Parameter Insensitive Disturbance-Rejection for Infinite-Dimensional Systems

Naohisa Otsuka† and Hiroshi Inaba‡

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† Institute of Information Sciences and Electronics, University of Tsukuba,Tennodai 1-1-1, Tsukuba city, 305 Ibaraki, Japan.

‡ Department of Information Sciences, Tokyo Denki University, Hatoyama-Machi, Hiki-Gun, Saitama 350-03, Japan.

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Abstract

This paper studies in the framework of the so-called geometric approach two parameter insensitive disturbance-rejection problems with state feedback and with incomplete-state feedback for linear systems defined in Hilbert spaces, and present necessary and / or sufficient conditions for these problems to be solvable under certain assumptions.

1. Introduction

Wonham and Morse[7] studied the disturbance-rejection problem with state feedback for finite-dimensional systems in the framework of the so-called geometric approach. On the other hand, for infinite-dimensional systems, the corresponding problem has been investigated by Curtain[2].

Ghosh[3] investigated two parameter insensitive disturbance-rejection problems with state feedback and with dynamic output feedback for finite-dimensional systems by introducing the notion of simultaneous feedback controlled invariant subspaces. The present authors[6] obtained solvability conditions for the parameter insensitive disturbance-rejection problem with static incomplete-state feedback for finite-dimensional systems.

The objective of this paper is to formulate an infinite-dimensional version of two parameter insensitive disturbance-rejection problems with state feedback and with static incomplete-state feedback, and to study their solvability.

This paper is organized as follows. Section 2 will give various notions of invariant subspaces and their properties. In Section 3, a Hilbert-space version of the parameter insensitive disturbance-rejection problem with state feedback will be formulated, and some necessary and / or sufficient conditions for its solvability will be presented. In Section 4, the static incomplete-state feedback version of the problem will be formulated and its solvability conditions will be presented. In Section 5, an illustrative example of our results will be presented. Finally, Section 6 will give some concluding remarks.

2 Preliminaries

In this section, we give some definitions of simultaneous invariant subspaces and their important properties.

First we give some notations used throughout this investigation. Let $\mathbf{B}(X;Y)$ denote the set of all bounded linear operators from a Hilbert space X into another Hilbert space Y; for notational simplicity, we write $\mathbf{B}(X)$ for $\mathbf{B}(X;X)$. The domain and the image of a linear operator A will be denoted by $D(A)$ and $\mathrm{Im}A$, respectively. Further we use the notations $\mathbf{r}_1:=\{1,\cdots,r_1\}$, $\mathbf{r}_2:=\{1,\cdots,r_2\}$ and $\mathbf{r}_3:=\{1,\cdots,r_3\}$.

Next, we consider the set $\{\Sigma_{ijk};\, i\in\mathbf{r}_1,\, j\in\mathbf{r}_2,\, k\in\mathbf{r}_3\}$ of $r_1\times r_2\times r_3$ systems defined in a real Hilbert space X :

$$(2.1)\quad \Sigma_{ijk}\left|\begin{array}{l}\dot{x}(t)=A_ix(t)+B_ju(t),\ \ x(0)=x_0\in X,\\ y(t)=C_kx(t)\end{array}\right.\qquad (i\in\mathbf{r}_1,\, j\in\mathbf{r}_2,\, k\in\mathbf{r}_3)$$

where A_i is the infinitesimal generator of a C_0-semigroup $\{S_{A_i}(t);\, t\geq 0\}$ on X, while B_j is a bounded linear operator from Euclidean space $\mathbf{R}^m$ into X (i.e., $B_j \in \mathbf{B}(\mathbf{R}^m;X)$, $j \in \mathbf{r}_2$), C_k is a bounded linear operator from X into $\mathbf{R}^p$ (i.e., $C_k \in \mathbf{B}(X;\mathbf{R}^p)$, $k \in \mathbf{r}_3$) and $x(t) \in X$, $u(t) \in \mathbf{R}^m$, $y(t) \in \mathbf{R}^p$ are the state, the input, the output, respectively. For a bounded linear operator $L \in \mathbf{B}(X)$, $\{S_{A_i+L}(t);\ t \geq 0\}$ denotes a semigroup generated by a linear operator A_i+L.

For these systems $\{\Sigma_{ijk}: i \in \mathbf{r}_1, j \in \mathbf{r}_2, k \in \mathbf{r}_3\}$, we give the following definitions.

(2.2)**Definition.** Let $V \subset X$ be a closed subspace.

(i) V is said to be feedback-(A_i,B_j)-invariant if there exists an $F_{ij} \in \mathbf{B}(X;\mathbf{R}^m)$ such that

$$(A_i+B_jF_{ij})(V \cap \mathrm{D}(A_i)) \subset V.$$

(ii) V is said to be $\mathrm{S}(A_i,B_j)$-invariant if there exists an $F_{ij} \in \mathbf{B}(X;\mathbf{R}^m)$ such that

$$S_{A_i+B_jF_{ij}}(t)V \subset V \text{ for all } t \geq 0.$$

(iii) V is said to be (C_k,A_i)-invariant if there exists a $G_{ik} \in \mathbf{B}(\mathbf{R}^q;X)$ such that

$$(A_i+G_{ik}C_k)(V \cap \mathrm{D}(A_i)) \subset V.$$

(iv) V is said to be $\mathrm{S}(C_k,A_i)$-invariant if there exists a $G_{ik} \in \mathbf{B}(\mathbf{R}^q;X)$ such that

$$S_{A_i+G_{ik}C_k}(t)V \subset V \text{ for all } t \geq 0.$$

(v) V is said to be (C_k,A_i,B_j)-invariant if there exists an $H_{ijk} \in \mathbf{B}(\mathbf{R}^q;\mathbf{R}^m)$ such that

$$(A_i+B_jH_{ijk}C_k)(V \cap \mathrm{D}(A_i)) \subset V.$$

(vi) V is said to be $\mathrm{S}(C_k,A_i,B_j)$-invariant if there exists an $H_{ijk} \in \mathbf{B}(\mathbf{R}^q;\mathbf{R}^m)$ such that

$$S_{A_i+B_jH_{ijk}C_k}(t)V \subset V \text{ for all } t \geq 0. \qquad //$$

The following definition is a simultaneous version of Definition (2.2).

