}(v_{km}) \text{ of a class } k \text{ customer; } (k,m)\in\mathcal{C}, \end{cases} \tag{3.8}$$

$$N_{kl}^j(v_{km}) \equiv \begin{cases} \text{the number of customers at station } l \text{ at a completion epoch} \\ \text{of } B^j \text{ initiated with an exceptional service } T_{kj}(v_{km}) \text{ of} \\ \text{a class } k \text{ customer; } (k,m)\in\mathcal{C}, \end{cases} \tag{3.9}$$

($0 \le j < l \le J$), which are generated during sub-busy periods that compose $B^j(\mathbf{Y};\mathcal{C})$. Then we have

$$E[V_l^j(\mathbf{Y};\mathcal{C})] = \sum_{m\in\{m:(l,m)\notin\mathcal{C}\}} v_{lm} + \sum_{(k,m)\in\mathcal{C}} E[V_{kl}^j(v_{km})], \tag{3.10}$$

$$E[N_l^j(\mathbf{Y};\mathcal{C})] = \sum_{m\in\{m:(l,m)\notin\mathcal{C}\}} 1 + \sum_{(k,m)\in\mathcal{C}} E[N_{kl}^j(v_{km})], \tag{3.11}$$

where $0 \le j < l \le J$. An empty sum, which often occurs at $j = 0$, is defined to be 0 from now on. Let $V_{kl}^j \equiv V_{kl}^j(S_k)$ and $N_{kl}^j \equiv N_{kl}^j(S_k)$. Then by conditioning on the completion epoch of current work v, we obtain

$$E[V_{kl}^j(v)] = \lambda_l v E[S_l] + \sum_{i=1}^{j} \lambda_i v E[V_{il}^j] + p_{kl}E[S_l] + \sum_{i=1}^{j} p_{ki}E[V_{il}^j], \quad k=1,\ldots,J, \tag{3.12}$$

$$E[N_{kl}^j(v)] = \lambda_l v + \sum_{i=1}^{j} \lambda_i v E[N_{il}^j] + p_{kl} + \sum_{i=1}^{j} p_{ki}E[N_{il}^j], \quad k=1,\ldots,J, \tag{3.13}$$

where $0 \le j < l \le J$. In (3.12), the first term denotes amounts of current works that are brought by customers arrived at station l during v, the second term denotes amounts of current works that are brought during class j busy periods initiated by customers arrived at station i during v, the third term denotes a current work that is brought by the initial customer with current work v by his feedback into station l, and the fourth term denotes amounts of current works that are brought during a class j busy period initiated by the initial customer feeds back into station i. Each term of (3.13) has the similar meanings. Further, $E[V_{kl}^j]$ and $E[N_{kl}^j]$ satisfy the following equations:

$$E[V_{kl}^j] = E[S_k]\left\{\lambda_l E[S_l] + \sum_{i=1}^{j} \lambda_i E[V_{il}^j]\right\} + p_{kl}E[S_l] + \sum_{i=1}^{j} p_{ki}E[V_{il}^j], \quad k = 1, \ldots, j, \tag{3.14}$$

$$E[N_{kl}^j] = E[S_k]\left\{\lambda_l + \sum_{i=1}^{j} \lambda_i E[N_{il}^j]\right\} + p_{kl} + \sum_{i=1}^{j} p_{ki}E[N_{il}^j], \quad k = 1, \ldots, j, \tag{3.15}$$

where $1 \le j < l \le J$. These equations are easily solved by the usual techniques in vector forms under **Assumption 2.**. We now define the following constants:

$$\overline{\xi}_l^j \equiv \lambda_l E[S_l] + \sum_{i=1}^{j} \lambda_i E[V_{il}^j] = \xi_l^j E[S_l], \tag{3.16}$$

$$\overline{\chi}_{kl}^j \equiv p_{kl}E[S_l] + \sum_{i=1}^{j} p_{ki}E[V_{il}^j] = \chi_{kl}^j E[S_l], \tag{3.17}$$

$$\xi_l^j \equiv \lambda_l + \sum_{i=1}^{j} \lambda_i E[N_{il}^j], \tag{3.18}$$

$$\chi_{kl}^j \equiv p_{kl} + \sum_{i=1}^{j} p_{ki}E[N_{il}^j], \tag{3.19}$$

where $0 \le j < l \le J$ and $k = 1, \ldots, J$. For convenience, let $\xi_l^j \equiv 0$ and $\chi_{kl}^j \equiv 0$ for $j \ge l$. $E[V_{il}^j]$ and $E[N_{il}^j]$ are the solutions of equations (3.14) and (3.15).

Let $V_l^j(v)$ and $N_l^j(v)$ $(0 \le j < l \le J)$ be, respectively, the total amount of current works and the number of customers at station l at a completion epoch of $B^j(v)$. It is assumed for these variables that the initial customer with his current work v is ignored from temporary considerations after receiving his service v as if it was rejected from the system. This is because it is convenient to consider states of the system just prior to his feedback arrivals. Then we can show

$$E[V_l^j(v)] = \lambda_l v E[S_l] + \sum_{i=1}^{j} \lambda_i v E[V_{il}^j] = v\overline{\xi}_l^j, \tag{3.20}$$

$$E[N_l^j(v)] = \lambda_l v + \sum_{i=1}^{j} \lambda_i v E[N_{il}^j], = v\xi_l^j. \tag{3.21}$$

These results are summarized in the next lemma.

Lemma 1. Consider the multiclass M/G/1 system with feedback defined in Section 2.. Let $\mathbf{Y} = (i, \mathbf{v}, \mathbf{n}, L) \in \mathcal{E}$ be any state at any transition epoch of the process $\mathcal{Q}$ and let $\mathcal{C} = \mathcal{C}(\mathbf{Y})$ be an arbitrary set of customers initially in the system. Then we have

$$E[V_l^j(\mathbf{Y};\mathcal{C})] = \sum_{m\in\{m:(l,m)\notin\mathcal{C}\}} v_{lm} + \sum_{(k,m)\in\mathcal{C}} \left\{v_{km}\overline{\xi}_l^j + \overline{\chi}_{kl}^j\right\}, \tag{3.22}$$

$$E[N_l^j(\mathbf{Y};\mathcal{C})] = \sum_{m\in\{m:(l,m)\notin\mathcal{C}\}} 1 + \sum_{(k,m)\in\mathcal{C}} \left\{v_{km}\xi_l^j + \chi_{kl}^j\right\}, \tag{3.23}$$

$$E[V_l^j(v)] = v\overline{\xi}_l^j, \tag{3.24}$$

$$E[N_l^j(v)] = v\xi_l^j, \tag{3.25}$$

where $j = 0, 1, \dots, J$ and $l = 1, \dots, J$. The constants ξ_l^j, χ_{kl}^j and $\overline{\xi}_l^j, \overline{\chi}_{kl}^j$ are given by (3.16)–(3.19).
□

These results are used to obtain expected values of system states after every initial stay of a tagged customer.

4. Initial cost functions.

In this section we derive the initial cost functions $W_j^I(\cdot)$ and $G_j^I(\cdot)$ of an e^{th} customer ($j = 1, \dots, J$). Since every scheduling algorithm has its own cost functions, we consider them individually for the stations with two specific service disciplines:

FCFS disciplines.

We first derive them for an e^{th} customer arrived at station i with an FCFS discipline ($i = 1, \dots, J$ and $e = 1, 2, \dots$). We distinguish the station with the FCFS discipline by a superscript F. Let $\mathbf{Y}(\sigma_l^e) = \mathbf{Y} = (i, \mathbf{v}, \mathbf{n}, L) \in \mathcal{E}$ be any state of the system at σ_l^e ($l = 0, 1, \dots$). The set $\mathcal{C}_i^F$ of customers is composed of customers from classes 1 through i (except for the e^{th} customer) who are in the system at time σ_l^e. The initial sojourn time of the class i customer is composed of a class $i-1$ busy period initiated with $\{\mathbf{Y}; \mathcal{C}_i^F\}$, and a class $i-1$ busy period initiated with the e^{th} customer's service S_i, regardless of the disciplines adopted by stations $1, \dots, i-1$. Hence,

$$W_j^I(\mathbf{Y}, e, l) = \begin{cases} E[B^{i-1}(\mathbf{Y}; \mathcal{C}_i^F) + B^{i-1}(S_i)], & j = i, \\ 0, & j \neq i, \end{cases} \qquad j = 1, \dots, J. \tag{4.1}$$

As we have shown in the last section, the expected value is

$$W_i^I(\mathbf{Y}, e, l) = \sum_{k=1}^{i} \sum_{m=1}^{n_k} \frac{E[T_{k,i-1}(v_{km})]}{1 - \rho_{i-1}^+} + \frac{E[S_i]}{1 - \rho_{i-1}^+}, \tag{4.2}$$

$$W_j^I(\mathbf{Y}, e, l) = 0, \quad j = 1, \dots, J \text{ and } j \neq i. \tag{4.3}$$

On the other hand, cumulative current works of customers gradually decrease to 0. Then we carefully calculate the expected value. Let us consider a tagged customer who arrives at station i. For the station with a FCFS discipline, his current work is equal to S_i until his service begins. Then it decreases at the rate of second by second until he is preempted by an arriving customer from classes 1 through $i-1$. His current work keeps its last level. After completing the class $i-1$ busy period initiated by the arriving customer, it then decreases at the rate of second by second until he is preempted by another arriving customer from classes 1 through $i-1$, and so on. Let $S_i(k)$ ($k = 1, 2, \dots$) denote his attained service time on his k^{th} preemption. We obtain the value of the cost function $G_i^I(\mathbf{Y}, e, l)$ by conditioning on service time S_i of the e^{th} customer, on the number $N_P^i \equiv \sum_{k=1}^{i-1} A_k(S_i)$ of customers who preempt him and on time $S_i(k), k = 1, \dots, N_P^i$. Then we obtain

$$\begin{aligned} & G_i^I(\mathbf{Y}, e, l | S_i, N_P^i = m, \{S_i(k)\}) \\ & = S_i \left\{ E[B^{i-1}(\mathbf{Y}; \mathcal{C}_i^F)] \right\} + \sum_{k=1}^{m} (S_i - S_i(k)) E[B_k^{i-1}] + \frac{S_i^2}{2} \end{aligned} \tag{4.4}$$

where B_k^{i-1} is the k^{th} class $i-1$ busy period. By the nature of the Poisson process, $\{S_i(k)\}$ have the same distribution as the order statistics corresponding to m independent random variables uniformly distributed on the interval S_i [16]. Hence we obtain

$$E[S_i(k) | S_i, N_P^i = m] = \frac{k}{m+1} S_i, \quad k = 1, \dots, m. \tag{4.5}$$

Then we have

$$G_i^I(\mathbf{Y},e,l) = E[S_i]\sum_{k=1}^{i}\sum_{m=1}^{n_k}\frac{E[T_{k,i-1}(v_{km})]}{1-\rho_{i-1}^+} + \frac{1}{2}\lambda_{i-1}^+E[S_i^2]E[B^{i-1}] + \frac{E[S_i^2]}{2}, \quad (4.6)$$

$$G_j^I(\mathbf{Y},e,l) = 0, \quad j=1,\ldots,J \text{ and } j\neq i. \quad (4.7)$$

PR-LCFS disciplines.

