A Primal-Dual Algorithm for Globally Solving a Production-Transportation Problem with Concave Production Cost

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Abstract. In this paper, we propose a primal-dual algorithm for solving a class of production-transportation problems. Among m (≥ 2) sources exist two factories, each of which produces certain goods at some concave cost and supplies n terminals with the product. We show that one can globally minimize the total cost of production and transportation by solving a Hitchcock transportation problem with m sources and n terminals and a minimum (linear-)cost flow problem with $m + n$ nodes. The number of arithmetic operations required by the algorithm is pseudo-polynomial in the problem input length.

Key words: Concave minimization, global optimization, production-transportation problem, primal-dual algorithm, minimum concave-cost flow problem.

1. Introduction

Suppose a corporation has m sources of certain goods, p of which are factories and the rest are warehouses. There are n terminal stores dealing in the goods. The decision maker of this corporation has to satisfy the demands of these terminals, so as to minimize the total cost of producing the goods and of shipping them to each terminal. This is the production-transportation problem which we consider in the paper.

The production cost is in general a nondecreasing and concave function of the output, which means that the production-transportation problem has multiple locally optimal solutions, many of which need not be globally optimal. Hence the problem belongs to a class of global optimization [8]. Although such a problem is well known to be difficult, a number of promising algorithms are proposed for some network problems (see [6, 5] for the current state-of-the-art of nonconvex network optimization).

In their recent articles [12, 13], Tuy et al. proposed a strongly polynomial-time algorithm for solving a special type of production-transportation problems, where the

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number p of factories is fixed at two or three and warehouses are absent, i.e., $m = p$. Their result sharply contrasts with general concave minimization problems, which are NP-hard even when just one variable is nonlinear [11]. They developed this algorithm further to solve the problem with any fixed $m = p$ in a subsequent article [14]. Another special type have been studied in our article [9], where warehouses are absent again but the number p of factories are not fixed. We assumed that terminals are partitioned into $p-1$ disjoint sets and each of $p-1$ factories is allowed to supply only its assigned set of terminals. We exploited this network structure and solved the problem in time $O(npb)$, where b represents the total demand of terminals.

In this paper, we assume that $m \geq p$, i.e., there can be some warehouses, each of which produces nothing but supplies a certain amount of the goods. Under this condition, we will concentrate on the case $p = 2$, i.e., among m (≥ 2) sources there are two factories, each of which can produce the goods and also supply any terminals with them. In Section 2, we reduce the problem to minimization of a univariate function F, each value of which is provided by solving an ordinary Hitchcock transportation problem with m sources and n terminals. In Section 3, we construct an auxiliary network with $m + n$ nodes associated with the transportation problem providing the values of F. We show that a global minimum of F can be obtained in the course of computing a minimum (linear-)cost flow in the auxiliary network. Section 4 devotes to the algorithm for globally minimizing the total cost of the original problem. The number of arithmetic operations required by the algorithm is pseudo-polynomial in the problem input length. We also discuss a class of minimum concave-cost flow problems related to our production-transportation problem in Section 5.

2. Reduction to Minimization of a Univariate Function

We have two factories, each of which can produce at most a_i units of the goods, $i = 1, 2$, and $m-2$ warehouses, each with a supply of a_i units, $i = 3, \ldots, m$. The cost of producing y_1 and y_2 units at factories 1 and 2 is given by $g(y_1, y_2)$. We assume that $g : \mathbb{R}^2 \to \mathbb{R}^1$ is a concave function. The production function g is often assumed to be separable, i.e., $g(y_1, y_2) = g_1(y_1) + g_2(y_2)$ for some concave functions $g_i : \mathbb{R}^1 \to \mathbb{R}^1$, $i = 1, 2$. However, since a non-separable g is more realistic as discussed in [13], we will not impose such an assumption throughout the paper. On the other hand, each of n terminals has a demand of b_j units, $j = 1, \ldots, n$. We also know the unit cost c_{ij} of shipping the goods from source i, which is either a factory or a warehouse, to terminal j. Our problem is then formulated as follows:

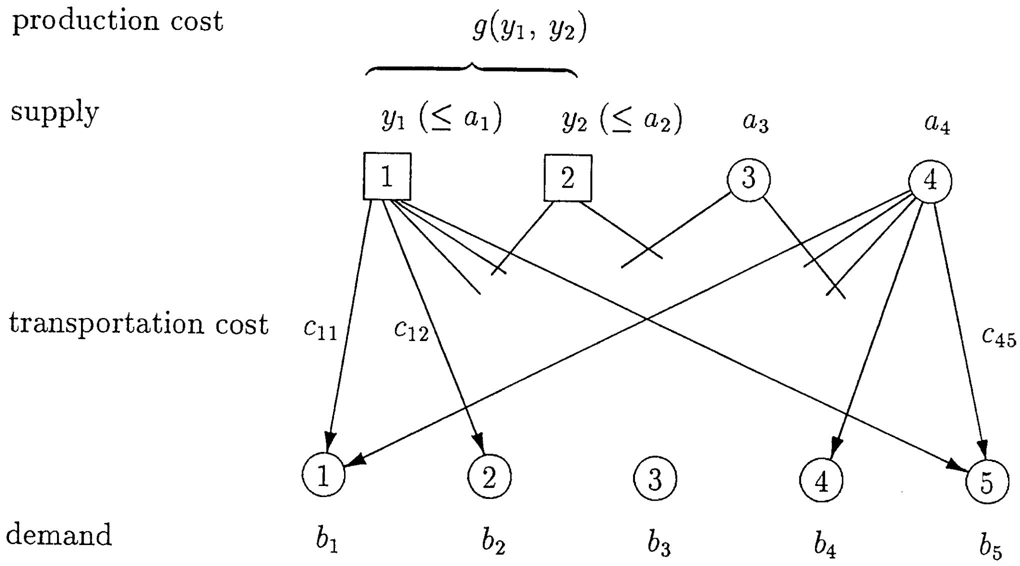

Figure 2.1. Example of the problem.