(2.3)**Definition.** Let $V \subset X$ be a closed subspace.

(i) V is said to be simultaneous feedback-$\{(A_i,B_j);\ i \in \mathbf{r}_1, j \in \mathbf{r}_2\}$-invariant if there exists an $F \in \mathbf{B}(X;\mathbf{R}^m)$ such that

$$(A_i+B_jF)(V \cap \mathrm{D}(A_i)) \subset V \quad \text{for all } i \in \mathbf{r}_1, j \in \mathbf{r}_2.$$

(ii) V is said to be simultaneous $\{\mathrm{S}(A_i,B_j); i \in \mathbf{r}_1, j \in \mathbf{r}_2\}$-invariant if there exists an $F \in \mathbf{B}(X;\mathbf{R}^m)$

such that

$$S_{A_i+B_jF}(t)V\subset V \text{ for all } t\geq 0 \text{ and all } i\in\mathbf{r}_1,\, j\in\mathbf{r}_2.$$

(iii) V is said to be simultaneous $\{(C_k,A_i);\, i\in\mathbf{r}_1,\, k\in\mathbf{r}_3\}$-invariant if there exists a $G\in\mathbf{B}(\mathbf{R}^q;X)$ such that

$$(A_i+GC_k)(V\cap D(A_i))\subset V \text{ for all } i\in\mathbf{r}_1,\, k\in\mathbf{r}_3.$$

(iv) V is said to be simultaneous $\{S(C_k,A_i);\, i\in\mathbf{r}_1,\, k\in\mathbf{r}_3\}$-invariant if there exists a $G\in\mathbf{B}(\mathbf{R}^q;X)$ such that

$$S_{A_i+GC_k}(t)V\subset V \text{ for all } t\geq 0 \text{ and all } i\in\mathbf{r}_1,\, k\in\mathbf{r}_3.$$

(v) V is said to be simultaneous $\{(C_k,A_i,B_j);\, i\in\mathbf{r}_1,\, j\in\mathbf{r}_2,\, k\in\mathbf{r}_3\}$-invariant if there exists an $H\in\mathbf{B}(\mathbf{R}^q;\mathbf{R}^m)$ such that

$$(A_i+B_jHC_k)(V\cap D(A_i))\subset V \text{ for all } i\in\mathbf{r}_1,\, j\in\mathbf{r}_2,\, k\in\mathbf{r}_3.$$

(vi) V is said to be simultaneous $\{S(C_k,A_i,B_j)\ i\in\mathbf{r}_1,\, j\in\mathbf{r}_2,\, k\in\mathbf{r}_3\}$-invariant if there exists an $H\in\mathbf{B}(\mathbf{R}^q;\mathbf{R}^m)$ such that

$$S_{A_i+B_jHC_k}(t)V\subset V \text{ for all } t\geq 0 \text{ and all } i\in\mathbf{r}_1,\, j\in\mathbf{r}_2,\, k\in\mathbf{r}_3. \quad //$$

(2.4)**Remark.**

(i) We note that, for each system $\Sigma_{ijk}=(A_i,B_j,C_k)$, an $S(C_k,A_i,B_j)$-invariant subspace V has the property that if $x(0)\in V$ then there exists an incomplete-state feedback input $u(t)=H_{ijk}y(t)$ such that $x(t)\in V$ for all $t\geq 0$. On the other hand, for a family $\{\Sigma_{ijk}=(A_i,B_j,C_k)\ i\in\mathbf{r}_1,\, j\in\mathbf{r}_2,\, k\in\mathbf{r}_3\}$ of systems, a simultaneous $\{S(C_k,A_i,B_j);i\in\mathbf{r}_1,\, j\in\mathbf{r}_2,\, k\in\mathbf{r}_3\}$-invariant subspace V has the property that if $x(0)\in V$ then there exists an incomplete-state feedback input $u(t)=Hy(t)$ which is independent on i, j and k such that $x(t)\in V$ for all $t\geq 0$.

(ii) We note that the semigroup invariance implies infinitesimal generator invariance in Definitions (2.2) and (2.3). //

The following lemma can be used to prove our main results.

(2.5)**Lemma.**

(i) If A_i ($i\in\mathbf{r}_1$) are bounded linear operators on X (i.e., $A_i\in\mathbf{B}(X)$), then the statements (i), (ii), the statements (iii), (iv) and the statements (v), (vi) in Definition (2.2) are equivalent, respectively.

(ii) If A_i ($i \in \mathbf{r}_1$) are bounded linear operators on X (i.e., $A_i \in \mathbf{B}(X)$), then the statements (i), (ii), the statements (iii), (iv) and the statements (v)-(vi) in Definition (2.3) are equivalent, respectively. //

(2.6)**Definition.** Let $\Lambda \subset X$ be a closed subspace.

(i) $\mathbf{V}(A_i, B_j; \Lambda) := \{V \subset \Lambda \mid V$ is $S(A_i, B_j)$-invariant subspace.$\}$

(ii) $\mathbf{V}_s(A_i, B_j; \Lambda) := \{V \subset \Lambda \mid V$ is simultaneous $\{S(A_i, B_j); i \in \mathbf{r}_1, j \in \mathbf{r}_2\}$-invariant subspace.$\}$ //

For the case where X is finite dimensional, both $\mathbf{V}(A_i, B_j; \Lambda)$ and $\mathbf{V}_s(A_i, B_j; \Lambda)$ have unique supremal element V_{ij}^* and V^*, respectively[3,7]. On the other hand, when X is infinite-dimensional, the families $\mathbf{V}(A_i, B_j; \Lambda)$ and $\mathbf{V}_s(A_i, B_j; \Lambda)$ are not necessarily closed under subspace addition, and thus there are in general no guarantee that V_{ij}^* and V^* exist. However, we remark that Curtain[2] and Zwart[8] gave some sufficient conditions for its existence.