Second, we derive the mean initial sojourn time of an e^{th} customer arrived at station i with a PR-LCFS discipline ($i = 1,\ldots,J$ and $e = 1,2,\ldots$). We distinguish the station with the PR-LCFS discipline by a superscript PL. Let $\mathbf{Y}(\sigma_l^e) = \mathbf{Y} = (i,\mathbf{v},\mathbf{n},L) \in \mathcal{E}$ be any state of the system at σ_l^e ($l = 0,1,\ldots$). The set $\mathcal{C}_i^{PL}$ of customers is composed of customers from classes 1 through $i-1$ who are initially in the system ($\mathcal{C}_1^{PL} = \emptyset$). Similarly, as in the FCFS case, the cost function W_j^I can be obtained by

$$W_i^I(\mathbf{Y},e,l) = E[B^i(\mathbf{Y};\mathcal{C}_i^{PL}) + B^i(S_i)] \quad (4.8)$$

$$= \sum_{k=1}^{i-1}\sum_{m=1}^{n_k}\frac{E[T_{ki}(v_{km})]}{1-\rho_i^+} + \frac{E[S_i]}{1-\rho_i^+}, \quad (4.9)$$

$$W_j^I(\mathbf{Y},e,l) = 0, \quad j=1,\ldots,J \text{ and } j\neq i. \quad (4.10)$$

Further we have

$$G_i^I(\mathbf{Y},e,l) = E[S_i]\sum_{k=1}^{i-1}\sum_{m=1}^{n_k}\frac{E[T_{ki}(v_{km})]}{1-\rho_i^+} + \frac{1}{2}\lambda_i^+E[S_i^2]E[B^i] + \frac{E[S_i^2]}{2}, \quad (4.11)$$

$$G_j^I(\mathbf{Y},e,l) = 0, \quad j=1,\ldots,J \text{ and } j\neq i. \quad (4.12)$$

General forms of the initial cost functions.

As we may see from the above expressions, the initial sojourn times and the initial cumulative current works are linear combinations of some components of states. We summarize it in the following lemma.

Lemma 2. Consider the multiclass M/G/1 system with feedback defined in Section 2.. By appropriately choosing nonnegative constants ϕ_{lk}^i, η_{lk}^i, w^i and g^i ($i,k=1,\ldots,J; l=1,2$), we can obtain the following expressions of cost functions W_i^I and G_i^I defined by (2.5) and (2.8), respectively.

$$W_i^I(\mathbf{Y},e,l) = \sum_{k=1}^{J}\{\phi_{1k}^i\sum_{m=1}^{n_k}v_{km} + \phi_{2k}^i n_k\} + w^i, \quad (4.13)$$

$$G_i^I(\mathbf{Y},e,l) = \sum_{k=1}^{J}\{\eta_{1k}^i\sum_{m=1}^{n_k}v_{km} + \eta_{2k}^i n_k\} + g^i, \quad (4.14)$$

for $\mathbf{Y} = (i,\mathbf{v},\mathbf{n},L) \in \mathcal{E}$, $e = 1,2,\ldots$ and $l = 0,1,\ldots$.

Proof. In the case of FCFS disciplines, we define

$$\phi_{1k}^i \equiv \begin{cases} 1/(1-\rho_{i-1}^+), & k=1,\ldots,i, \\ 0, & k=i+1,\ldots,J, \end{cases}$$

$$\phi_{2k}^i \equiv \begin{cases} \sum_{l=1}^{i-1}p_{kl}E[T_{l,i-1}]/(1-\rho_{i-1}^+), & k=1,\ldots,i, \\ 0, & k=i+1,\ldots,J, \end{cases}$$

$$w^i \equiv E[S_i]/(1-\rho_{i-1}^+),$$

$$\eta^i_{1k} \equiv \begin{cases} E[S_i]/(1-\rho^+_{i-1}), & k=1,\ldots,i, \\ 0, & k=i+1,\ldots,J, \end{cases}$$

$$\eta^i_{2k} \equiv \begin{cases} E[S_i]\sum_{l=1}^{i-1} p_{kl}E[T_{l,i-1}]/(1-\rho^+_{i-1}), & k=1,\ldots,i, \\ 0, & k=i+1,\ldots,J, \end{cases}$$

$$g^i \equiv E[S_i^2](\lambda^+_{i-1}E[B^{i-1}]+1)/2 = E[S_i^2]/\{2(1-\rho^+_{i-1})\},$$

where $i=1,\ldots,J$. The expected values $E[T_{l,i-1}]$ and $E[B^{i-1}]$ are given by (2.2) and (3.2). Then we can show (4.13) and (4.14) from (2.1), (4.2) and (4.6). In the same manner, we can define these coefficients for the stations with PR-LCFS disciplines. □

For notational convenience, we define the following vectors:

$$\mathbf{w}^i \equiv (\phi^i_{11},\ldots,\phi^i_{1J},\phi^i_{21},\ldots,\phi^i_{2J})' \in \mathcal{R}^{2J\times 1}, \quad i=1,\ldots,J, \tag{4.15}$$

$$\mathbf{g}^i \equiv (\eta^i_{11},\ldots,\eta^i_{1J},\eta^i_{21},\ldots,\eta^i_{2J})' \in \mathcal{R}^{2J\times 1}, \quad i=1,\ldots,J, \tag{4.16}$$

where $'$ denotes a transposition of a matrix. Then expressions (4.13) and (4.14) are generally arranged as follows:

$$W^I_j(\mathbf{Y},e,l) = \begin{cases} (\mathbf{v},\mathbf{n})\mathbf{w}^i + w^i, & j=i, \\ 0, & j\neq i, \end{cases} \quad j=1,\ldots,J, \tag{4.17}$$

$$G^I_j(\mathbf{Y},e,l) = \begin{cases} (\mathbf{v},\mathbf{n})\mathbf{g}^i + g^i, & j=i, \\ 0, & j\neq i, \end{cases} \quad j=1,\ldots,J, \tag{4.18}$$

where $\mathbf{Y}=(i,\mathbf{v},\mathbf{n},L)\in\mathcal{E}$, $e=1,2,\ldots$ and $l=0,1,\ldots$. The expressions will be cited later to derive the cost functions $W_j(\cdot)$ and $G_j(\cdot)$.

The important things to consider about these expressions are: first that elements $X(\sigma^e_l)$, $\mathbf{v}(\sigma^e_l)$ and $\mathbf{n}(\sigma^e_l)$ of state of the system should be sufficient for estimating the cost functions $W^I_j(\mathbf{Y},e,l)$ and $G^I_j(\mathbf{Y},e,l)$, and, second that these cost functions should be linear functions of components $(\mathbf{v},\mathbf{n})$ of the state of the system. Of course, every scheduling algorithm has its own coefficients $\mathbf{w}^i,\mathbf{g}^i,w^i$ and g^i $(i=1,\ldots,J)$.

5. States at completion epochs of initial stays.

In this section we derive expected values of states of the system at completion epochs of customer's initial stays.

FCFS disciplines.

First we consider that an e^{th} customer arrives station i with an FCFS discipline ($e=1,2,\ldots$ and $i=1,\ldots,J$). We recall the relation (4.1) between an initial sojourn time and busy periods. Then the expected values of elements of states of the system at completion epochs of e^{th} customer's initial stays are obtained by

$$E^F[v_j(\sigma^e_{l+1})|\mathbf{Y}(\sigma^e_l)=\mathbf{Y}] = \begin{cases} E[V^{i-1}_j(\mathbf{Y};\mathcal{C}^F_i)+V^{i-1}_j(S_i)], & i\neq 0, \\ 0, & i=0, \end{cases} \quad j=1,\ldots,J, \tag{5.1}$$

$$E^F[n_j(\sigma^e_{l+1})|\mathbf{Y}(\sigma^e_l)=\mathbf{Y}] = \begin{cases} E[N^{i-1}_j(\mathbf{Y};\mathcal{C}^F_i)+N^{i-1}_j(S_i)], & i\neq 0, \\ 0, & i=0, \end{cases} \quad j=1,\ldots,J, \tag{5.2}$$

for $\mathbf{Y}=(i,\mathbf{v},\mathbf{n},L)\in\mathcal{E}$, $l=0,1,\ldots$ and $e=1,2,\ldots$.

PR-LCFS disciplines.

Next, we consider station i with PR-LCFS discipline ($i = 1, \ldots, J$). We also recall the relation (4.8). Then the expected values of elements of states of the system at completion epochs of e^{th} customer's initial stays are obtained by

$$E^{PL}[v_j(\sigma_{l+1}^e)|\mathbf{Y}(\sigma_l^e) = \mathbf{Y}] = \begin{cases} E[V_j^i(\mathbf{Y}; \mathcal{C}_i^{PL}) + V_j^i(S_i)], & i \neq 0, \\ 0, & i = 0, \end{cases} \quad j = 1, \ldots, J, \tag{5.3}$$

$$E^{PL}[n_j(\sigma_{l+1}^e)|\mathbf{Y}(\sigma_l^e) = \mathbf{Y}] = \begin{cases} E[N_j^i(\mathbf{Y}; \mathcal{C}_i^{PL}) + N_j^i(S_i)], & i \neq 0, \\ 0, & i = 0, \end{cases} \quad j = 1, \ldots, J, \tag{5.4}$$

for $\mathbf{Y} = (i, \mathbf{v}, \mathbf{n}, L) \in \mathcal{E}$, $l = 0, 1, \ldots$ and $e = 1, 2, \ldots$.

General forms of the quantities.