$$\begin{array}{lll} \text{minimize} & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + g(y_1, y_2) & \\ \text{subject to} & \sum_{j=1}^{n} x_{ij} = y_i, & i = 1, 2, \\ & \sum_{j=1}^{n} x_{ij} = a_i, & i = 3, \ldots, m, \\ & \sum_{i=1}^{m} x_{ij} = b_j, & j = 1, \ldots, n, \\ & x_{ij} \ge 0, & i = 1, \ldots, m, \quad j = 1, \ldots, n, \\ & 0 \le y_i \le a_i, & i = 1, 2, \end{array} \tag{2.1}$$

where x_{ij}'s, y_1 and y_2 are variables to be determined. We assume that all constants a_i's, b_j's and c_{ij}'s are nonnegative integers. Figure 2.1 shows problem (2.1) with 4 sources and 5 terminals.

Any feasible solution of (2.1) has to satisfy

$$y_1 + y_2 + \sum_{i=3}^{m} a_i = \sum_{j=1}^{n} b_j. \tag{2.2}$$

Hence, letting

$$\bar{g}(y) = g(y, \sum_{j=1}^{n} b_j - \sum_{i=3}^{m} a_i - y), \tag{2.3}$$

we can rewrite (2.1) as follows:

$$
\text{(TP)} \quad \left| \begin{array}{lll}
\text{minimize} & \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + \bar{g}(y) & \\
\text{subject to} & \displaystyle\sum_{j=1}^{n} x_{1j} = y, \quad \sum_{j=1}^{n} x_{2j} = d - y, & \\
& \displaystyle\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 3, \ldots, m, & \\
& \displaystyle\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, \ldots, n, & \\
& x_{ij} \geq 0, \qquad i = 1, \ldots, m, \quad j = 1, \ldots, n, & \\
& \ell \leq y \leq u, \qquad i = 1, 2, &
\end{array} \right.
$$

where

$$
d = \sum_{j=1}^{n} b_j - \sum_{i=3}^{m} a_i, \quad \ell = \max\{0, \, d - a_2\}, \quad u = \min\{a_1, \, d\}. \tag{2.4}
$$

Note that (TP) can still have multiple locally optimal solutions, since $\bar{g} : \mathbb{R}^1 \to \mathbb{R}^1$ is concave. To exclude trivial cases, we assume in the sequel that $\ell < u$. For any fixed y, let us consider a subproblem:

$$
\text{(TP}(y)\text{)} \quad \left| \begin{array}{lll}
\text{minimize} & \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} & \\
\text{subject to} & \displaystyle\sum_{j=1}^{n} x_{1j} = y, \quad \sum_{j=1}^{n} x_{2j} = d - y, & \\
& \displaystyle\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 3, \ldots, m, & \\
& \displaystyle\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, \ldots, n, & \\
& x_{ij} \geq 0, \qquad i = 1, \ldots, m, \quad j = 1, \ldots, n. &
\end{array} \right.
$$

This is just an ordinary Hitchcock transportation problem. We can obtain an optimal solution in polynomial time if $\ell \leq y \leq u$. We denote it by a vector $\boldsymbol{x}^*(y)$, whose components are $x^*_{ij}(y)$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. Let

$$
f(y) = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x^*_{ij}(y). \tag{2.5}
$$

Then we see that solving the original problem (2.1) amounts to finding a global minimum of a function:

$$
F(y) = f(y) + \bar{g}(y). \tag{2.6}
$$

To be precise, (TP) can be solved if we solve a minimization problem of a single variable:

(MP) $\quad$: $\text{minimize}\{F(y) \mid \ell \leq y \leq u\}$, which we call the *master problem* of (TP).

The above observation is summarized into the following:

Lemma 2.1. *If y^* is a globally optimal solution of (MP), then $(\boldsymbol{x}^*(y^*), y^*)$ solves (TP), where $\boldsymbol{x}^*(y^*)$ is an optimal solution of (TP(y^*)).* $\square$

3. Characteristics of the Univariate Function

Since the objective function F of (MP) is univariate, we can obtain a globally optimal solution y^* by enumerating local minima of $F(y)$ successively from $y = \ell$ to u. To do this efficiently, we need to know the shape of F exactly. As seen in the previous section, F consists of two functions f and $\bar{g}$. While the latter is given beforehand, the former requires solving the transportation problem (TP(y)) for all y in the interval $[\ell, u]$. This fact, however, tells us the shape of F in outline.