(2.7)**Lemma**[1]. Let $V \subset X$ be a closed subspace, and let $Q_1 \in \mathbf{B}(X)$. If there exists a $Q_2 \in \mathbf{B}(X)$ such that $S_{A+Q_2}(t)V \subset V$ for all $t \geq 0$ and $(Q_1 - Q_2)(V \cap D(A)) \subset V$, then $S_{A+Q_1}(t)V \subset V$ for all $t \geq 0$. //

(2.8)**Lemma**[4]. Let $\{V_1, \cdots, V_s\}$ be a set of closed subspaces of X and W be a any closed subspace of X. If $V_{i+1} \subset V_i$ ($i=1,\cdots,s$-1), then, there exists a set $\{X_1, \cdots, X_s\}$ of closed subspaces of X such that

$$V_i = X_i \oplus (V_i \cap W), \ X_{i+1} \subset X_i \ (i=0,\ 1,\cdots,s\text{-}1) \text{ and } X = X_0 \oplus W. \quad //$$

(2.9)**Lemma**[5]. Let U_1, U_2 be real Hilbert spaces, and $F_i \in \mathbf{B}(X; U_i)$ $(i=1,2)$ be given. If $\mathrm{Im} F_2$ is closed in U_2, then the following statements are equivalent.

(i) $\mathrm{Ker} F_1 \supset \mathrm{Ker} F_2$.

(ii) There exists a $K \in \mathbf{B}(U_2; U_1)$ such that $F_1 = K F_2$. //

(2.10)**Proposition.** Let V be a closed subspace of X, $C \in \mathbf{B}(X; \mathbf{R}^p)$ be given, and suppose that

(i) V is simultaneous $\{S(A_i, B_j); i \in \mathbf{r}_1, j \in \mathbf{r}_2\}$-invariant,

(ii) V is simultaneous $\{(C, A_i); i \in \mathbf{r}_1\}$-invariant, and

(iii) $\overline{V\cap \mathrm{Ker}C\cap \mathrm{D}(A_{i_0})} = V\cap \mathrm{Ker}C$ for some $i_0\in \mathbf{r}_1$.

Then, V is simultaneous $\{S(C,A_i,B_j);\ i\in \mathbf{r}_1, j\in \mathbf{r}_2\}$-invariant.

Proof. Suppose that V satisfies (i), (ii) and (iii). Then, there exists an $F\in \mathbf{B}(X;\mathbf{R}^m)$ such that

$$S_{A_i+B_jF}(t)V\subset V \text{ for all } t\geq 0 \text{ and all } i\in \mathbf{r}_1,\ j\in \mathbf{r}_2,$$

and there exists a $G\in \mathbf{B}(\mathbf{R}^q;X)$ such that

$$(A_i+GC)(V\cap \mathrm{D}(A_i))\subset V \text{ for all } i\in \mathbf{r}_1.$$

Since $V\subset X$, it follows from Lemma (2.8) that there exists a set $\{X_0, X_1\}$ of subspaces of X such that

$$V=X_1\oplus(V\cap \mathrm{Ker}C),\ \ X_1\subset X_0,\ \text{ and } X=X_0\oplus \mathrm{Ker}C.$$

Further, denote by P the projection operator of X onto X_0 along Ker$C$, and set $F_0:=FP$. Then since $Px=0$ for any $x\in \mathrm{Ker}C$, one has

$$F_0x=F(Px)=0 \text{ for all } x\in \mathrm{Ker}C,$$

which shows that

$$\mathrm{Ker}C\subset \mathrm{Ker}F_0. \tag{1}$$

Hence, it follows, from Lemma (2.9) with (1) and closedness of ImC, that there exists an $H\in \mathbf{B}(\mathbf{R}^q;\mathbf{R}^m)$ such that $F_0=HC$.

Now, we claim that this H satisfies

$$S_{A_i+B_jHC}(t)V\subset V \text{ for all } t\geq 0 \text{ and all } i\in \mathbf{r}_1,\ j\in \mathbf{r}_2. \tag{2}$$

In order to verify this claim, it suffices, by virtue of Lemma (2.7), to show

$$(B_jF - B_jF_0)V\subset V. \tag{3}$$

To show (3), first let $x\in V=X_1\oplus(V\cap \mathrm{Ker}C)$. Then x can be written uniquely in the form $x=y+z$ with $y\in X_1$ and $z\in V\cap \mathrm{Ker}C$. By the hypothesis (iii), there exists a sequence $\{z_n^{i_0}\}$ from $(V\cap \mathrm{Ker}C\cap \mathrm{D}(A_{i_0}))$ such that $\lim_{n\to\infty} z_n^{i_0}=z$. Now, by the definition of F_0, one obtain

$$(B_jF - B_jF_0)x=(B_jF - B_jF_0)z=B_jFz. \tag{4}$$

On the other hand, since $(A_{i_0}+B_jF)z_n^{i_0}\in V$ and $(A_{i_0}+GC)z_n^{i_0}\in V$, it follows that

$$B_jFz_n^{i_0}=(A_{i_0}+B_jF)z_n^{i_0} - (A_{i_0}+GC)z_n^{i_0}\in V \text{ for all } j\in \mathbf{r}_2. \tag{5}$$

Thus, by continuity of B_jF, closedness of V and (4), (5), one obtains

$$(B_jF - B_jF_0)x=B_jFz=\lim_{n\to\infty} B_jFz_n^{i_0}\in V \text{ for all } j\in \mathbf{r}_2.$$