As we may see from the above expressions, expected values of states at completion epochs of customer's initial stays are linear combinations of some components of the state of the system. We summarize it in the following lemma:

Lemma 3. Consider the multiclass M/G/1 system with feedback defined in Section 2.. By appropriately choosing nonnegative constants $\overline{\alpha}_{kj}^i$, $\overline{\beta}_{kj}^i$, $\overline{\gamma}_j^i$, and α_{kj}^i, β_{kj}^i, γ_j^i, ($i = 0, 1, \ldots, J$ and $k, j = 1, \ldots, J$), we can obtain the following expressions:

$$E[v_j(\sigma_{l+1}^e)|\mathbf{Y}(\sigma_l^e) = \mathbf{Y}] = \sum_{k=1}^{J} \{\overline{\alpha}_{kj}^i v_k + \overline{\beta}_{kj}^i n_k\} + \overline{\gamma}_j^i, \quad j = 1, \ldots, J, \tag{5.5}$$

$$E[n_j(\sigma_{l+1}^e)|\mathbf{Y}(\sigma_l^e) = \mathbf{Y}] = \sum_{k=1}^{J} \{\alpha_{kj}^i v_k + \beta_{kj}^i n_k\} + \gamma_j^i, \quad j = 1, \ldots, J, \tag{5.6}$$

for $\mathbf{Y} = (i, \mathbf{v}, \mathbf{n}, L) \in \mathcal{E}$, $l = 0, 1, \ldots$ and $e = 1, 2, \ldots$.

Proof. In the case of a FCFS discipline, we can obtain from (3.22), (3.23), (3.24) and (3.25):

$$E[v_j(\sigma_{l+1}^e)|\mathbf{Y}(\sigma_l^e) = \mathbf{Y}] = E[V_j^{i-1}(\mathbf{Y}; \mathcal{C}_i^F) + V_j^{i-1}(S_i)]$$
$$= \begin{cases} 0, & j = 1, \ldots, i-1, \\ \sum_{k=1}^{i} \left\{\overline{\xi}_i^{i-1} v_k + \overline{\chi}_{ki}^{i-1} n_k\right\} + \overline{\xi}_i^{i-1} E[S_i], & j = i, \\ v_j + \sum_{k=1}^{i} \left\{\overline{\xi}_j^{i-1} v_k + \overline{\chi}_{kj}^{i-1} n_k\right\} + \overline{\xi}_j^{i-1} E[S_i], & j = i+1, \ldots, J, \end{cases} \tag{5.7}$$

$$E[n_j(\sigma_{l+1}^e)|\mathbf{Y}(\sigma_l^e) = \mathbf{Y}] = E[N_j^{i-1}(\mathbf{Y}; \mathcal{C}_i^F) + N_j^{i-1}(S_i)]$$
$$= \begin{cases} 0, & j = 1, \ldots, i-1, \\ \sum_{k=1}^{i} \left\{\xi_i^{i-1} v_k + \chi_{ki}^{i-1} n_k\right\} + \xi_i^{i-1} E[S_i], & j = i, \\ n_j + \sum_{k=1}^{i} \left\{\xi_j^{i-1} v_k + \chi_{kj}^{i-1} n_k\right\} + \xi_j^{i-1} E[S_i], & j = i+1, \ldots, J, \end{cases} \tag{5.8}$$

where $i = 1, \ldots, J$. For $i = 0$, all coefficients are defined to be 0. Then we can show that eqs. (5.5) and (5.6) hold. Similar results hold for PR-LCFS disciplines. □

For notational conveniences, we define following matrices and vectors.

$$\begin{array}{ll} \overline{\mathbf{A}}^i \equiv (\overline{\alpha}_{kj}^i : k, j = 1, \ldots, J) \in \mathcal{R}^{J \times J}, & \overline{\mathbf{B}}^i \equiv (\overline{\beta}_{kj}^i : k, j = 1, \ldots, J) \in \mathcal{R}^{J \times J}, \\ \mathbf{A}^i \equiv (\alpha_{kj}^i : k, j = 1, \ldots, J) \in \mathcal{R}^{J \times J}, & \mathbf{B}^i \equiv (\beta_{kj}^i : k, j = 1, \ldots, J) \in \mathcal{R}^{J \times J}, \\ \overline{\gamma}^i \equiv (\overline{\gamma}_1^i, \ldots, \overline{\gamma}_J^i), & \gamma^i \equiv (\gamma_1^i, \ldots, \gamma_J^i), \end{array} \tag{5.9}$$

where $i = 1, \ldots, J$. We further arrange these matrices and these vectors as follows:

$$\mathbf{U}^i \equiv \begin{pmatrix} \overline{\mathbf{A}}^i & \mathbf{A}^i \\ \overline{\mathbf{B}}^i & \mathbf{B}^i \end{pmatrix}, \quad i = 1, \ldots, J, \quad \text{and } \mathbf{U}^0 \equiv \mathbf{O}, \tag{5.10}$$

$$\mathbf{u}^i \equiv (\overline{\gamma}^i, \gamma^i), \quad i = 1, \ldots, J, \quad \text{and } \mathbf{u}^0 \equiv \mathbf{0}. \tag{5.11}$$

Therefore, expressions (5.5) and (5.6) can be generally arranged as follows:

$$E[(\mathbf{v}(\sigma_{l+1}^e), \mathbf{n}(\sigma_{l+1}^e)) | \mathbf{Y}(\sigma_l^e) = \mathbf{Y}] = (\mathbf{v}, \mathbf{n})\mathbf{U}^i + \mathbf{u}^i \in \mathcal{R}^{1\times 2J}, \tag{5.12}$$

for $\mathbf{Y} = (i, \mathbf{v}, \mathbf{n}, L) \in \mathcal{E}$, $l = 0, 1, \ldots$ and $e = 1, 2, \ldots$. These expressions will be cited later when we derive the cost functions.

Similarly, as in the initial cost functions, the important things to consider about these expressions are: first that $X(\sigma_l^e)$, $\mathbf{v}(\sigma_l^e)$ and $\mathbf{n}(\sigma_l^e)$ should be sufficient for estimating expected values of elements $\mathbf{v}(\sigma_{l+1}^e)$ and $\mathbf{n}(\sigma_{l+1}^e)$ of states and, second that they should be a linear function of $\mathbf{v}(\sigma_l^e)$ and $\mathbf{n}(\sigma_l^e)$. Of course, every scheduling algorithm has its own coefficients $\mathbf{U}^i$ and $\mathbf{u}^i$ $(i = 1, \ldots, J)$.

6. Expressions of the cost functions.

In this section and the next section, we derive the explicit formulae of the cost functions $W_j(\cdot)$ and $G_j(\cdot)$ under some assumptions. Let us consider the system operated under some fixed scheduling algorithm. As we have defined, $W_j(\mathbf{Y}, e, l)$ is the mean sojourn time of an e^{th} customer, who arrives at station i at time σ_l^e just when the system is in state $\mathbf{Y} = (i, \mathbf{v}, \mathbf{n}, L) \in \mathcal{E}$, spent at station j until he departs from the system, and $G_j(\mathbf{Y}, e, l)$ is the expected value of cumulative current works of the customer accumulated at station j until he departs from the system $(j = 1, \ldots, J)$.

We make the expressions, which are derived from the analysis in the previous sections, as a set of assumptions.

Assumption 6..

$$W_j^I(\mathbf{Y}, e, l) = \begin{cases} (\mathbf{v}, \mathbf{n})\mathbf{w}^i + w^i, & j = i, \\ 0, & j \neq i, \end{cases} \quad j = 1, \ldots, J, \tag{6.1}$$

$$G_j^I(\mathbf{Y}, e, l) = \begin{cases} (\mathbf{v}, \mathbf{n})\mathbf{g}^i + g^i, & j = i, \\ 0, & j \neq i, \end{cases} \quad j = 1, \ldots, J, \tag{6.2}$$

$$E[(\mathbf{v}(\sigma_{l+1}^e), \mathbf{n}(\sigma_{l+1}^e)) | \mathbf{Y}(\sigma_l^e) = \mathbf{Y}] = (\mathbf{v}, \mathbf{n})\mathbf{U}^i + \mathbf{u}^i, \tag{6.3}$$

hold for $\mathbf{Y} = (i, \mathbf{v}, \mathbf{n}, L) \in \mathcal{E}$, $l = 0, 1, \ldots$ and $e = 1, 2, \ldots$ where $\mathbf{w}^i, \mathbf{g}^i \geq \mathbf{0}$ and $w^i, g^i \geq 0$ is given in Section 4. and where $\mathbf{U}^i \geq \mathbf{O}$ and $\mathbf{u}^i \geq \mathbf{0}$ are given in Section 5.. □

Of course, the assumption is satisfied by priority queueing systems with FCFS disciplines and PR-LCFS disciplines analyzed so far.

Let $J_2 \equiv 2J^2$ and define the following matrice:

$$\begin{aligned}
\phi_0^j &\equiv (\mathbf{0}, \ldots, \mathbf{0}, \mathbf{w}^{j\prime}, \mathbf{0}, \ldots, \mathbf{0})' \quad \in \mathcal{R}^{J_2 \times 1}, \\
\eta_0^j &\equiv (\mathbf{0}, \ldots, \mathbf{0}, \mathbf{g}^{j\prime}, \mathbf{0}, \ldots, \mathbf{0})' \quad \in \mathcal{R}^{J_2 \times 1}, \\
\mathbf{I}_0 &\equiv \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \in \mathcal{R}^{2J \times 2J}, \\
\mathbf{Q} &\equiv \begin{pmatrix} p_{11}\mathbf{I}_0 & p_{12}\mathbf{I}_0 & \cdots & p_{1J}\mathbf{I}_0 \\ p_{21}\mathbf{I}_0 & p_{22}\mathbf{I}_0 & \cdots & p_{2J}\mathbf{I}_0 \\ \vdots & \vdots & \ddots & \vdots \\ p_{J1}\mathbf{I}_0 & p_{J2}\mathbf{I}_0 & \cdots & p_{JJ}\mathbf{I}_0 \end{pmatrix} \in \mathcal{R}^{J_2 \times J_2}, \\
\mathbf{U} &\equiv \begin{pmatrix} \mathbf{U}^1 & 0 & \cdots & 0 \\ 0 & \mathbf{U}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{U}^J \end{pmatrix} \in \mathcal{R}^{J_2 \times J_2},
\end{aligned}$$

where $'$ denotes transposition of a vector. Then we have

$$\mathbf{UQ} = \begin{pmatrix} \mathbf{U}^1 p_{11} & \mathbf{U}^1 p_{12} & \cdots & \mathbf{U}^1 p_{1J} \\ \mathbf{U}^2 p_{21} & \mathbf{U}^2 p_{22} & \cdots & \mathbf{U}^2 p_{2J} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{U}^J p_{J1} & \mathbf{U}^J p_{J2} & \cdots & \mathbf{U}^J p_{JJ} \end{pmatrix} \in \mathcal{R}^{J_2 \times J_2}. \tag{6.4}$$