Proposition 3.1. *Function $F : [\ell, u] \rightarrow \mathbb{R}^1$ is continuous and piecewise concave.*

Proof: Since (TP(y)) is a parametric right-hand-side linear program, its optimal value $f(y)$ is a piecewise affine and convex function (see e.g. [3]). The sum of affine and concave functions is concave [10], and hence F is concave on each affine piece of f. $\square$

We immediately see from the proposition that among extreme points of affine pieces of f exists a global minimizer y^* of F.

Let us suppose $f(y')$ is given for an arbitrary $y' \in [\ell, u)$. Hence we have an optimal solution $\boldsymbol{x}^*(y')$ of (TP(y')). In the rest of the section, we will develop a procedure for computing $f(y' + \delta)$ for sufficiently small $\delta \geq 0$. Using the procedure we will identify the affine piece of f containing y'.

3.1. Minimum cost flow in an auxiliary network

We first construct an auxiliary graph $G(y') = (M, N, A(y'))$ associated with (TP(y')), where $M = \{1, \ldots, m\}$ and $N = \{1, \ldots, n\}$ are the sets of sources and terminals of (TP) respectively, and $A(y')$ is a set of directed arcs. Based on the optimal solution $\boldsymbol{x}^*(y')$ of (TP(y')), we define $A(y')$ and capacity $u_{ij}(y')$ of each arc $(i, j) \in A(y')$ as follows (see also Figure 3.1): For each pair (i, j) such that $i \in M$ and $j \in N$, let

$$(i, j) \in A(y'), \quad u_{ij}(y') = +\infty, \tag{3.1}$$

$$(j, i) \in A(y'), \quad u_{ji}(y') = x^*_{ij}(y') \quad \text{if} \quad x^*_{ij}(y') > 0. \tag{3.2}$$

In addition, for each $(i, j) \in A(y')$ we define a cost:

$$c_{ij}(y') = \begin{cases} c_{ij} & \text{if } i \in M, \ j \in N, \\ -c_{ij} & \text{if } j \in M, \ i \in N. \end{cases} \tag{3.3}$$

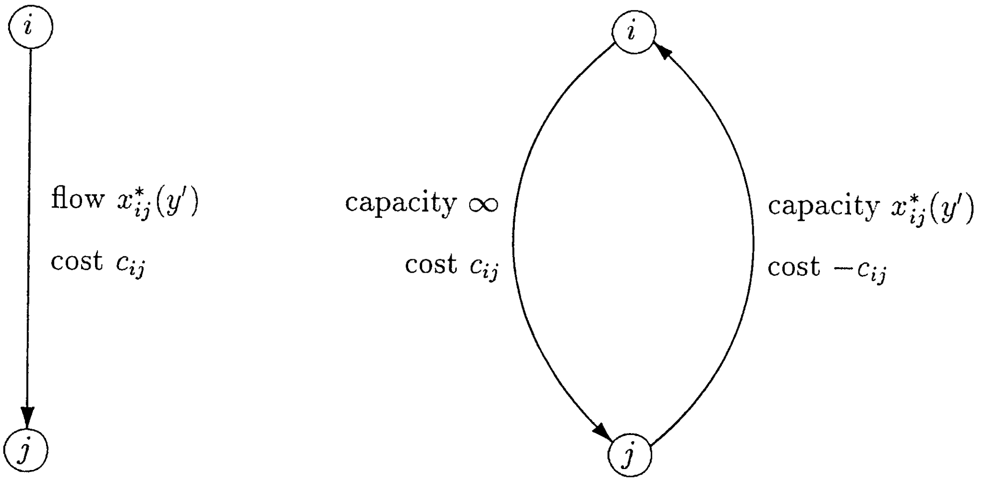

Figure 3.1. Arcs in the auxiliary network $\mathcal{N}(y')$.

In Figure 3.1, the right arcs are constructed from the left one. Denote by $\boldsymbol{c}(y')$ and $\boldsymbol{u}(y')$ the vectors of $c_{ij}(y')$'s and $u_{ij}(y')$'s respectively. Then we have the following problem in network $\mathcal{N}(y') = (G(y'),\ \boldsymbol{c}(y'),\ \boldsymbol{u}(y'))$:

$$(\mathrm{P}(\delta;\ y'))\quad \left|\begin{array}{lll} \text{minimize} & \displaystyle\sum_{(i,j)\in A(y')} c_{ij}(y') z_{ij} & \\ \text{subject to} & \displaystyle\sum_{j\in V(1)} z_{1j} - \sum_{j\in W(1)} z_{j1} = \delta, & \\ & \displaystyle\sum_{j\in V(2)} z_{2j} - \sum_{j\in W(2)} z_{j2} = -\delta, & \\ & \displaystyle\sum_{j\in V(i)} z_{ij} - \sum_{j\in W(i)} z_{ji} = 0, & i \in M \cup N \setminus \{1,\ 2\}, \\ & 0 \le z_{ij} \le u_{ij}(y'), & (i,\ j) \in A(y'), \end{array}\right.$$

where z_{ij}'s are variables, $V(i) = \{j \in M \cup N \mid (i,\ j) \in A(y')\}$ and $W(i) = \{j \in M \cup N \mid (j,\ i) \in A(y')\}$. Since $(\mathrm{P}(\delta;\ y'))$ is a minimum linear-cost flow problem in $\mathcal{N}(y')$ with source $s = 1 \in M$ and sink $t = 2 \in M$, we can solve it efficiently using any one of existing algorithms. Let us denote an optimal solution of $(\mathrm{P}(\delta;\ y'))$ by $\boldsymbol{z}^*(\delta;\ y')$, whose components are $z^*_{ij}(\delta;\ y')$, $(i,\ j) \in A(y')$.