Finally, it follows that $(B_jF - B_jF_0)V \subset V$ for all $j \in \mathbf{r}_2$, showing (3). Thus, (2) is satisfied, and V is simultaneous $\{S(C,A_i,B_j);\ i \in \mathbf{r}_1, j \in \mathbf{r}_2\}$-invariant. This completes the proof. //

3. The Problem with State Feedback

In this section, we will first formulate our parameter insensitive disturbance-rejection problem with state feedback for infinite-dimensional systems, and then give some solvability conditions for this problem. The linear control system to be considered is given by

$$(3.1)\qquad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + K\xi(t), & x(0)=x_0 \in X \\ z(t) = Dx(t) \end{cases}$$

where $x(t) \in X$, $u(t) \in \mathbf{R}^m$ and $z(t) \in \mathbf{R}^p$ are the state, the input and the controlled output, respectively. $\xi(\cdot)$ represents a disturbance which is a locally integrable function from $(0,\infty)$ to a Hilbert space Q (i.e., $\xi(\cdot) \in L_1^{\rm loc}(0,\infty;Q)$), and $K \in \mathbf{B}(Q;X)$ and $D \in \mathbf{B}(X;\mathbf{R}^p)$.

We assume that operators A, B, K and D are unknown, but they are assumed to have the following forms :

$$(3.2)\qquad \begin{cases} A = \alpha A_1 + (1-\alpha)A_2, & B = \beta B_1 + (1-\beta)B_2 \\ K = \gamma K_1 + (1-\gamma)K_2, & D = \sigma D_1 + (1-\sigma)D_2 \end{cases}$$

where parameters α, β, γ, $\sigma \in [0,1]$ are unknown and operator A_i is the infinitesimal generator of a C_0-semigroup $\{S_{A_i}(t);\ t \geq 0\}$ on X, B_1, $B_2 \in \mathbf{B}(\mathbf{R}^m;X)$, K_1, $K_2 \in \mathbf{B}(Q;X)$ and D_1, $D_2 \in \mathbf{B}(X;\mathbf{R}^p)$, and all these operators are known. We note that, even if A_1 and A_2 are infinitesimal generators, $A = \alpha A_1 + (1-\alpha)A_2$ ($\alpha \in [0,1]$) may not be an infinitesimal generator. Therefore, whenever considering an operator $A = \alpha A_1 + (1-\alpha)A_2$ of (3.2), it is always assumed that A is the infinitesimal generator of a C_0-semigroup $\{S_A(t);\ t \geq 0\}$ on X. We remark that if A_1 and A_2 are bounded linear operators, $A = \alpha A_1 + (1-\alpha)A_2$ is always infinitesimal generator.

Now, for a subset V of X, introduce the notation

$$< S_A(\cdot) \mid V > := \overline{\mathrm{L}(\bigcup_{t \geq 0} S_A(t)V)},$$

where $\mathrm{L}(E)$ means the linear subspace generated by the set E and the over bar indicates the closure in

Hilbert space X.

Next, we consider a state feedback of the form

(3.3) $\qquad u(t)=Fx(t)$

where $F\in\mathbf{B}(X;\mathbf{R}^m)$. Then we have the following closed loop system

(3.4) $$\begin{cases} \dot{x}(t) = (A+BF)x(t) + K\xi(t), & x(0)=x_0\in X \\ z(t) = Dx(t). \end{cases}$$

Our parameter insensitive disturbance-rejection problem with state feedback is to find a state feedback of (3.3) such that output $z(t)$ in system (3.4) is not affected by disturbance $\xi(t)$ for all $\alpha,\ \beta,\ \gamma,\ \sigma\in[0,1]$. To achieve this control requirement we must solve the following problem : Given $A_1, A_2, B_1, B_2, K_1, K_2, D_1$ and D_2 of (3.2), find (if possible) an $F\in\mathbf{B}(X;\mathbf{R}^m)$ such that

$$D\int_0^t S_{A+BF}(t-\tau)K\xi(\tau)\mathrm{d}\tau = 0$$

for all $\xi\in L_1^{\mathrm{loc}}(0,\infty;\mathcal{Q})$, all $t\geq 0$ and all $\alpha,\ \beta,\ \gamma,\ \sigma\in[0,1]$, or equivalently $<S_{A+BF}(\cdot)|\mathrm{Im}K>\subset\mathrm{Ker}D$ for all $\alpha,\ \beta,\ \gamma,\ \sigma\in[0,1]$.

This problem can be formulated as follows.

(3.5)**Parameter Insensitive Disturbance-Rejection Problem with State Feedback (PIDRPSF).** Given $A_1, A_2, B_1, B_2, K_1, K_2, D_1$ and D_2 of (3.2), find (if possible) an $F\in\mathbf{B}(X;\mathbf{R}^m)$ such that $<S_{A+BF}(\cdot)|\mathrm{Im}K>\subset\mathrm{Ker}D$ for all $\alpha,\ \beta,\ \gamma,\ \sigma\in[0,1]$. //

(3.6) **Assumption.**

(i) It is assumed that each $\mathbf{V}(A_i,B_j;\mathrm{Ker}D_1\cap\mathrm{Ker}D_2)$ has a unique supremal element V_{ij}^*, $i, j\in\{1,2\}$.

(ii) It is assumed that $\mathbf{V}_s(A_i,B_j;\mathrm{Ker}D_1\cap\mathrm{Ker}D_2)$ has a unique supremal element V^*. //

The following lemmas play important roles to prove our main theorems.

(3.7)**Lemma.** Suppose that system (3.1) satisfies Assumption(3.6,ii), and let V^* denote the supremal subspace in $\mathbf{V}_s(A_i,B_j;\mathrm{Ker}D_1\cap\mathrm{Ker}D_2)$. Then, the following two statements hold.