We now suppose $(\mathbf{I} - \mathbf{UQ})^{-1}$ exists, where $\mathbf{I}$ is an identity matrix in $\mathcal{R}^{J_2 \times J_2}$. Then we can define

$$\begin{pmatrix} \mathbf{w}_{1j} \\ \vdots \\ \mathbf{w}_{Jj} \end{pmatrix} \equiv (\mathbf{I} - \mathbf{UQ})^{-1} \phi_0^j \in \mathcal{R}^{J_2 \times 1}, \tag{6.5}$$

$$\begin{pmatrix} \mathbf{g}_{1j} \\ \vdots \\ \mathbf{g}_{Jj} \end{pmatrix} \equiv (\mathbf{I} - \mathbf{UQ})^{-1} \eta_0^j \in \mathcal{R}^{J_2 \times 1}, \tag{6.6}$$

and

$$\mathbf{w}_{0j} \equiv \mathbf{0}, \tag{6.7}$$

$$\mathbf{g}_{0j} \equiv \mathbf{0}. \tag{6.8}$$

Further we define

$$\phi_1^j \equiv \begin{pmatrix} \mathbf{u}^1 \sum_{k=1}^J p_{1k} \mathbf{w}_{kj} \\ \vdots \\ \mathbf{u}^{j-1} \sum_{k=1}^J p_{j-1k} \mathbf{w}_{kj} \\ w^j + \mathbf{u}^j \sum_{k=1}^J p_{jk} \mathbf{w}_{kj} \\ \mathbf{u}^{j+1} \sum_{k=1}^J p_{j+1k} \mathbf{w}_{kj} \\ \vdots \\ \mathbf{u}^J \sum_{k=1}^J p_{Jk} \mathbf{w}_{kj} \end{pmatrix} \in \mathcal{R}^{J \times 1}, \quad \eta_1^j \equiv \begin{pmatrix} \mathbf{u}^1 \sum_{k=1}^J p_{1k} \mathbf{g}_{kj} \\ \vdots \\ \mathbf{u}^{j-1} \sum_{k=1}^J p_{j-1k} \mathbf{g}_{kj} \\ g^j + \mathbf{u}^j \sum_{k=1}^J p_{jk} \mathbf{g}_{kj} \\ \mathbf{u}^{j+1} \sum_{k=1}^J p_{j+1k} \mathbf{g}_{kj} \\ \vdots \\ \mathbf{u}^J \sum_{k=1}^J p_{Jk} \mathbf{g}_{kj} \end{pmatrix} \in \mathcal{R}^{J \times 1}.$$

From **Assumption 2.**, $(\mathbf{I} - \mathbf{P}_J)^{-1}$ exists. Then we can define

$$\begin{pmatrix} w_{1j} \\ \vdots \\ w_{Jj} \end{pmatrix} \equiv (\mathbf{I} - \mathbf{P}_J)^{-1} \phi_1^j \in \mathcal{R}^{J \times 1}, \tag{6.9}$$

$$\begin{pmatrix} g_{1j} \\ \vdots \\ g_{Jj} \end{pmatrix} \equiv (\mathbf{I} - \mathbf{P}_J)^{-1} \eta_1^j \in \mathcal{R}^{J \times 1}, \tag{6.10}$$

and

$$w_{0j} \equiv 0, \tag{6.11}$$

$$g_{0j} \equiv 0. \tag{6.12}$$

As stated in the previous sections, every scheduling algorithm has its own coefficients $\mathbf{w}^i, w^i$, $\mathbf{g}^i$ and g^i. So these vectors and matrices are also different for all scheduling algorithms.

The following theorem is now derived:

Theorem 1. We make **Assumption 2.** and **Assumption 6.**. Further, if we assume that $(\mathbf{I}-\mathbf{UQ})^{-1}$ exists, then for any $j = 1, \ldots, J$,

$$\begin{aligned} \hat{W}_j(\mathbf{Y}, e, l) &\equiv W_{ij}(\mathbf{v}, \mathbf{n}) \equiv (\mathbf{v}, \mathbf{n})\mathbf{w}_{ij} + w_{ij}, & (6.13)\\ \hat{G}_j(\mathbf{Y}, e, l) &\equiv G_{ij}(\mathbf{v}, \mathbf{n}) \equiv (\mathbf{v}, \mathbf{n})\mathbf{g}_{ij} + g_{ij}, & (6.14) \end{aligned}$$

for $\mathbf{Y} = (i, \mathbf{v}, \mathbf{n}, L) \in \mathcal{E}$, $l = 0, 1, \ldots$ and $e = 1, 2, \ldots$ are solutions of equations (2.6) and (2.9), respectively.

Proof. We show that (6.13) and (6.14) satisfy equations (2.6) and (2.9), respectively. For $\mathbf{Y} = (i, \mathbf{v}, \mathbf{n}, L) \in \mathcal{E}$ and $j \neq i$,

$$\begin{aligned} & E[\hat{W}_j(\mathbf{Y}(\sigma_{l+1}^e), e, l+1)|\mathbf{Y}] \\ = & \sum_{k=1}^{J} p_{ik} E[(\mathbf{v}(\sigma_{l+1}^e), \mathbf{n}(\sigma_{l+1}^e))\mathbf{w}_{kj} + w_{kj}|\mathbf{Y}] \\ = & \sum_{k=1}^{J} p_{ik} \left[E[(\mathbf{v}(\sigma_{l+1}^e), \mathbf{n}(\sigma_{l+1}^e))|\mathbf{Y}]\mathbf{w}_{kj} + w_{kj}\right] \qquad (6.15)\\ = & \sum_{k=1}^{J} p_{ik} \left[\left\{(\mathbf{v}, \mathbf{n})\mathbf{U}^i + \mathbf{u}^i\right\} \mathbf{w}_{kj} + w_{kj}\right] \\ = & (\mathbf{v}, \mathbf{n})\mathbf{U}^i \sum_{k=1}^{J} p_{ik}\mathbf{w}_{kj} + \left\{\mathbf{u}^i \sum_{k=1}^{J} p_{ik}\mathbf{w}_{kj} + \sum_{k=1}^{J} p_{ik} w_{kj}\right\} \\ = & \hat{W}_j(\mathbf{Y}, e, l). \end{aligned}$$

The last equation follows from the definition of the constants $\mathbf{w}_{ij}$ and w_{ij}, that is,

$$\begin{aligned} \mathbf{w}_{ij} &= \mathbf{U}^i \sum_{k=1}^{J} p_{ik}\mathbf{w}_{kj}, & (6.16)\\ w_{ij} &= \mathbf{u}^i \sum_{k=1}^{J} p_{ik}\mathbf{w}_{kj} + \sum_{k=1}^{J} p_{ik} w_{kj}. & (6.17) \end{aligned}$$

Hence, $\hat{W}_j(\mathbf{Y}, e, l)$ satisfies equation (2.6). In the same manner, we can show that $\hat{G}_j(\mathbf{Y}, e, l)$ satisfies equation (2.9) for $i \neq j$.

For $\mathbf{Y} = (i, \mathbf{v}, \mathbf{n}, L) \in \mathcal{E}$ and $j = i$,

$$\begin{aligned} & W_i^I(\mathbf{Y}, e, l) + E[\hat{W}_i(\mathbf{Y}(\sigma_{l+1}^e), e, l+1)|\mathbf{Y}] \\ = & (\mathbf{v}, \mathbf{n})\mathbf{w}^i + w^i + \sum_{k=1}^{J} p_{ik} E[(\mathbf{v}(\sigma_{l+1}^e), \mathbf{n}(\sigma_{l+1}^e))\mathbf{w}_{ki} + w_{ki}|\mathbf{Y}] \\ = & (\mathbf{v}, \mathbf{n})\mathbf{w}^i + w^i + \sum_{k=1}^{J} p_{ik} \left\{E[(\mathbf{v}(\sigma_{l+1}^e), \mathbf{n}(\sigma_{l+1}^e))|\mathbf{Y}]\mathbf{w}_{ki} + w_{ki}\right\} \qquad (6.18)\\ = & (\mathbf{v}, \mathbf{n})\mathbf{w}^i + w^i + \sum_{k=1}^{J} p_{ik} \left[\left\{(\mathbf{v}, \mathbf{n})\mathbf{U}^i + \mathbf{u}^i\right\} \mathbf{w}_{ki} + w_{ki}\right] \\ = & (\mathbf{v}, \mathbf{n})\left\{\mathbf{w}^i + \mathbf{U}^i \sum_{k=1}^{J} p_{ik}\mathbf{w}_{ki}\right\} + \left\{w^i + \mathbf{u}^i \sum_{k=1}^{J} p_{ik}\mathbf{w}_{ki} + \sum_{k=1}^{J} p_{ik} w_{ki}\right\} \\ = & \hat{W}_i(\mathbf{Y}, e, l). \end{aligned}$$

The last equation follows from the definition of the constants $\mathbf{w}_{ii}$ and w_{ii}, that is,

$$\mathbf{w}_{ii} = \mathbf{w}^i + \mathbf{U}^i \sum_{k=1}^{J} p_{ik}\mathbf{w}_{ki}, \tag{6.19}$$

$$w_{ii} = w^i + \mathbf{u}^i \sum_{k=1}^{J} p_{ik}\mathbf{w}_{ki} + \sum_{k=1}^{J} p_{ik}w_{ki}. \tag{6.20}$$

Hence, $\hat{W}_i(\mathbf{Y}, e, l)$ satisfies equation (2.6). In the same manner, we can show that $\hat{G}_i(\mathbf{Y}, e, l)$ satisfies equation (2.9). □

7. Uniqueness of the solutions.

In Theorem 1, we have obtained the solutions $\hat{W}_j$ and $\hat{G}_j$ that satisfy equations (2.6) and (2.9), respectively ($j = 1, \ldots, J$). Now we prove uniqueness of these solutions under appropriate assumptions.

Since the following set of assumptions are often used, we arrange them as

Assumption 7..