Lemma 3.2. *For each $(i,\ j)$ such that $i \in M$ and $j \in N$, let*

$$x^*_{ij}(\delta;\ y') = \begin{cases} z^*_{ij}(\delta;\ y') & \text{if } x^*_{ij}(y') = 0, \\ x^*_{ij}(y') + z^*_{ij}(\delta;\ y') - z^*_{ji}(\delta;\ y') & \text{otherwise.} \end{cases} \tag{3.4}$$

Then $\boldsymbol{x}^*(\delta;\ y')$, whose components are $x^*_{ij}(\delta;\ y')$'s, is an optimal solution of $(TP(y'+\delta))$.

Proof: For an arbitrary feasible solution $\boldsymbol{z}$ of (P(δ; y')), let

$$x'_{ij} = \begin{cases} z_{ij} & \text{if } x^*_{ij}(y') = 0, \\ x^*_{ij}(y') + z_{ij} - z_{ji} & \text{otherwise.} \end{cases}$$

Then we have

$$\begin{aligned} \sum_{j=1}^{n} x'_{ij} &= \sum_{j=1}^{n} x^*_{ij}(y') + \Big(\sum_{j \in V(i)} z_{ij} - \sum_{j \in W(i)} z_{ji}\Big) \\ &= \begin{cases} y' + \delta, & i = 1, \\ d - y' - \delta, & i = 2, \\ a_i, & i = 3, \ldots, m, \end{cases} \\ \sum_{i=1}^{m} x'_{ij} &= \sum_{i=1}^{m} x^*_{ij}(y') - \Big(\sum_{i \in V(j)} z_{ji} - \sum_{i \in W(j)} z_{ij}\Big) = b_j, \quad j = 1, \ldots, n, \end{aligned}$$

and

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x'_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x^*_{ij}(y') + \sum_{(i,j) \in A(y')} c_{ij}(y')z_{ij}.$$

Hence the feasible set of (P(δ; y')) represents possible adjustments of $\boldsymbol{x}^*(y')$ to a slight change in y'. Among such adjustments $\boldsymbol{z}^*(\delta;\ y')$ requires the minimum cost, which apparently implies that $\boldsymbol{x}^*(\delta;\ y')$ defined by (3.4) is optimal to (TP($y' + \delta$)). □

3.2. Application of the primal-dual algorithm

We next try applying the primal-dual algorithm [4] to the auxiliary problem (P(δ; y')). The algorithm begins with a zero flow in $\mathcal{N}(y')$, and augments it by adding some flow along a directed path from source s to sink t with the least cost in $\mathcal{N}(y')$. To find such an augmenting path, we need to solve a shortest path problem in $G(y')$ with arc length $\mathbf{c}(y')$. It follows from (3.2) that there exists some $j \in N$ such that $(j,\ 2) \in A(y')$ as long as $y' < d$. Hence we can obtain a shortest path $\pi(y') \subset A(y')$ from s to t ($= 2 \in M$). Let

$$\bar{\delta} = \min\{u_{ij} \mid (i,\ j) \in \pi(y')\}. \tag{3.5}$$

Lemma 3.3. *If* $0 \le \delta \le \bar{\delta}$, *then*

$$z^*_{ij}(\delta;\ y') = \begin{cases} \delta & \textit{if } (i,\ j) \in \pi(y'), \\ 0 & \textit{otherwise}, \end{cases} \tag{3.6}$$

is an optimal solution of (P(δ; y')).

Proof: Follows from a well-known result on the primal-dual algorithm for minimum cost flow problems (see e.g. [2]). □

It follows from (3.4) and (3.6) that

$$\begin{aligned} f(y' + \delta) &= \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x^*_{ij}(\delta;\, y') \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x^*_{ij}(y') + \sum_{(i,\,j) \in A(y')} c_{ij}(y') z^*_{ij}(\delta; y') \\ &= f(y') + \delta \sum_{(i,\,j) \in \pi(y')} c_{ij}(y'). \end{aligned} \tag{3.7}$$

This implies that f is an affine function over the interval $[y',\, y' + \bar{\delta}]$.

We are now ready to show the exact shape of F. Let $y_0 = \ell$ and let $(\mathrm{P}(\delta;\, y_0))$ be the auxiliary problem of $(\mathrm{TP}(y_0))$. Then, by Lemmas 3.2 and 3.3, we can obtain an interval $[y_0,\, y_0 + \bar{\delta}]$, where f is affine, in the first step of solving $(\mathrm{P}(\delta;\, y_0))$ by the primal-dual algorithm. If we let $y_1 = y_0 + \bar{\delta}$ and construct $(\mathrm{P}(\delta;\, y_1))$, then an alternative interval $[y_1,\, y_2]$ will be obtained in the same way as before. Repeating this process, we can generate a sequence of intervals $[y_0\, (= \ell),\, y_1]$, $[y_1,\, y_2]$, $\ldots$, $[y_{q-1},\, y_q\, (= u)]$ such that f is affine on each $[y_{k-1},\, y_k]$, $k = 1,\, \ldots, q$. Since F is concave on each $[y_{k-1},\, y_k]$, its minimum over the interval is achieved at either y_{k-1} or y_k. Hence a globally optimal solution of the master problem (MP) is given by

$$y^* \in \mathrm{argmin}\{F(y) \mid y = y_0,\, y_1,\, \ldots,\, y_q\}. \tag{3.8}$$