(i) If $A = A_1 = A_2$, then there exists an $F\in\mathbf{B}(X;\mathbf{R}^m)$ such that

$$S_{A+BF}(t)V^*\subset V^* \text{ for all } t\geq 0 \text{ and all } \beta\in[0,1].$$

(ii) If A_1 and A_2 are bounded linear operators on X (i.e., $A_1, A_2 \in \mathbf{B}(X)$), then there exists an $F \in \mathbf{B}(X;\mathbf{R}^m)$ such that

$$S_{A+BF}(t)\,V^* \subset V^* \quad \text{for all } t \geq 0 \text{ and all } \alpha,\ \beta \in [0,1].$$

Proof. First, we will prove (i). Since $A = A_1 = A_2$ and V^* is a supremal element in $\mathbf{V}_s(A_i, B_j; \mathrm{Ker}D_1 \cap \mathrm{Ker}D_2)$, there exists an $F \in \mathbf{B}(X;\mathbf{R}^m)$ such that

$$S_{A+B_jF}(t)\,V^* \subset V^* \quad \text{for all } t \geq 0 \text{ and all } j \in \{1,2\}, \tag{1}$$

and hence Remark (2.4,ii) gives

$$(A+B_jF)(V^* \cap \mathrm{D}(A)) \subset V^* \quad \text{for all } j \in \{1,2\}. \tag{2}$$

Now, note that

$$(A+BF) = (A+B_1F) - (1-\beta)(A+B_1F) + (1-\beta)(A+B_2F). \tag{3}$$

By virtue of (2) and (3), for arbitrary $x \in (V^* \cap \mathrm{D}(A))$, we have

$$(A+BF)x = (A+B_1F)x - (1-\beta)(A+B_1F)x + (1-\beta)(A+B_2F)\,x \in V^* \quad \text{for all } \beta \in [0,1],$$

which proves

$$(A+BF)(V^* \cap \mathrm{D}(A)) \subset V^* \quad \text{for all } \beta \in [0,1]. \tag{4}$$

Further, it follows from Lemma (2.7) and (1) that, if we show

$$(B_jF - BF)(V^* \cap \mathrm{D}(A)) \subset V^* \quad \text{for all } j \in \{1,2\} \text{ and } \beta \in [0,1] \tag{5}$$

then the desired relations

$$S_{A+BF}(t)\,V^* \subset V^* \quad \text{for all } t \geq 0 \text{ and all } \beta \in [0,1]$$

obtain. To prove (5), let $x \in (V^* \cap \mathrm{D}(A))$. Then, from (2) and (4) we obtain

$$(B_jF - BF)x = (A+B_jF)x - (A+BF)x \in V^* \quad \text{for all } j \in \{1,2\} \text{ and } \beta \in [0,1]$$

which proves (5). This proves assertion (i).

Next, we will prove (ii). Since that A_1 and A_2 are bounded linear operators on X and V^* is a supremal element of $\mathbf{V}_s(A_i, B_j; \mathrm{Ker}D_1 \cap \mathrm{Ker}D_2)$, there exists an $F \in \mathbf{B}(X;\mathbf{R}^m)$ such that

$$S_{A_i+B_jF}(t)\,V^* \subset V^* \quad \text{for all } t \geq 0 \text{ and all } i, j \in \{1,2\}, \tag{6}$$

and hence from Remark (2.4,ii) gives

$$(A_i+B_jF)\,V^* \subset V^* \quad \text{for all } i,\ j \in \{1,2\}. \tag{7}$$

Note that

$$(A+BF) = \alpha(A_1+B_1F) + (1-\alpha)(A_2+B_1F) - (1-\beta)(A_1+B_1F) + (1-\beta)(A_1+B_2F). \tag{8}$$

Hence, by virtue of (7) and (8), for arbitrary $x \in V^*$, we have

$$(A+BF)x = \alpha(A_1+B_1F)x + (1-\alpha)(A_2+B_1F)x - (1-\beta)(A_1+B_1F)x + (1-\beta)(A_1+B_2F)x \in V^*$$

which proves

$$(A+BF)V^* \subset V^* \quad \text{for all } \alpha,\ \beta \in [0,1].$$

Since $(A+BF)$ is a bounded linear operator, it follows from Lemma(2.5,ii) that

$$S_{A+BF}(t)V^* \subset V^* \quad \text{for all } t \geq 0 \text{ and all } \alpha,\ \beta \in [0,1].$$

Thus this proves assertion (ii). //

The following lemma can be easily obtained and its proof is omitted.

(3.8)**Lemma.** Suppose that system (3.1) satisfies Assumption(3.6,ii), and let V^* denote the supremal subspace in $\mathbf{V}_s(A_i,B_j;\mathrm{Ker}D_1 \cap \mathrm{Ker}D_2)$. Then, the following assertions hold.

(i) If $\mathrm{Im}K_1 + \mathrm{Im}K_2 \subset V^*$, then $\mathrm{Im}K \subset V^*$ for all $\gamma \in [0,1]$.

(ii) $V^* \subset \mathrm{Ker}D$ for all $\sigma \in [0,1]$. //

The following two theorems give sufficient conditions for PIDRPSF (3.5) to be solvable.

(3.9) **Theorem.** Suppose that system (3.1) satisfies $A = A_1 = A_2$ and Assumption (3.6,ii), and let V^* denote the supremal subspace in $\mathbf{V}_s(A_i,B_j;\mathrm{Ker}D_1 \cap \mathrm{Ker}D_2)$. If

$$\mathrm{Im}K_1 + \mathrm{Im}K_2 \subset V^*$$

then PIDRPSF (3.5) is solvable.