- **Assumption 2.** and **Assumption 6.** hold, and
- $(\mathbf{UQ})^m \to \mathbf{O}$ as $m \to \infty$ where $\mathbf{UQ}$ is the state transition matrix given by (6.4). □

The purpose of this section is to prove the following theorem:

Theorem 2. Let W_j and G_j be the cost functions defined by (2.4) and (2.7), respectively ($j = 1, \ldots, J$). We make **Assumption 7..** Then, W_j and G_j are equivalent to $\hat{W}_j$ and $\hat{G}_j$ defined by (6.13) and (6.14), respectively. □

As a first step, let $\mathcal{V}$ be a set of functions defined on $\mathcal{E} \times \{1, 2, \ldots\} \times \mathcal{I}_+$ into $\mathcal{R}$. Then a set of functions is defined as follows:

$$\mathcal{V}_B \equiv \left\{ f \in \mathcal{V} : \begin{array}{l} \text{There exist } \mathbf{c}_f \geq \mathbf{0} \text{ in } \mathcal{R}^{2J\times 1} \text{ and } c_f \geq 0 \text{ in } \mathcal{R} \text{ such that} \\ \quad |f(\mathbf{Y}, e, l)| \leq (\mathbf{v}, \mathbf{n})\mathbf{c}_f + c_f, \\ \text{for all } \mathbf{Y} = (i, \mathbf{v}, \mathbf{n}, L) \in \mathcal{E}, e \in \{1, 2, \ldots\} \text{ and } l \in \mathcal{I}_+. \text{ Moreover,} \\ \quad f(\mathbf{Y}_0, e, l) = 0, \\ \text{for all } \mathbf{Y}_0 = (0, \mathbf{v}, \mathbf{n}, L) \in \mathcal{E}, e \in \{1, 2, \ldots\} \text{ and } l \in \mathcal{I}_+. \end{array} \right\}. \tag{7.1}$$

By introducing addition and multiplication by scalars into $\mathcal{V}_B$ as usual, it becomes a vector space [17]. Then we arrange equations (2.6) and (2.9) as follows: for any $f^I \in \mathcal{V}$, we consider a set of equations:

$$f(\mathbf{Y}, e, l) = f^I(\mathbf{Y}, e, l) + E[f(\mathbf{Y}(\sigma^e_{l+1}), e, l+1) | \mathbf{Y}(\sigma^e_l) = \mathbf{Y}], \tag{7.2}$$

for all $\mathbf{Y} = (i, \mathbf{v}, \mathbf{n}, L) \in \mathcal{E}$, $e \in \{1, 2, \ldots\}$ and $l \in \mathcal{I}_+$, where f is an unknown function in $\mathcal{V}$. Let

$$Z^e_{Cl} \equiv \begin{cases} 1, & \text{if } X(\sigma^e_l) \neq 0, \\ 0, & \text{otherwise,} \end{cases} \tag{7.3}$$

for $l = 0, 1, \ldots$ and $e = 1, 2, \ldots$.

As a second step, we need the following lemmas.

Lemma 4. We make **Assumption 7.**. Then there exists a matrix $\mathbf{C}_0 \geq \mathbf{O}$ in $\mathcal{R}^{2J\times 2J}$ and a vector $\mathbf{c}_0 \geq \mathbf{0}$ in $\mathcal{R}^{1\times 2J}$ such that

$$E\left[\sum_{k=l}^{M^e-1}(\mathbf{v}(\sigma_k^e),\mathbf{n}(\sigma_k^e))|\mathbf{Y}(\sigma_l^e)=\mathbf{Y}\right] \leq (\mathbf{v},\mathbf{n})\mathbf{C}_0+\mathbf{c}_0, \tag{7.4}$$

for any $\mathbf{Y}=(i,\mathbf{v},\mathbf{n},L)\in\mathcal{E}$, $l=0,1,\ldots$ and $e=1,2,\ldots$.

Proof. From **Assumption 6.**, it can be easily shown that

$$\begin{aligned}
& E[Z_{C,k-1}^e(\mathbf{v}(\sigma_k^e),\mathbf{n}(\sigma_k^e))|\mathbf{Y}(\sigma_l^e)=\mathbf{Y}] \\
= \; & (\mathbf{v},\mathbf{n})\mathbf{R}^i(\mathbf{QU})^{k-1-l}\mathbf{I}_1+\mathbf{e}_i\sum_{j=0}^{k-1-l}\mathbf{P}_J^{k-1-l-j}\mathbf{R}^0(\mathbf{QU})^j\mathbf{I}_1\in\mathcal{R}^{1\times 2J}, \quad k>l=0,1,\ldots,
\end{aligned} \tag{7.5}$$

where

$$\begin{aligned}
\mathbf{R}^i &= (\mathbf{O},\ldots,\mathbf{O},\mathbf{U}^i,\mathbf{O},\ldots,\mathbf{O})\in\mathcal{R}^{2J\times J_2}, \quad i=1,\ldots,J,\\
\mathbf{R}^0 &= \begin{pmatrix} \mathbf{u}^1 & \mathbf{0} & \cdots & \cdots & \mathbf{0}\\ \mathbf{0} & \mathbf{u}^2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots\\ \vdots & & \ddots & \ddots & \mathbf{0}\\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{u}^J\end{pmatrix}\in\mathcal{R}^{J\times J_2},\\
\mathbf{e}_i &= (0,\ldots,0,1,0,\ldots,0)\in\mathcal{R}^{1\times J}, \quad i=1,\ldots,J,\\
\mathbf{I}_1 &= \begin{pmatrix}\mathbf{I}_0\\ \vdots \\ \mathbf{I}_0\end{pmatrix}\in\mathcal{R}^{J_2\times 2J},
\end{aligned}$$

and the other constants are defined in Section 6.. Since $M^e\geq k \Longleftrightarrow X(\sigma_{k-1}^e)\neq 0$, we have

$$\begin{aligned}
& E[\sum_{k=l+1}^{M^e-1}(\mathbf{v}(\sigma_k^e),\mathbf{n}(\sigma_k^e))|\mathbf{Y}] \leq E[\sum_{k=l+1}^{\infty} Z_{C,k-1}^e(\mathbf{v}(\sigma_k^e),\mathbf{n}(\sigma_k^e))|\mathbf{Y}]\\
&= \sum_{k=l+1}^{\infty} E[Z_{C,k-1}^e(\mathbf{v}(\sigma_k^e),\mathbf{n}(\sigma_k^e))|\mathbf{Y}]\\
&= (\mathbf{v},\mathbf{n})\mathbf{R}^i\sum_{k=l+1}^{\infty}(\mathbf{QU})^{k-1-l}\mathbf{I}_1+\mathbf{e}_i\sum_{k=l+1}^{\infty}\sum_{j=0}^{k-1-l}\mathbf{P}_J^{k-1-l-j}\mathbf{R}^0(\mathbf{QU})^j\mathbf{I}_1\\
&= \{(\mathbf{v},\mathbf{n})\mathbf{R}^i+\mathbf{e}_i(\mathbf{I}-\mathbf{P}_J)^{-1}\mathbf{R}^0\}\{\mathbf{I}+\mathbf{Q}(\mathbf{I}-\mathbf{UQ})^{-1}\mathbf{U}\}\mathbf{I}_1.
\end{aligned}$$

The last equality holds from **Assumption 2.** and the assumption that $(\mathbf{UQ})^m\to\mathbf{O}$ as $m\to\infty$. □

From Lemma 4, for any $f\in\mathcal{V}_B$, we have

$$\begin{aligned}
& E[|f(\mathbf{Y}(\sigma_l^e),e,l)||\mathbf{Y}(\sigma_0^e)=\mathbf{Y}]\leq E[Z_{Cl}^e\{(\mathbf{v}(\sigma_l^e),\mathbf{n}(\sigma_l^e))\mathbf{c}_f+c_f\}|\mathbf{Y}(\sigma_0^e)=\mathbf{Y}]\\
&\leq E[\sum_{l=0}^{M^e-1}(\mathbf{v}(\sigma_l^e),\mathbf{n}(\sigma_l^e))|\mathbf{Y}(\sigma_0^e)=\mathbf{Y}]\mathbf{c}_f+c_f\\
&\leq \{(\mathbf{v},\mathbf{n})\mathbf{C}_0+\mathbf{c}_0\}\mathbf{c}_f+c_f<\infty, \quad \mathbf{Y}\in\mathcal{E}, e=1,2,\ldots \text{ and } l=0,1,\ldots.
\end{aligned}$$

Lemma 5. We make **Assumption 7.**. Suppose that $f \in \mathcal{V}_B$ satisfies the following inequality:

$$|f(\mathbf{Y},e,l)| \leq E[|f(\mathbf{Y}(\sigma_{l+1}^e),e,l+1)|\,|\mathbf{Y}(\sigma_l^e)=\mathbf{Y}] \tag{7.6}$$

for all $\mathbf{Y}=(i,\mathbf{v},\mathbf{n},L)\in\mathcal{E}$, $e=1,2,\ldots$ and $l=0,1,\ldots$. Then

$$f=0. \tag{7.7}$$

Proof. Let $\mathbf{Y}=(i,\mathbf{v},\mathbf{n},L)\in\mathcal{E}$, $l=0,1,\ldots$ and $e=1,2,\ldots$. By recursively applying (7.6), we have

$$|f(\mathbf{Y},e,l)| \leq E[|f(\mathbf{Y}(\sigma_k^e),e,k)|\,|\mathbf{Y}(\sigma_l^e)=\mathbf{Y}], \quad k>l.$$

Since $f\in\mathcal{V}_B$, there exist $\mathbf{c}_f\in\mathcal{R}^{2J\times 1}$ and $c_f\in\mathcal{R}$ such that

$$|f(\mathbf{Y},e,l)| \leq E[Z_{Ck}^e\{(\mathbf{v}(\sigma_k^e),\mathbf{n}(\sigma_k^e))\mathbf{c}_f+c_f\}\,|\mathbf{Y}(\sigma_l^e)=\mathbf{Y}]\to 0, \text{ as } k\to\infty.$$

The right-hand side of the above expression goes to 0 because

$$\begin{aligned}\sum_{k=l}^{\infty}E[Z_{Ck}^e(\mathbf{v}(\sigma_k^e),\mathbf{n}(\sigma_k^e))|\mathbf{Y}(\sigma_l^e)=\mathbf{Y}] &= E\left[\sum_{k=l}^{M^e-1}(\mathbf{v}(\sigma_k^e),\mathbf{n}(\sigma_k^e))|\mathbf{Y}(\sigma_l^e)=\mathbf{Y}\right]\\ &\leq (\mathbf{v},\mathbf{n})\mathbf{C}_0+\mathbf{c}_0<\infty,\end{aligned}$$

$$E[Z_{Ck}^e|\mathbf{Y}(\sigma_l^e)=\mathbf{Y}]=\mathbf{e}_i\mathbf{P}_J^{k-l}\mathbf{1},$$

where $\mathbf{1}\equiv(1,\ldots,1)'$ and $\mathbf{e}_0\equiv\mathbf{0}$. □

Lemma 5 shows that any solution on $\mathcal{V}_B$ of the system:

$$f(\mathbf{Y},e,l)=E[f(\mathbf{Y}(\sigma_{l+1}^e),e,l+1)|\mathbf{Y}(\sigma_l^e)=\mathbf{Y}], \quad \mathbf{Y}\in\mathcal{E}, e\in\{1,2,\ldots\} \text{ and } l\in\mathcal{I}_+, \tag{7.8}$$

must be 0 under the assumptions of the lemma. Uniqueness of solutions of inhomogeneous systems (7.2) can be led from the lemma. Then we have

Lemma 6. We make **Assumption 7.**. Let f^I be any function in $\mathcal{V}$. Then there exists at most one solution on $\mathcal{V}_B$ that satisfies equation (7.2) for f^I.