4. Solution Method for the Problem

In the previous section, to generate each interval $[y_{k-1},\, y_k]$, we solved a minimum cost flow problem $(\mathrm{P}(\delta;\, y_{k-1}))$ from scratch. In practice, however, we need not do so. The whole sequence of $[y_{k-1},\, y_k]$'s will be generated if we solve a single problem $(\mathrm{P}(u - \ell;\, \ell))$. Recall that the primal-dual algorithm [4] builds up a flow step by step, by adding flows along augmenting paths with the least cost in some auxiliary network $\mathcal{N}'$. At each iteration we find a least-cost augmenting path π' in $\mathcal{N}'$. If π' exists, we augment the flow along π' until the flow reaches the capacity of π', and then update the auxiliary network $\mathcal{N}'$. When we apply this algorithm entirely to $(\mathrm{P}(u - \ell;\, \ell))$, the auxiliary network $\mathcal{N}'$ and the augmenting path π' at the kth iteration just correspond to $\mathcal{N}(y_{k-1})$ and $\pi(y_{k-1})$ respectively, and the capacity of π' is given by $\min\{u_{ij} \mid \pi(y_{k-1})\}$.

4.1. Algorithm for the original problem

According to the above observation, let us describe the algorithm for solving the original problem (TP).

Algorithm PDM.

Step 1. Solve a transportation problem (TP(ℓ)) and let $\boldsymbol{x}^*(\ell)$ be an optimal solution. Let $f(\ell) = \sum_{i=1}^m \sum_{i=1}^n c_{ij} x_{ij}^*(\ell)$. Initialize the incumbent:

$$\boldsymbol{x}^* = \boldsymbol{x}^*(\ell), \quad y^* = \ell, \quad F^* = f(\ell) + \bar{g}(\ell).$$

Step 2. Construct the auxiliary network $\mathcal{N}(\ell) = (G(\ell) = (M, N, A(\ell)), \boldsymbol{c}(\ell), \boldsymbol{u}(\ell))$ of (TP(ℓ)) according to (3.1) – (3.3). Let $s = 1 \in M$ and $t = 2 \in M$.

Step 3. Let $y_0 = \ell$, $k = 0$ and do the following:

1° Compute a shortest path $\pi(y_k)$ from s to t in $G(y_k)$ with arc length $\boldsymbol{c}(y_k)$. Let $\bar{\delta} = \min\{u_{ij} \mid (i, j) \in \pi(y_k)\}$. If $y_k + \bar{\delta} > u$, then let $\bar{\delta} = u - y_k$.

2° Let $y_{k+1} = y_k + \bar{\delta}$. For each (i, j) such that $i \in M$ and $j \in N$, let

$$x_{ij}^*(y_{k+1}) = \begin{cases} x_{ij}^*(y_k) + \bar{\delta} & \text{if} \quad (i, j) \in \pi(y_k), \\ x_{ij}^*(y_k) - \bar{\delta} & \text{if} \quad (j, i) \in \pi(y_k), \\ x_{ij}^*(y_k) & \text{otherwise.} \end{cases}$$

Also let $f(y_{k+1}) = f(y_k) + \bar{\delta} \sum_{(i,j) \in \pi(y_k)} c_{ij}(y_k)$.

3° If $f(y_{k+1}) + \bar{g}(y_{k+1}) < F^*$, then revise the incumbent:

$$\boldsymbol{x}^* = \boldsymbol{x}^*(y_{k+1}), \quad y^* = y_{k+1}, \quad F^* = f(y_{k+1}) + \bar{g}(y_{k+1}).$$

4° If $y_{k+1} = u$, then terminate. Otherwise, update the auxiliary network $\mathcal{N}(y_k)$ according to (3.1) – (3.3) and let $\mathcal{N}(y_{k+1})$ be the resultant network. Let $k = k + 1$ and return to 1°. □

Note that Step 3 of this algorithm is nothing but the primal-dual algorithm for solving the minimum cost flow problem (P($u - \ell$; ℓ)) if it lacks substep 3°, which computes a global minimizer of F.

Let us denote by $S(m, n)$ the running time to solve a shortest path problem with mn arcs and $m + n$ nodes, and by $T(m, n)$ that to solve a Hitchcock transportation problem with m sources and n terminals. As well known (see e.g. [1]), both $S(m, n)$ and $T(m, n)$ are lower-order polynomial functions of m and n.

Theorem 4.1. *Algorithm PDM yields a globally optimal solution* $(\boldsymbol{x}^*, y^*)$ *of (TP) in* $O(cS(m, n) + T(m, n))$ *arithmetic operations and* $O(c)$ *evaluations of* g, *where* $c = u - \ell$.