Proof. Suppose that $\mathrm{Im}K_1 + \mathrm{Im}K_2 \subset V^*$. Then, it follows from Lemma (3.8) that

$$\mathrm{Im}K \subset V^* \subset \mathrm{Ker}D \tag{9}$$

for all $\gamma,\ \sigma \in [0,1]$. On the other hand, since $A = A_1 = A_2$, it follows from Lemma (3.7,i) that there exists an $F \in \mathbf{B}(X;\mathbf{R}^m)$ such that

$$S_{A+BF}(t)V^* \subset V^* \quad (t \geq 0) \quad \text{for all } \beta \in [0,1]. \tag{10}$$

Thus, (9) and (10) give that

$$< S_{A+BF}(\cdot) \mid \mathrm{Im}K > \subset < S_{A+BF}(\cdot) \mid V^* > = V^* \subset \mathrm{Ker}D \quad \text{for all } \beta,\ \gamma,\ \sigma \in [0,1].$$

Hence, PIDRPSF (3.5) is solvable. //

(3.10)**Theorem.** Suppose that system (3.1) satisfies A_1 and A_2 are bounded linear operators on X (i.e., $A_1, A_2 \in \mathbf{B}(X)$) and satisfies Assumption (3.6,ii), and let V^* denote the supremal subspace in $\mathbf{V}_s(A_i, B_j; \mathrm{Ker}D_1 \cap \mathrm{Ker}D_2)$. If

$$\mathrm{Im}K_1 + \mathrm{Im}K_2 \subset V^*$$

then PIDRPSF (3.5) is solvable.

Proof. Suppose that $\mathrm{Im}K_1 + \mathrm{Im}K_2 \subset V^*$. Then, it follows from Lemma (3.8) that

$$\mathrm{Im}K \subset V^* \subset \mathrm{Ker}D \tag{11}$$

for all γ, $\sigma \in [0,1]$. On the other hand, since A_1 and A_2 are bounded linear operators on X, it follows from Lemma (3.7,ii) that there exists an $F \in \mathbf{B}(X;\mathbf{R}^m)$ such that

$$S_{A+BF}(t)\, V^* \subset V^* \quad \text{for all } t \geq 0 \text{ and all } \alpha,\ \beta \in [0,1]. \tag{12}$$

Thus, (11) and (12) give that

$$< S_{A+BF}(\cdot) \mid \mathrm{Im}K > \ \subset \ < S_{A+BF}(\cdot) \mid V^* > = V^* \subset \mathrm{Ker}D \quad \text{for all } \alpha,\ \beta,\ \gamma,\ \sigma \in [0,1].$$

Hence, PIDRPSF (3.5) is solvable. //

(3.11)**Theorem.** Suppose that system (3.1) satisfies Assumption (3.6,i), and let V_{ij}^* denote the supremal subspace in $\mathbf{V}(A_i, B_j; \mathrm{Ker}D_1 \cap \mathrm{Ker}D_2)$. If PIDRPSF (3.5) is solvable then

$$\mathrm{Im}K_1 + \mathrm{Im}K_2 \subset \bigcap_{i,j \in \{1,2\}} V_{ij}^*.$$

Proof. Suppose that PIDRPSF (3.5) is solvable. Then, there exists an $F \in \mathbf{B}(X;\mathbf{R}^m)$ such that

$$< S_{A+BF}(\cdot) \mid \mathrm{Im}K > \subset \mathrm{Ker}D \quad \text{for all } \alpha,\ \beta,\ \gamma,\ \sigma \in [0,1].$$

Since α, β, γ, $\sigma \in [0,1]$ are arbitrary, we have

$$< S_{A_i+B_jF}(\cdot) \mid \mathrm{Im}K_1 > \ \in \mathbf{V}(A_i, B_j; \mathrm{Ker}D_1 \cap \mathrm{Ker}D_2).$$

Thus,

$$\mathrm{Im}K_1 \subset < S_{A_i+B_jF}(\cdot) \mid \mathrm{Im}K_1 > \ \subset V_{ij}^* \quad \text{for all } i,\ j \in \{1,2\}.$$

Similarly, we obtain

$$\mathrm{Im}K_2 \subset < S_{A_i+B_jF}(\cdot) \mid \mathrm{Im}K_2 > \ \subset V_{ij}^* \quad \text{for all } i,\ j \in \{1,2\}.$$

From (13) and (14), the desired result

$$\mathrm{Im}K_1 + \mathrm{Im}K_2 \subset \bigcap_{i,j \in \{1,2\}} V_{ij}^*.$$

follows. //

4. The Problem with Incomplete-State Feedback

In this section, a parameter insensitive disturbance-rejection problem with incomplete-state feedback will be studied. We will first formulate the problem and then we will present some solvability conditions.

Consider System (3.1) with an incomplete-state $y(t)\in\mathbf{R}^p$, i.e.,

$$\text{(4.1)}\quad \begin{cases} \dot{x}(t)=Ax(t)+Bu(t)+K\xi(t), \quad x(0)=x_0\in X \\ y(t)=Cx(t) \\ z(t)=Dx(t). \end{cases}$$

We assume that operator C is also unknown, but has the following form:

$$\text{(4.2)}\quad C=\delta C_1+(1-\delta)C_2$$

where parameter $\delta\in[0,1]$ is unknown and operators $C_1, C_2\in\mathbf{B}(X;\mathbf{R}^p)$ are known.

Now, we consider an incomplete-state feedback of the form

$$\text{(4.3)}\quad u(t)=Hy(t)$$

where $H\in\mathbf{B}(\mathbf{R}^p;\mathbf{R}^m)$. Then we have the following closed loop system

$$\text{(4.4)}\quad \begin{cases} \dot{x}(t)=(A+BHC)x(t)+K\xi(t), \quad x(0)=x_0\in X \\ z(t)=Dx(t). \end{cases}$$

Our problem is to find $H\in\mathbf{B}(\mathbf{R}^p;\mathbf{R}^m)$ such that the output $z(t)$ of System (4.4) is unaffected by the disturbance $\xi(t)$ for all $\alpha, \beta, \gamma, \delta, \sigma\in[0,1]$, as in PIDRPSF(3.5), it is not difficult to see that this problem can be formulated as follows.