Proof. Let $f^1\in\mathcal{V}_B$ and $f^2\in\mathcal{V}_B$ be any two solutions that satisfy equation (7.2) for f^I. Then

$$|f^1(\mathbf{Y},e,l)-f^2(\mathbf{Y},e,l)| \leq E\left[\left|f^1(\mathbf{Y}(\sigma_{l+1}^e),e,l+1)-f^2(\mathbf{Y}(\sigma_{l+1}^e),e,l+1)\right|\,|\mathbf{Y}(\sigma_l^e)=\mathbf{Y}\right],$$

for $\mathbf{Y}\in\mathcal{E}$, $e\in\{1,2,\ldots\}$ and $l\in\mathcal{I}_+$. Since $f^1-f^2\in\mathcal{V}_B$, we have from Lemma 5,

$$f^1=f^2. \quad \square \tag{7.9}$$

The above lemma ensures uniqueness of solutions of a set of equation (7.2) on $\mathcal{V}_B$. We next show that the cost functions defined by (2.4) and (2.7) are indeed elements of $\mathcal{V}_B$.

Lemma 7. Let W_j and G_j be the cost functions defined by (2.4) and (2.7), respectively ($j=1,\ldots,J$). We make **Assumption 7.**. Then, $W_j\in\mathcal{V}_B$ and $G_j\in\mathcal{V}_B$.

Proof. Let $\mathbf{Y}\in\mathcal{E}$, $e\in\{1,2,\ldots\}$ and $l\in\mathcal{I}_+$. Then we have

$$\begin{aligned}W_j(\mathbf{Y},e,l) &= E[\sum_{k=l+1}^{M^e}\int_{\sigma_{k-1}^e}^{\sigma_k^e}C_{Wj}^e(t)dt|\mathbf{Y}(\sigma_l^e)=\mathbf{Y}]\\ &= \sum_{k=l+1}^{\infty}E\left[Z_{C,k-1}^e\int_{\sigma_{k-1}^e}^{\sigma_k^e}C_{Wj}^e(t)dt|\mathbf{Y}(\sigma_l^e)=\mathbf{Y}\right], \quad j=1,\ldots,J.\end{aligned}$$

From **Assumption 6..1**, each term in the above sum is calculated as follows:

$$
\begin{aligned}
& E\left[Z^e_{C,k-1}\int_{\sigma^e_{k-1}}^{\sigma^e_k} C^e_{Wj}(t)dt|\mathbf{Y}(\sigma^e_l)=\mathbf{Y}\right] \\
= & E\left[Z^e_{C,k-1}E[\int_{\sigma^e_{k-1}}^{\sigma^e_k} C^e_{Wj}(t)dt|\mathbf{Y}(\sigma^e_{k-1})]|\mathbf{Y}(\sigma^e_l)=\mathbf{Y}\right] \\
= & E\left[Z^e_{C,k-1}W^I_j(\mathbf{Y}(\sigma^e_{k-1}),e,k-1)|\mathbf{Y}(\sigma^e_l)=\mathbf{Y}\right] \\
\leq & E\left[Z^e_{C,k-1}\{(\mathbf{v}(\sigma^e_{k-1}),\mathbf{n}(\sigma^e_{k-1}))\mathbf{w}^0+w^0\}|\mathbf{Y}(\sigma^e_l)=\mathbf{Y}\right], \quad k>l,
\end{aligned}
$$

where $\mathbf{w}^0 \equiv (\max_k \phi^k_{11},\ldots,\max_k \phi^k_{1J},\max_k \phi^k_{21},\ldots,\max_k \phi^k_{2J})'$ and $w^0 \equiv \max_k w^k$. Hence

$$
\begin{aligned}
W_j(\mathbf{Y},e,l) & \leq E[\sum_{k=l+1}^{M^e}\{(\mathbf{v}(\sigma^e_{k-1}),\mathbf{n}(\sigma^e_{k-1}))\mathbf{w}^0+w^0\}|\mathbf{Y}(\sigma^e_l)=\mathbf{Y}] \\
& \leq E[\sum_{k=l}^{M^e-1}(\mathbf{v}(\sigma^e_k),\mathbf{n}(\sigma^e_k))|\mathbf{Y}(\sigma^e_l)=\mathbf{Y}]\mathbf{w}^0+E[M]w^0, \quad j=1,\ldots,J,
\end{aligned}
$$

where $E[M] \equiv \max_i \mathbf{e}_i(\mathbf{I}-\mathbf{P}_J)^{-1}\mathbf{1}$. Then, from Lemma 4, we have

$$
0 \leq W_j(\mathbf{Y},e,l) \leq \{(\mathbf{v},\mathbf{n})\mathbf{C}_0+\mathbf{c}_0\}\mathbf{w}^0+E[M]w^0, \quad j=1,\ldots,J.
$$

Further, from the definition of the cost functions, we have

$$
W_j(\mathbf{Y}_0,e,l)=0, \quad j=1,\ldots,J,
$$

for all $\mathbf{Y}_0=(0,\mathbf{v},\mathbf{n},L)$, $e=\{1,2,\ldots\}$ and $l \in \mathcal{I}_+$.

The similar argument holds for G_j $(j=1,\ldots,J)$ if we replace function C^e_{Wj} into function C^e_{Gj}.
□

We are now in a position to prove Theorem 2.

Proof of Theorem 2. For any j $(j=1,\ldots,J)$, we have shown that the cost function W_j defined by (2.4) satisfies the set of equations (7.2) for $f^I \equiv W^I_j$ (or equivalently eq. (2.6)). Further under the assumptions of this theorem, W_j is an element of $\mathcal{V}_B$ by Lemma 7. Then W_j is a unique solution of equations (7.2) on $\mathcal{V}_B$ by Lemma 6.

On the other hand, for any j $(j=1,\ldots,J)$, $\hat{W}_j$ defined by (6.13) is obviously an element of $\mathcal{V}_B$. From theorem 1, each function $\hat{W}_j$ also satisfies the set of equations (7.2) for $f^I \equiv W^I_j$ under the assumptions of this theorem, since $(\mathbf{I}-\mathbf{UQ})^{-1}$ exists. Hence the function W_j must be equivalent to $\hat{W}_j$.

The similar argument holds for G_j $(j=1,\ldots,J)$. □

Remark. In the theory of integral equations various results on existence and uniqueness of solutions for linear systems are investigated ([13]). We modify them to fit our problem. By considering **Assumption 7.**, we can ensure simultanuously that solutions of equations (2.6) and (2.9) are unique on $\mathcal{V}_B$ and that W_j and G_j defined by (2.4) and (2.7) are elements of $\mathcal{V}_B$ $(j=1,\ldots,J)$.

8. Steady state values of the cost functions.

In this section we evaluate steady state values of the cost functions W_j and G_j.

Let us consider the system operated under some fixed scheduling algorithm defined in Section 2.. Let

$$W_j^e \equiv \int_0^\infty C_{Wj}^e(s)ds, \quad j = 1,\ldots,J, \tag{8.1}$$

$$G_j^e \equiv \int_0^\infty C_{Gj}^e(s)ds, \quad j = 1,\ldots,J, \tag{8.2}$$

be respectively the sojourn times of the e^{th} customer and the cumulative values of his current works ($e = 1,2,\ldots$). We define the following customer average values:

$$\bar{W}_{\cdot j} \equiv \lim_{N\to\infty} \frac{1}{N}\sum_{e=1}^{N} W_j^e, \quad j = 1,\ldots,J, \tag{8.3}$$

$$\bar{G}_{\cdot j} \equiv \lim_{N\to\infty} \frac{1}{N}\sum_{e=1}^{N} G_j^e, \quad j = 1,\ldots,J, \tag{8.4}$$

if these limits shall exist. We are willing to assume that

[A1] the process $\mathcal{Q}$ is regenerative [16].

Let N_B be the number of customers served during a regenerative cycle. We assume that

[A2] the system is initially empty, and

[A3] $E[N_B] < \infty$.

Then these cost functions may be represented as:

$$\bar{W}_{\cdot j} = \frac{E[\sum_{e=1}^{N_B} W_j^e]}{E[N_B]}, \quad j = 1,\ldots,J, \tag{8.5}$$

$$\bar{G}_{\cdot j} = \frac{E[\sum_{e=1}^{N_B} G_j^e]}{E[N_B]}, \quad j = 1,\ldots,J, \tag{8.6}$$

if we may assume that every numerator in the right-hand side of the above expressions is finite. Further the customer average values $(\bar{\mathbf{v}}, \bar{\mathbf{n}}) = (\bar{v}_1,\ldots,\bar{v}_J,\bar{n}_1,\ldots,\bar{n}_J)$ of components of states are defined by:

$$\bar{n}_j \equiv \lim_{N\to\infty} \frac{1}{N}\sum_{e=1}^{N} n_j(\sigma_0^e), \quad j = 1,\ldots,J, \tag{8.7}$$

$$\bar{v}_j \equiv \lim_{N\to\infty} \frac{1}{N}\sum_{e=1}^{N} v_j(\sigma_0^e), \quad j = 1,\ldots,J, \tag{8.8}$$

if these limits shall exist. Now we assume that

[A4] $E[\sum_{e=1}^{N_B} n_j(\sigma_0^e)] < \infty$ and $E[\sum_{e=1}^{N_B} v_j(\sigma_0^e)] < \infty, \quad j = 1,\ldots,J.$

Then we have

$$\bar{n}_j = \frac{E[\sum_{e=1}^{N_B} n_j(\sigma_0^e)]}{E[N_B]}, \quad j = 1,\ldots,J, \tag{8.9}$$

$$\bar{v}_j = \frac{E[\sum_{e=1}^{N_B} v_j(\sigma_0^e)]}{E[N_B]}, \quad j = 1,\ldots,J. \tag{8.10}$$

The time average values $(\tilde{\mathbf{v}}, \tilde{\mathbf{n}}) = (\tilde{v}_1, \ldots, \tilde{v}_J, \tilde{n}_1, \ldots, \tilde{n}_J)$ of components of states are defined by:

$$\tilde{n}_j \equiv \lim_{t\to\infty} \frac{1}{t}\int_0^t n_j(s)ds, \quad j = 1, \ldots, J, \tag{8.11}$$

$$\tilde{v}_j \equiv \lim_{t\to\infty} \frac{1}{t}\int_0^t v_j(s)ds, \quad j = 1, \ldots, J. \tag{8.12}$$

In the following discussions, we consider the system with the steady state.