Proof: The main parts of PDM are Steps 1 and 3. Step 1 requires $T(m, n)$ arithmetic operations to solve a transportation problem (TP(ℓ)). In Step 3, we compute a shortest path $\pi(y_k)$ and the value $\bar{g}(y_{k+1})$ at each iteration. The total number of iterations in Step 3 is bounded by $c = u - \ell$, since $\bar{\delta} \geq 1$ on the assumption that all constants are integral in (TP). Thus the assertion follows. □

The worst-case time complexity of Algorithm PDM is not polynomial in the problem input length even though the value of g is provided by an oracle. However, all the problems solved in PDM are essentially two network optimization problems, i.e., one is a Hitchcock transportation problem and the other a minimum cost flow problem. By many experiments performed so far, it is known that even non-polynomial algorithms can solve both the problems quite efficiently. Therefore, we may affirm that PDM is also practically efficient unless evaluations of g are extremely expensive.

Remarks. 1) In Step 3. 1° of PDM, we cannot use Dijkstra's algorithm immediately to compute the shortest path $\pi(y_k)$ because some components of arc length $\boldsymbol{c}(y_k)$ are negative. However, on the assumption that all c_{ij}'s are nonnegative in (TP), it is possible to transform $\boldsymbol{c}(y_k)$ into a nonnegative vector in every $\mathcal{N}(y_k)$ if we introduce node potentials. Then we can compute $\pi(y_k)$ in time $S(m, n) = O(mn + (m+n)\log(m+n))$. The readers are referred to any textbook on network flows, e.g. [1] for further details.

2) We have assumed that the production cost $\bar{g}$ is a concave and hence continuous function. However, one might reasonably expect $\bar{g}$ to be piecewise concave but discontinuous (e.g. a fixed-charge cost function). So long as $\bar{g}$ is lower semi-continuous, we can handle a discontinuous $\bar{g}$ using PDM with a minor modification. Let us divide each $[y_{k-1}, y_k]$ at discontinuous points of $\bar{g}$. Then $[\ell, u]$ is partitioned into r $(\geq q)$ subintervals $[\eta_{k'-1}, \eta_{k'}]$, $k' = 1, \ldots, r$, where $\eta_{k'}$ is either a y_k or a discontinuous point of $\bar{g}$. Since $F = f + \bar{g}$ is concave on the interior of each $[\eta_{k'-1}, \eta_{k'}]$, it achieves the minimum on $[\ell, u]$ at some $\eta_{k'}$ by the lower semi-continuity. Hence, to locate y^* in $[\ell, u]$, we need only to compute the values of F at discontinuous points of $\bar{g}$ as well as y_k's. □

4.2. Numerical example

Before concluding this section, let us illustrate Algorithm PDM using a simple instance of (TP) given by the table below:

source \ terminal	t_1	t_2	t_3	t_4	supply	capacity
s_1	12	1	3	4	y	200
s_2	4	9	6	2	$300 - y$	200
s_3	2	6	2	10	150	—
demand	80	180	120	70	450	—

where each entry (s_i, t_j) represents the transportation cost c_{ij}. The production cost of factories s_1 and s_2 is assumed to be

$$\bar{g}(y) = 100.0 \cdot \sqrt{y}.$$

The lower and the upper bounds of y are respectively

$$\ell = 100, \quad u = 200.$$

In Step 1 of PDM, we solve a transportation problem (TP(100)). Then an optimal solution $\boldsymbol{x}^*(100)$ is as follows:

	t_1	t_2	t_3	t_4	supply
s_1	0	100	0	0	100
s_2	80	50	0	70	200
s_3	0	30	120	0	150

We also initialize the incumbent:

$$\boldsymbol{x}^* = \boldsymbol{x}^*(100), \quad y^* = 100,$$
$$F^* = f(100) + \bar{g}(100) = 1430 + 1000.00 = 2430.00.$$

In Step 2, for each arc (i, j) with $x^*_{ij}(100) > 0$ we put a reverse arc (j, i) with capacity $x^*_{ij}(100)$ and cost $-c_{ij}$, i.e.,

arc (t_2, s_1)	with capacity	100	and cost	-1,
arc (t_1, s_2)	with capacity	80	and cost	-4,
arc (t_2, s_2)	with capacity	50	and cost	-9,
arc (t_4, s_2)	with capacity	70	and cost	-2,
arc (t_2, s_3)	with capacity	30	and cost	-6,
arc (t_3, s_3)	with capacity	120	and cost	-2,

and denote by $\mathcal{N}(100)$ the resultant network. Letting $y_0 = 100$, we proceed to Step 3.

In Step 3, we first compute a shortest path $\pi(100)$ from $s = s_1$ to $t = s_2$ in $\mathcal{N}(100)$ and obtain:

$$\pi(100) = (s_1, t_2, s_2), \quad \bar{\delta} = \min\{\infty, 50\} = 50.$$

Then we let $y_1 = 150$ and compute $\boldsymbol{x}^*(150)$, which is given by

	t_1	t_2	t_3	t_4	supply
s_1	0	150	0	0	150
s_2	80	0	0	70	150
s_3	0	30	120	0	150

Since

$$f(150) + \bar{g}(150) = 1030 + 1224.75 = 2254.75 < F^*,$$

we revise the incumbent as follows:

$$x^* = x^*(150), \quad y^* = 150, \quad F^* = f(150) + \bar{g}(150) = 2254.75.$$

According to the same rule as before, we update $\mathcal{N}(100)$ based on $x^*(150)$, and denote by $\mathcal{N}(150)$ the resultant network.