(4.5)**Parameter Insensitive Disturbance-Rejection Problem with Incomplete-State Feedback (PIDRPISF).** Given A_1, A_2, B_1, B_2, K_1, K_2, C_1, C_2, D_1 and D_2 of (3.2) and (4.2), find (if possible) an $H\in\mathbf{B}(\mathbf{R}^p;\mathbf{R}^m)$ such that

$$< S_{A+BHC}(\cdot) \mid \mathrm{Im}K > \subset \mathrm{Ker}D \quad \text{for all } \alpha, \beta, \gamma, \delta, \sigma\in[0,1]. \quad //$$

First, the following theorem will be proved.

(4.6)**Theorem.** If PIDRPISF (4.5) is solvable, then there exists (C_k, A_i, B_j)-invariant subspaces $\{V_{ijk}, i, j, k\in\{1, 2\}\}$ such that

(i) $(\mathrm{Im}K_1 + \mathrm{Im}K_2) \subset \bigcap_{i,j,k \in \{1,2\}} V_{ijk}$,

(ii) $\sum_{i,j,k \in \{1,2\}} V_{ijk} \subset (\mathrm{Ker}D_1 \cap \mathrm{Ker}D_2)$ and

(iii) $\bigcap_{i,j,k \in \{1,2\}} \mathbf{H}(A_i, B_j, C_k; V_{ijk}) \neq \varnothing$,

where $\mathbf{H}(A_i, B_j, C_k; V_{ijk}) := \{H \in \mathbf{B}(\mathbf{R}^p; \mathbf{R}^m) \mid S_{A_i+B_jHC_k}(t) V_{ijk} \subset V_{ijk} \text{ for all } t \geq 0\}$.

Proof. Suppose that Problem (4.5) is solvable. Then, there exists an $H \in \mathbf{B}(\mathbf{R}^p; \mathbf{R}^m)$ such that

$$< S_{A+BHC}(\cdot) \mid \mathrm{Im}K > \subset \mathrm{Ker}D \quad \text{for all } \alpha, \beta, \gamma, \delta, \sigma \in [0,1]. \tag{1}$$

For each $i, j, k \in \{1,2\}$, set

$$V_{ijk}^1 := < S_{A_i+B_jHC_k}(\cdot) \mid \mathrm{Im}K_1 > \text{ and } V_{ijk}^2 := < S_{A_i+B_jHC_k}(\cdot) \mid \mathrm{Im}K_2 >. \tag{2}$$

Then, from the definition of V_{ijk}^m (m=1,2), each V_{ijk}^m is an $S_{A_i+B_jHC_k}(t)$-invariant subspace, and hence so is $V_{ijk} := V_{ijk}^1 + V_{ijk}^2$. Therefore, it follows from (1) and (2) that

$$\mathrm{Im}K_1 + \mathrm{Im}K_2 \subset V_{ijk} \subset (\mathrm{Ker}D_1 \cap \mathrm{Ker}D_2) \text{ and}$$

$$S_{A_i+B_jHC_k}(t) V_{ijk} \subset V_{ijk} \quad \text{for all } t \geq 0 \text{ and all } i, j, k \in \{1,2\},$$

showing (i), (ii) and (iii). This completes the proof. //

Now, it is ready to show our main results.

(4.7)**Theorem.** Suppose that system (4.1) satisfies $A = A_1 = A_2$. If there exists a simultaneous $\{\mathrm{S}(C_k, A, B_j); j, k \in \{1,2\}\}$-invariant subspace V satisfying

$$\mathrm{Im}K_1 + \mathrm{Im}K_2 \subset V \subset (\mathrm{Ker}D_1 \cap \mathrm{Ker}D_2), \tag{3}$$

then PIDRPISF (4.5) is solvable.

Proof. Suppose that (3) holds. So, there exists an $H \in \mathbf{B}(\mathbf{R}^p; \mathbf{R}^m)$ such that

$$S_{A+B_jHC_k}(t) V \subset V \text{ for all } t \geq 0 \text{ and all } j, k \in \{1,2\}. \tag{4}$$

Then, (4) implies

$$(A + B_jHC_k) V \subset V \text{ for all } j, k \in \{1,2\}. \tag{5}$$

Note that

$$(A+BHC) = \beta\delta(A+B_1HC_1) + \beta(1-\delta)(A+B_1HC_2)$$
$$-(1-\beta)\delta(A+B_2HC_1) + (1-\beta)(1-\delta)(A+B_2HC_2). \tag{6}$$

Now, by virtue of (5) and (6), for arbitrary $x \in V \cap D(A)$, we have

$$(A+BHC)x = \beta\delta(A+B_1HC_1)x + \beta(1-\delta)(A+B_1HC_2)x - (1-\beta)\delta(A+B_2HC_1)x$$
$$+(1-\beta)(1-\delta)(A+B_2HC_2)x \in V$$

which proves

$$(A+BHC)(V \cap D(A)) \subset V \quad \text{for all } \beta, \delta \in [0,1]. \tag{7}$$

Further, it follows from Lemma (2.7) and (4) that, if we show

$$(B_jHC_k - BHC)(V \cap D(A)) \subset V \quad \text{for all } j,\ k \in \{1,2\} \text{ and all } \beta,\ \delta \in [0,1] \tag{8}$$

then the inclusion

$$S_{A+BHC}(t)V \subset V \quad \text{for all } t \geq 0 \text{ and all } \beta,\ \delta \in [0,1] \tag{9}$$

is obtained. To prove (8), let $x \in (V \cap D(A))$. Then, from (5) and (7) we obtain

$$(B_jHC_k - BHC)x = (A+B_jHC_k)x - (A+BHC)x \in V \quad \text{for all } j,\ k \in \{1,2\} \text{ and all } \beta,\ \delta \in [0,1]$$

which proves (8).