Now we show the following lemma concerned with representations of steady state values of the cost functions.

Lemma 8. Put **Assumption 7.** and the steady state assumptions from [A1] through [A4]. Then $E[\sum_{e=1}^{N_B} W_j^e] < \infty$ and $E[\sum_{e=1}^{N_B} G_j^e] < \infty$ $(j = 1, \ldots, J)$. Further we have the following representations:

$$\bar{W}_{\cdot j} = \sum_{i=1}^{J} \frac{\lambda_i}{\lambda}\{(\bar{\mathbf{v}}, \bar{\mathbf{n}})\mathbf{w}_{ij} + w_{ij}\}, \quad j = 1, \ldots, J, \tag{8.13}$$

$$\bar{G}_{\cdot j} = \sum_{i=1}^{J} \frac{\lambda_i}{\lambda}\{(\bar{\mathbf{v}}, \bar{\mathbf{n}})\mathbf{g}_{ij} + g_{ij}\}, \quad j = 1, \ldots, J. \tag{8.14}$$

Proof. First we consider $\bar{W}_{\cdot j}$ $(j = 1, \ldots, J)$. The main difficulty of the lemma lies in the dependence between the variables N_B and W_j^e. Let

$$Z_B^e \equiv \begin{cases} 1 & \text{if } N_B + 1 \geq e, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$E\left[\sum_{e=1}^{N_B+1} W_j^e\right] = E\left[\sum_{e=1}^{\infty} Z_B^e W_j^e\right] = \sum_{e=1}^{\infty} E\left[Z_B^e W_j^e\right].$$

Each term in the above sum is calculated as follows:

$$\begin{aligned} E[Z_B^e W_j^e] &= E[Z_B^e E[W_j^e | Z_B^e, \mathbf{Y}(\sigma_0^e)]] \\ &= E[Z_B^e E[W_j^e | \mathbf{Y}(\sigma_0^e)]] \\ &= \sum_{i=1}^{J} \frac{\lambda_i}{\lambda}\{E[Z_B^e(\mathbf{v}(\sigma_0^e), \mathbf{n}(\sigma_0^e))]\mathbf{w}_{ij} + E[Z_B^e]w_{ij}\}. \end{aligned}$$

The second equality in the above expression comes from the definition of Z_B^e and the Markov property of the process embeded in $\mathcal{Q}$, since Z_B^e is determined by $\{\mathbf{Y}(\sigma_0^s) : s < e\}$. The last equality comes from (6.13) and uniqueness of the function. Hence we have

$$\begin{aligned} E\left[\sum_{e=1}^{N_B+1} W_j^e\right] &= \sum_{e=1}^{\infty}\sum_{i=1}^{J} \frac{\lambda_i}{\lambda}\{E[Z_B^e(\mathbf{v}(\sigma_0^e), \mathbf{n}(\sigma_0^e))]\,\mathbf{w}_{ij} + E[Z_B^e]\,w_{ij}\} \\ &= \sum_{i=1}^{J} \frac{\lambda_i}{\lambda}\left\{E\left[\sum_{e=1}^{N_B+1}(\mathbf{v}(\sigma_0^e), \mathbf{n}(\sigma_0^e))\right]\mathbf{w}_{ij} + E[N_B+1]w_{ij}\right\}. \end{aligned}$$

Further we have

$$E[W_j^{N_B+1}] = E[W_j^1] = \sum_{i=1}^{J} \frac{\lambda_i}{\lambda} w_{ij},$$

$$(\mathbf{v}(\sigma_0^{N_B+1}), \mathbf{n}(\sigma_0^{N_B+1})) = (\mathbf{0}, \mathbf{0}).$$

Hence, from (8.7) and (8.8), we have

$$E\left[\sum_{e=1}^{N_B} W_j^e\right] = \sum_{i=1}^{J}\frac{\lambda_i}{\lambda}\left\{E\left[\sum_{e=1}^{N_B}(\mathbf{v}(\sigma_0^e),\mathbf{n}(\sigma_0^e))\right]\mathbf{w}_{ij} + E[N_B]w_{ij}\right\} < \infty.$$

By deviding both sizes of the above expression by $E[N_B]$, the desired result for $\bar{W}_{\cdot j}$ is obtained from (8.5), (8.7) and (8.8). In a similar way, we can obtain the desired result for $\bar{G}_{\cdot j}$ $(j = 1, \dots, J)$. □

We use the generalized version of Little's formula ($H = \lambda G$) stated by Glynn and Whitt [7] and Whitt [20] that equates time average values of costs with customer average values of costs to obtain

$$\tilde{n}_j = \lambda \bar{W}_{\cdot j}, \quad j = 1, \dots, J, \tag{8.15}$$
$$\tilde{v}_j = \lambda \bar{G}_{\cdot j}, \quad j = 1, \dots, J. \tag{8.16}$$

For a Poisson arrival, a fraction of time that the system is in any state is equal to a fraction of arrivals when the system is in the state. This is the Poisson Arrivals See Time Averages (PASTA) property investigated by Melamed and Whitt [14] and Wolff [22]. Then, from (8.13) and (8.14), we have

$$\bar{W}_{\cdot j} = \sum_{i=1}^{J}\frac{\lambda_i}{\lambda}\left\{(\tilde{\mathbf{v}},\tilde{\mathbf{n}})\mathbf{w}_{ij} + w_{ij}\right\}, \quad j = 1, \dots, J, \tag{8.17}$$
$$\bar{G}_{\cdot j} = \sum_{i=1}^{J}\frac{\lambda_i}{\lambda}\left\{(\tilde{\mathbf{v}},\tilde{\mathbf{n}})\mathbf{g}_{ij} + g_{ij}\right\}, \quad j = 1, \dots, J. \tag{8.18}$$

From the equations between (8.15) and (8.18), we obtain

$$\tilde{n}_j = \sum_{i=1}^{J}\lambda_i\left\{(\tilde{\mathbf{v}},\tilde{\mathbf{n}})\mathbf{w}_{ij} + w_{ij}\right\}, \quad j = 1, \dots, J, \tag{8.19}$$
$$\tilde{v}_j = \sum_{i=1}^{J}\lambda_i\left\{(\tilde{\mathbf{v}},\tilde{\mathbf{n}})\mathbf{g}_{ij} + g_{ij}\right\}, \quad j = 1, \dots, J. \tag{8.20}$$

Define

$$\mathbf{S} \equiv \sum_{i=1}^{J}\lambda_i(\mathbf{g}_{i1}, \dots, \mathbf{g}_{iJ}, \mathbf{w}_{i1}, \dots, \mathbf{w}_{iJ}), \tag{8.21}$$
$$\mathbf{s} \equiv \sum_{i=1}^{J}\lambda_i(g_{i1}, \dots, g_{iJ}, w_{i1}, \dots, w_{iJ}). \tag{8.22}$$

Then we arrive at the equation that determines the steady state expected value $(\tilde{\mathbf{v}},\tilde{\mathbf{n}})$:

$$(\tilde{\mathbf{v}},\tilde{\mathbf{n}}) = (\tilde{\mathbf{v}},\tilde{\mathbf{n}})\mathbf{S} + \mathbf{s}. \tag{8.23}$$

Now we assume that the inverse matrix $(\mathbf{I} - \mathbf{S})^{-1}$ exists. Then we have

$$(\tilde{\mathbf{v}},\tilde{\mathbf{n}}) = \mathbf{s}(\mathbf{I} - \mathbf{S})^{-1}. \tag{8.24}$$

Finally let $\bar{W}_{ij}$ and $\bar{G}_{ij}$ be the steady state value of the sojourn time of a customer spend at station j and the steady state value of the total amount of current works of the customer accumulated at station j, respectively, given that the customer arrives at station i from outside the system $(i, j = 1, \ldots, J)$. Then we can get the steady state values:

$$\bar{W}_{ij} = \mathbf{s}(\mathbf{I} - \mathbf{S})^{-1}\mathbf{w}_{ij} + w_{ij}, \quad (8.25)$$
$$\bar{G}_{ij} = \mathbf{s}(\mathbf{I} - \mathbf{S})^{-1}\mathbf{g}_{ij} + g_{ij}. \quad (8.26)$$

These results are arranged in the following theorem:

Theorem 3. Assume that the multiclass M/G/1 system with feedback defined in Section 2. satisfy the steady state assumptions from [A1] through [A4]. Let $\tilde{\mathbf{n}} = (\tilde{n}_1, \ldots, \tilde{n}_J)$ and $\tilde{\mathbf{v}} = (\tilde{v}_1, \ldots, \tilde{v}_J)$ be the vectors of the steady state mean *number of customers* and the steady state expected *total amount of current works* in the system defined by (8.11) and (8.12), respectively. Further, let $\bar{W}_{ij}$ $(i, j = 1, \ldots, J)$ be the steady state value of *sojourn time* of customers, who initially arrive at station i from outside the system, spend at station j until their departure from the system. We make **Assumption 7.**. Further we assume that the inverse matrix $(\mathbf{I} - \mathbf{S})^{-1}$ exists where $\mathbf{S}$ is defined in (8.21). Then

$$(\tilde{\mathbf{v}}, \tilde{\mathbf{n}}) = \mathbf{s}(\mathbf{I} - \mathbf{S})^{-1}, \quad (8.27)$$
$$\bar{W}_{ij} = \mathbf{s}(\mathbf{I} - \mathbf{S})^{-1}\mathbf{w}_{ij} + w_{ij}, \quad (8.28)$$

where $\mathbf{s}$ is defined in (8.22), and where $\mathbf{w}_{ij}$ and other constants are defined in Section 6.. □

Of course, the total mean sojourn time $\bar{W}_{i\cdot}$ of a customer who arrives at station i from outside the system spend until his departure is given by

$$\bar{W}_{i\cdot} \equiv \sum_{j=1}^{J} \bar{W}_{ij} = \mathbf{s}(\mathbf{I} - \mathbf{S})^{-1} \sum_{j=1}^{J} \mathbf{w}_{ij} + \sum_{j=1}^{J} w_{ij}. \quad (8.29)$$

Remark. It is worth noting that equation (8.16) is a variation of the formula relating a steady state (time average) work, say $\tilde{v}$, in a general single class queue to a steady state (customer average) waiting time, say $\bar{W}$, i.e.,

$$\tilde{v} = \lambda\{E[S\bar{W}] + E[S^2]/2\} \quad (8.30)$$

where λ denotes the arrival rate and S denotes the service time. Our method employed in the paper can be considered to be a *supplementary variable method* [2], where the supplementary variables are current works in the system instead of attained service times.