At the next iteration in Step 2, we compute a shortest path in $\mathcal{N}(150)$:

$$\pi(150) = (\mathrm{s}_1, \mathrm{t}_2, \mathrm{s}_3, \mathrm{t}_1, \mathrm{s}_2), \quad \bar{\delta} = \{\infty, 30, \infty, 80\} = 30.$$

We let $y_2 = 180$ and compute $x^*(180)$:

	t_1	t_2	t_3	t_4	supply
s_1	0	180	0	0	180
s_2	50	0	0	70	120
s_3	30	0	120	0	150

Since

$$f(180) + \bar{g}(180) = 820 + 1341.64 = 2161.64,$$

we revise the incumbent again:

$$x^* = x^*(180), \quad y^* = 180, \quad F^* = f(180) + \bar{g}(180) = 2161.64.$$

We also update $\mathcal{N}(150)$ based on $x^*(180)$ and obtain $\mathcal{N}(180)$.

At the third iteration, we compute a shortest path in $\mathcal{N}(180)$:

$$\pi(180) = (\mathrm{s}_1, \mathrm{t}_3, \mathrm{s}_3, \mathrm{t}_1, \mathrm{s}_2), \quad \bar{\delta} = \min\{\infty, 120, \infty, 50\} = 50.$$

Since $y_2 + \bar{\delta} = 230 > u = 200$, we modify $\bar{\delta} = 20$. We let $y_3 = 200$ and compute $x^*(200)$:

	t_1	t_2	t_3	t_4	supply
s_1	0	180	20	0	200
s_2	30	0	0	70	100
s_3	50	0	100	0	150

Since

$$f(200) + \bar{g}(200) = 800 + 1414.21 = 2214.21 > F^*$$

and $y_3 = u$, we find that the current (x^*, y^*) is a globally optimal solution of our instance.

5. A Minimum Concave-Cost Flow Problem

In both combinatorial and global optimization, one of the most attractive but most difficult problems is the minimum concave-cost flow problem. To solve this NP-hard problem, many algorithms have been developed so far, and some of them turned out to be promising for some special cases (see [5, 6] and references therein). Especially when both the numbers of sources and nonlinear-cost arcs are fixed, uncapacitated problems can be solved in polynomial time [7, 12, 15]. In this section, we will show that capacitated problems with a single nonlinear-cost arc can be transformed into the class (TP) of production-transportation problems and hence solved by Algorithm PDM in pseudo-polynomial time.

Let $G = (N, A)$ be a directed graph consisting of a set N of nodes and a set A of directed arcs. We associate with each arc $(i, j) \in A$ concave cost $g_{ij} : \mathbb{R}^1 \to \mathbb{R}^1$ and capacity $u_{ij} \geq 0$, and with each node $i \in N$ a number b_i, which indicates its supply or demand depending on whether $b_i > 0$ or $b_i < 0$. Then the minimum concave-cost flow problem is formulated as follows:

$$\text{(FP)} \quad \left| \begin{array}{lll} \text{minimize} & \displaystyle\sum_{(i,j)\in A} g_{ij}(x_{ij}) & \\ \text{subject to} & \displaystyle\sum_{j\in V(i)} x_{ij} - \sum_{j\in W(i)} x_{ji} = b_i, & i \in N, \\ & 0 \leq x_{ij} \leq u_{ij}, & (i, j) \in A, \end{array} \right.$$

where x_{ij}'s are variables, $V(i) = \{j \in N \mid (i, j) \in A\}$ and $W(i) = \{j \in N \mid (j, i) \in A\}$. We assume that all constants are integral, and for simplicity that (FP) has a feasible flow. In this problem, we are concerned with the case where all g_{ij}'s except one, say, g_{vw}, are linear functions, i.e., for some nonnegative integers c_{ij}'s,

$$g_{ij}(x_{ij}) = c_{ij}x_{ij}, \quad (i, j) \in A \setminus \{(v, w)\}. \tag{5.1}$$

Given such an instance of (FP), we will construct an instance of (TP).

If flow x_{vw} of the nonlinear-cost arc (v, w) is fixed at any value y, we have a minimum linear-cost flow problem:

$$\text{(FP}(y)) \quad \left| \begin{array}{lll} \text{minimize} & \displaystyle\sum_{(i,j)\in A'} c_{ij}x_{ij} & \\ \text{subject to} & \displaystyle\sum_{j\in V'(v)} x_{vj} - \sum_{j\in W'(v)} x_{jv} = -y, & \\ & \displaystyle\sum_{j\in V'(w)} x_{wj} - \sum_{j\in W'(w)} x_{jw} = y, & \\ & \displaystyle\sum_{j\in V'(i)} x_{ij} - \sum_{j\in W'(i)} x_{ji} = b_i, & i \in N \setminus \{v, w\}, \\ & 0 \leq x_{ij} \leq u_{ij}, & (i, j) \in A', \end{array} \right.$$