Now, finally it follows from Lemma (3.8), (3) and (9) that

$$< S_{A+BHC}(\cdot) \mid \mathrm{Im}K > \subset < S_{A+BHC}(\cdot) \mid V >$$
$$\subset V$$
$$\subset \mathrm{Ker}D \quad \text{for all } \beta,\ \gamma, \delta,\ \sigma \in [0,1].$$

Hence, PIDRPISF (4.5) is solvable. //

(4.8)**Theorem.** Suppose that A_1 and A_2 in system (4.1) are bounded linear operators on X (i.e., A_1, $A_2 \in \mathbf{B}(X)$). If there exists a simultaneous $\{S(C_k, A_i, B_j);\ i, j,\ k \in \{1,2\}\}$-invariant subspace V satisfying

$$\mathrm{Im}K_1 + \mathrm{Im}K_2 \subset V \subset (\mathrm{Ker}D_1 \cap \mathrm{Ker}D_2), \tag{10}$$

then PIDRPISF (4.5) is solvable.

Proof. Suppose that (10) holds. So, there exists an $H \in \mathbf{B}(\mathbf{R}^p; \mathbf{R}^m)$ such that

$$S_{A_i+B_jHC_k}(t)V \subset V \quad \text{for all } t \geq 0 \text{ and all } i, j,\ k \in \{1,2\}. \tag{11}$$

Then, (11) implies

$$(A_i+B_jHC_k)V \subset V \text{ for all } i, j,\ k \in \{1,2\}. \tag{12}$$

Note that

$$(A+BHC) = \alpha(A_1+B_1HC_1) + (\beta\delta - \alpha)(A_2+B_1HC_1) + \beta(1-\delta)(A_2+B_1HC_2)$$
$$+(1-\beta)\delta(A_2+B_2HC_1) + (1-\beta)(1-\delta)(A_2+B_2HC_2). \tag{13}$$

Now, by virtue of (12) and (13), for arbitrary $x \in V$, we have

$$(A+BHC)x = \alpha(A_1+B_1HC_1)x + (\beta\delta-\alpha)(A_2+B_1HC_1)x+\beta(1-\delta)(A_2+B_1HC_2)x$$
$$+(1-\beta)\delta(A_2+B_2HC_1)x+(1-\beta)(1-\delta)(A_2+B_2HC_2)x\in V$$

which proves

$$(A+BHC)V\subset V \quad \text{for all } \alpha,\ \beta,\ \delta\in[0,1]. \tag{14}$$

Since $(A+BHC)$ is a bounded linear operator on X, it follows from Lemma (2.5,ii) and (14) that

$$S_{A+BHC}(t)V\subset V \quad \text{for all } t\geq 0 \text{ and all } \alpha,\ \beta,\ \delta\in[0,1]. \tag{15}$$

Now, finally it follows from Lemma (3.8), (10) and (15) that

$$\begin{aligned} < S_{A+BHC}(\cdot) \mid \mathrm{Im}K > &\subset\ < S_{A+BHC}(\cdot) \mid V > \\ &\subset V \\ &\subset \mathrm{Ker}D \quad \text{for all } \alpha,\ \beta\ \gamma\ \delta,\ \sigma\in[0,1]. \end{aligned}$$

Hence, PIDRPISF (4.5) is solvable. //

5. Example

Consider the following system S :

$$\frac{\partial x(t,\eta)}{\partial t} = \frac{\partial^2 x(t,\eta)}{\partial\eta^2} + b(\eta)u(t) + k(\eta)\xi(t),\ \ (0<\eta<1),\qquad x(t,0)=0=x(t,1),$$

$$z(t) = \int_0^1 d(\eta)x(t,\eta)\mathrm{d}\eta\ ,$$

where $x(t,\eta)$ is the temperature distribution of a bar of the unit length at position η and time t, $u(t)\in\mathbf{R}$ is the input, $\xi(t)\in\mathbf{R}$ is the disturbance and $z(t)\in\mathbf{R}$ is the controlled output. Suppose that functions $b(\eta)$ and $k(\eta)$ are unknown, but have the following form :

$$b(\eta) = \beta\, b_1(\eta) + (1-\beta)b_2(\eta),\quad k(\eta) = \gamma\, k_1(\eta) + (1-\gamma)k_2(\eta),\quad \beta,\ \gamma\in[0,1]$$

where $b_i(\eta)$ and $k_i(\eta)$ $(i=1,2)$ are some given known functions in $L^2[0,1]$ and β, γ are unknown parameters.

Let $X = L^2[0,1]$, and define various operators as follows :

$$A := \frac{d^2}{d\eta^2} \text{ where } \mathrm{D}(A) = \{\ x(\cdot)\in X \mid x''\in X,\ x(0)=x(1)=0\ \},$$

$$B_1:=b_1(\eta)\in X,\ B_2:=b_2(\eta)\in X,\ K_1:=k_1(\eta)\in X,\ K_2:=k_2(\eta)\in X \text{ and } D := <\cdot \mid d(\eta)>_X.$$

Then, the given system S can be described by the following evolution equation on X :

$$\dot{x}(t) = Ax(t) + Bu(t) + K\xi(t), \quad z(t) = Dx(t).$$

Now, suppose that $d \in D(A)$, $<b_1, d> \neq 0$, $<b_2, d> \neq 0$, $b_1 = b_2 + \lambda$ where $\lambda \in \mathrm{Ker}D$. Then using the results of [8] it can be shown that Ker$D$ is a simultaneous $\{S(A, B_j); i \in \{1,2\}\}$-invariant subspace. Further, if $k_1(\eta)$ and $k_2(\eta)$ are assumed to be elements of Ker$D$, it follows from Theorem (3.9) that PIDRPSF (3.5) is solvable.

6. Conclusions

The infinite-dimensional version of two parameter insensitive disturbance-rejection problems with state feedback and with incomplete-state feedback were studied, and necessary and / or sufficient conditions for these problems to be solvable were obtained under certain assumptions. These results are extension of the finite-dimensional results of Ghosh[3] and of present authors[6] to the infinite-dimensional case. A general method for checking the conditions given in this paper has not been known, and this should be studied as a future problem. Further, it would be interesting to investigate necessary and sufficient conditions for Problems (3.5) and (4.5) to be solvable.

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