Numerical examples and the graphs.

Now we give numerical examples of the model. The number of the stations J is equal to 5. We calculate the values of the mean sojourn times of the following systems:

- System 1: All stations adopt FCFS disciplines.
- System 2: All stations adopt PR-LCFS disciplines.

The system parameters are listed below:

- $\lambda_j = 1/20.0$: the arrival rates $(j = 1, \ldots, 5)$.
- The service time distributions are the 5 stage Erlang distributions with the means varying from 0.1 to 1.5.

- The feedback probabilities are as follows:

$$\begin{aligned}
(p_{11}, p_{12}, p_{13}, p_{14}, p_{15}) &= (0.10, 0.10, 0.05, 0.05, 0.10), \\
(p_{21}, p_{22}, p_{23}, p_{24}, p_{25}) &= (0.10, 0.10, 0.15, 0.10, 0.10), \\
(p_{31}, p_{32}, p_{33}, p_{34}, p_{35}) &= (0.15, 0.10, 0.10, 0.10, 0.20), \\
(p_{41}, p_{42}, p_{43}, p_{44}, p_{45}) &= (0.15, 0.15, 0.15, 0.15, 0.15), \\
(p_{51}, p_{52}, p_{53}, p_{54}, p_{55}) &= (0.20, 0.20, 0.10, 0.10, 0.15).
\end{aligned}$$

We make the graphs (Figure 1. and Figure 2.) for these systems in which the mean sojourn times $\bar{W}_{i\cdot}$ of customers who initially arrive at station i ($i = 1, \ldots, 5$) are individually plotted.

9. Conclusions.

We have concerned with the multiclass M/G/1 system with feedback. Preemptive priority scheduling algorithms are considered. Every station adopts either FCFS discipline or PR-LCFS discipline. First we define the cost functions W_j and G_j of customers ($j = 1, \ldots, J$) which denote their *mean sojourn times* and their expected *cumulative current works*, respectively. We then obtain sets of equations that are satisfied by these cost functions. It is shown in Section 6. that these equations can be solved explicitly. Further these solutions are shown to be unique under some assumptions. Finally, we evaluate steady state expected values of these functions. The average *number of customers* and the average *current work* in each class are simultaneously obtained by solving a set of linear equations. Since these average values can be expressed in matrix forms, a numerical algorithm that yields these values can be easily constructed. The methodology given in the paper will be widely applicable to analysis of multiclass M/G/1 queueing systems.

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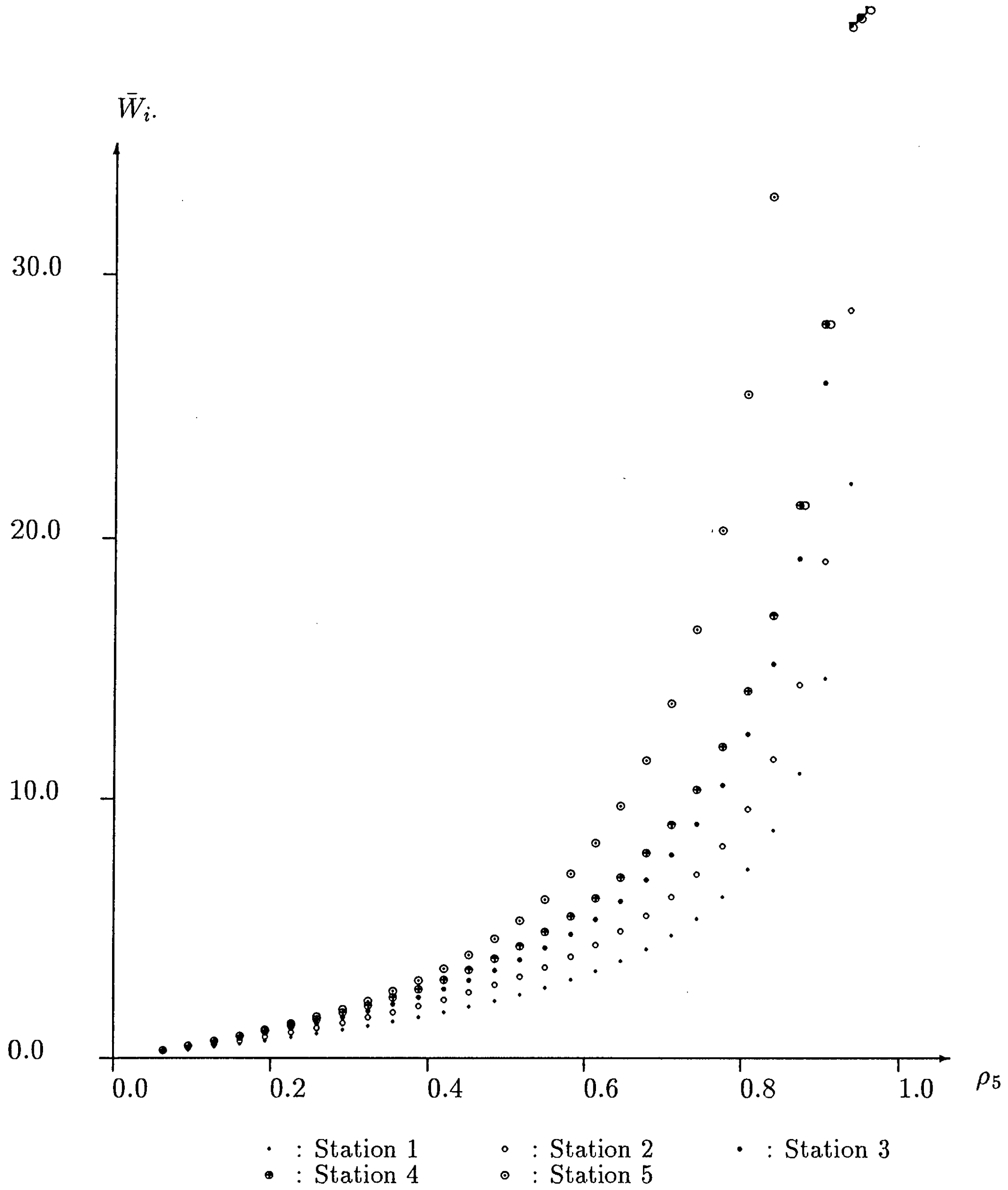

Figure 1. System 1.

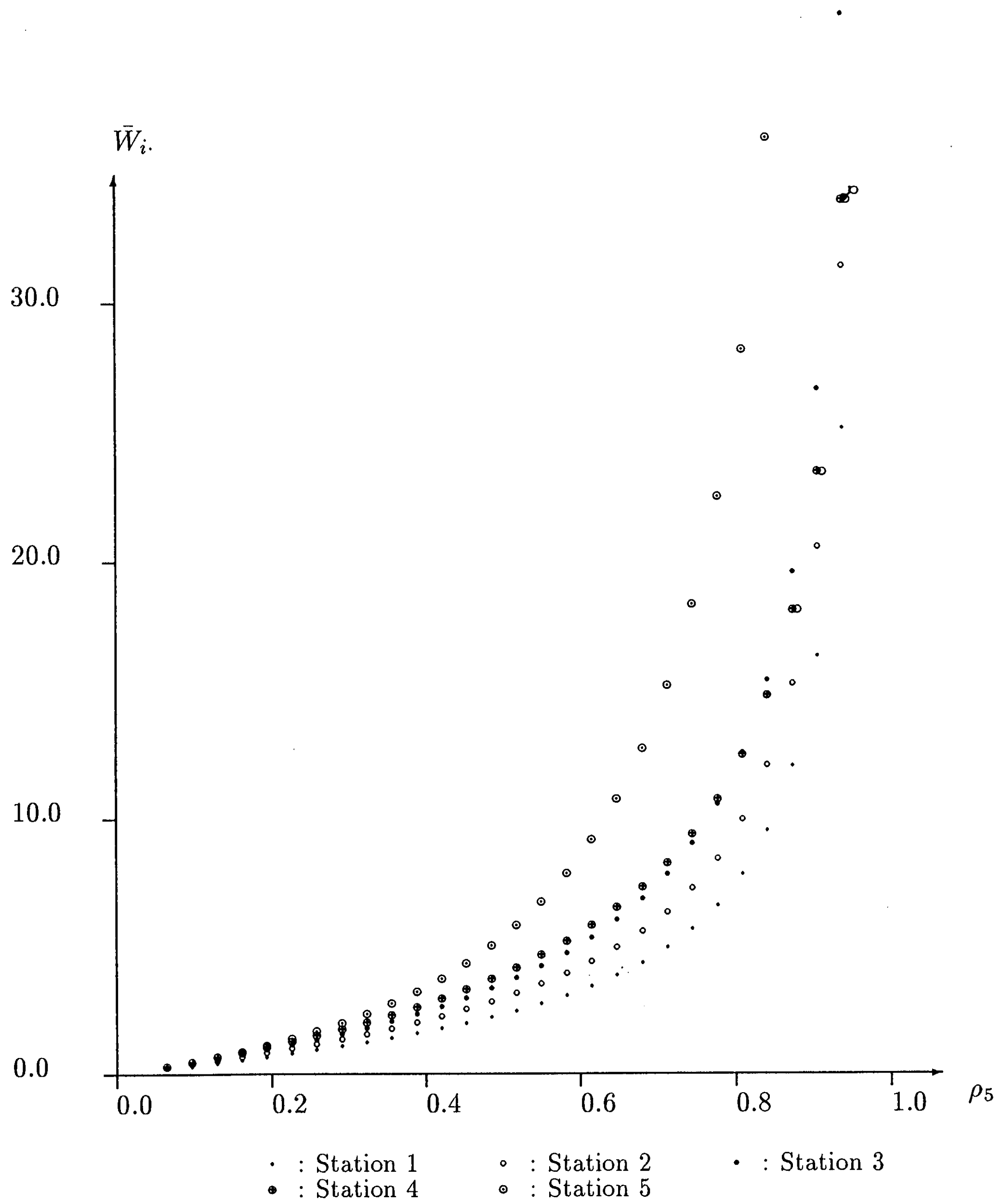

Figure 2. System 2.