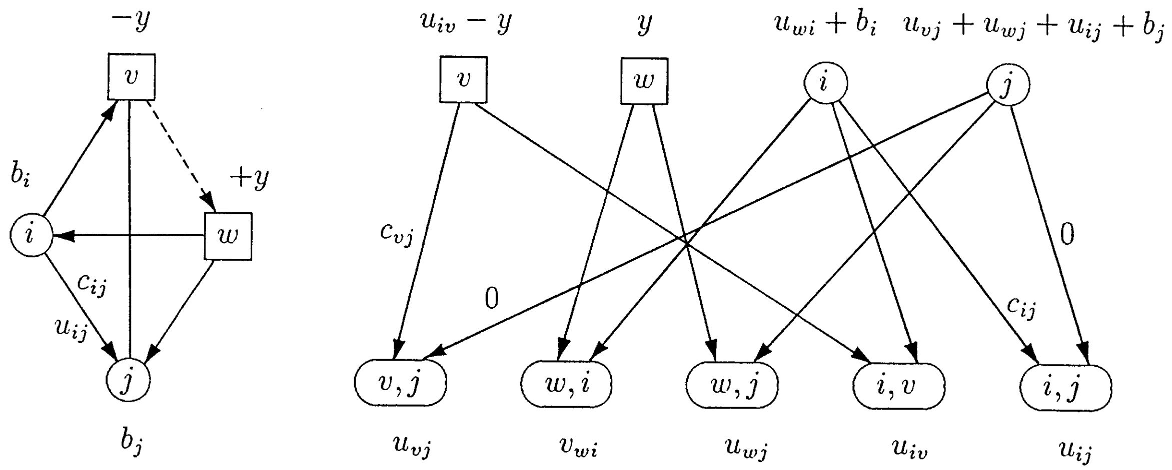

Figure 5.1. Transformation from (FP(y)) to (FP$'$(y)).

where $A' = A\backslash\{(v, w)\}$, $V'(i) = \{j \in N \mid (i, j) \in A'\}$ and $W'(i) = \{j \in N \mid (j, i) \in A'\}$. As well known, we can transform (FP(y)) into a Hitchcock transportation problem in the following manner (see e.g. [1] for details): Let us regard N as the set of sources and A' as the set of terminals. For each $(i, j) \in A'$ we first define two directed arcs $(i, (i, j))$ and $(j, (i, j))$, and assign cost c_{ij} to the former and cost zero to the latter. We next let $\sum_{j\in W'(i)} u_{ji} + b_i$ be the supply of source $i \in N$ and u_{ij} be the demand of terminal $(i, j) \in A'$. Figure 5.1 shows the transformation, where the right network is transformed from the left.

Now we have the following problem equivalent to (FP(y)):

$$
(\mathrm{FP}'(y))\quad
\begin{array}{|ll}
\text{minimize} & \displaystyle\sum_{(i,j)\in A'} c_{ij}\xi_{i(i,j)} \\
\text{subject to} & \displaystyle\sum_{(v,j)\in A'} \xi_{v(v,j)} + \sum_{(j,v)\in A'} \xi_{v(j,v)} = a_v - y, \\
& \displaystyle\sum_{(w,j)\in A'} \xi_{w(w,j)} + \sum_{(j,w)\in A'} \xi_{w(j,w)} = a_w + y, \\
& \displaystyle\sum_{(i,j)\in A'} \xi_{i(i,j)} + \sum_{(j,i)\in A'} \xi_{i(j,i)} = a_i + b_i, \quad i \in N \setminus \{v, w\}, \\
& \xi_{i(i,j)} + \xi_{j(i,j)} = u_{ij}, \qquad (i, j) \in A', \\
& \xi_{i(j,k)} \geq 0, \qquad i \in N \quad (j, k) \in A',
\end{array}
$$

where $\xi_{i(j,k)}$'s are variables and

$$a_i = \sum_{j\in W'(i)} u_{ji}, \quad i \in N. \tag{5.2}$$

It is easy to see that our instance of (FP) can be solved if we minimize the sum of the optimal value of $(\mathrm{FP}'(y))$ and $g_{vw}(y)$ subject to $0 \leq y \leq u_{vw}$. In other words, a globally optimal solution of (FP) with a single nonlinear-cost arc can be obtained if we solve a production-transportation problem:

$$\left|\begin{array}{lll} \text{minimize} & \displaystyle\sum_{(i,j)\in A'} c_{ij}\xi_{i(i,j)} + g_{vw}(y) & \\ \text{subject to} & \displaystyle\sum_{(v,j)\in A'} \xi_{v(v,j)} + \sum_{(j,v)\in A'} \xi_{v(j,v)} = a_v - y, & \\ & \displaystyle\sum_{(w,j)\in A'} \xi_{w(w,j)} + \sum_{(j,w)\in A'} \xi_{w(j,w)} = a_w + y, & \\ & \displaystyle\sum_{(i,j)\in A'} \xi_{i(i,j)} + \sum_{(j,i)\in A'} \xi_{i(j,i)} = a_i + b_i, & i \in N \setminus \{v,\ w\}, \\ & \xi_{i(i,j)} + \xi_{j(i,j)} = u_{ij}, & (i,\ j) \in A', \\ & \xi_{i(j,k)} \geq 0, & i \in N,\ \ (j,\ k) \in A', \\ & 0 \leq y \leq u_{vw}, & \end{array}\right. \tag{5.3}$$

which apparently belongs to (TP).

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