The Eigenvalue Problem for Infinite Complex Symmetric Tridiagonal Matrices with Application

Running Title:

The Eigenvalue Problem for Infinite Matrices

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Abstract

We consider an infinite complex symmetric (not necessarily Hermitian) tridiagonal matrix $\boldsymbol{T}$ whose diagonal elements diverge to ∞ in modulus and whose off-diagonal elements are bounded. We regard $\boldsymbol{T}$ as a linear operator mapping a maximal domain in the Hilbert space ℓ^2 into ℓ^2. Assuming the existence of $\boldsymbol{T}^{-1}$ we consider the problem of approximating a given *simple* eigenvalue λ of $\boldsymbol{T}$ by an eigenvalue λ_n of $\boldsymbol{T}_n$, the n–th order principal submatrix of $\boldsymbol{T}$. Let $\mathbf{x} = [x^{(1)},\ x^{(2)},\ \ldots]^T$ be an eigenvector corresponding to λ. Assuming $\mathbf{x}^T\mathbf{x} \neq 0$ and $f_{n+1}x^{(n+1)}/x^{(n)} \to 0$ as $n \to \infty$, we will show that there exists a sequence $\{\lambda_n\}$ of $\boldsymbol{T}_n$ such that $\lambda - \lambda_n = f_{n+1}x^{(n)}x^{(n+1)}[1 + o(1)]/(\mathbf{x}^T\mathbf{x}) \to 0$, where f_{n+1} represents the $(n,\ n+1)$ element of $\boldsymbol{T}$. Application to the following problems is included: (a) solve $J_\nu(z) = 0$ for ν, given $z \neq 0$, (b) compute the eigenvalues of the Mathieu equation, and (c) compute the eigenvalues of the spheroidal wave equation. Fortunately, the existence of $\boldsymbol{T}^{-1}$ need not be verified for these examples since we may show that $\boldsymbol{T} + \alpha\boldsymbol{I}$ with α taken appropriately has an inverse.

1 Introduction

In this paper we consider the eigenvalue problem

$$Tx = \lambda x, \tag{1}$$

where

$$T = \begin{bmatrix} d_1 & f_2 & & 0 \\ f_2 & d_2 & f_3 & \\ & f_3 & d_3 & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix}, \tag{2}$$

with

$$0 < |d_n| \to \infty, \text{ and } 0 < |f_n| \le \alpha \ (say, \text{ a constant}), \tag{3}$$

and the eigenvector

$$\mathbf{x} = [x^{(1)},\ x^{(2)},\ \ldots]^T \neq 0 \tag{4}$$

is sought in the Hilbert space ℓ^2, the well-known Hilbert space of all square-summable complex sequences (written as a column vector). The domain of T is defined to be the maximal domain

$$D(T) = \{ y = [y^{(1)},\ y^{(2)},\ \ldots]^T :\ [d_1 y^{(1)},\ d_2 y^{(2)},\ \ldots]^T \in \ell^2 \}. \tag{5}$$

In our earlier paper [10], we study the eigenvalue problem for a compact complex symmetric matrix, especially for a tridiagonal matrix (i.e., the case $d_n \to 0$ and $f_n \to 0$ as $n \to \infty$, to use the same notation as in (2)). The present paper is complementary to this earlier paper of ours. Also related to the present paper is [16], which is concerned with the localization of eigenvalues of a matrix acting in the ℓ_1 or ℓ_∞ space.

Our particular concern in this paper is the problem of approximating a given nonzero *simple* eigenvalue λ of T (see below for definition) by an eigenvalue λ_n of T_n, the n-th order principal submatrix of T ($n = 1,\ 2,\ \ldots$). We will assume the existence of T^{-1}, namely, that $Tu = 0$ has only the trivial solution $u = 0$. The operator T^{-1} will then be compact (see the proof of Theorem 1 in Section 2). The eigenvalue λ of T, or equivalently λ^{-1} of T^{-1}, is simple if the corresponding eigenvector is unique (up to the scalar multiplication, of course) and if no corresponding generalized eigenvectors of rank 2 exist, namely, if no vectors $\boldsymbol{y} \neq \boldsymbol{0}$ satisfy $(\boldsymbol{T} - \lambda \boldsymbol{I})^2 \boldsymbol{y} = \boldsymbol{0}$.

Let $\mathbf{x} = [x^{(1)},\ x^{(2)},\ \ldots]^T$ be an eigenvector corresponding to λ, the simple eigenvalue of $\boldsymbol{T}$ under study. Assuming $\mathbf{x}^T\mathbf{x} \neq 0$ and $f_{n+1}x^{(n+1)}/x^{(n)} \to 0$ as $n \to \infty$, we will be able to show that there exists a sequence $\{\lambda_n\}$ of $\boldsymbol{T}_n$ such that $\lambda - \lambda_n = f_{n+1}x^{(n)}x^{(n+1)}[1 + o(1)]/(\mathbf{x}^T\mathbf{x}) \to 0$. This is the essential theoretical result of this paper. See Theorem 1 in Section 2. It should be remarked that this expression for the error $\lambda - \lambda_n$ is identical in appearance with the expression for the same error $\lambda - \lambda_n$ for the case of $d_n \to 0$ and $f_n \to 0$ [10, Theorem 1.4].

Our study in this paper is motivated again by our wish to apply operator-theoretic techniques to problems in special function computation. In fact, we include in Sections 3 – 5 examples of application of Theorem 1. They are (a) the solution of $J_\nu(z) = 0$ for ν, given a generally complex $z \neq 0$, where $J_\nu(z)$ denotes the Bessel function of the first kind of order ν; (b) the computation of the eigenvalues of the Mathieu equation

$$(6) \qquad w'' + (\lambda - 2q\cos 2z)w = 0,$$

given $q \neq 0$, generally complex; and (c) the computation of the eigenvalues of the wave equation in prolate or oblate spheroidal coordinates

$$(7) \qquad \{(1 - z^2)w'\}' + (\lambda \mp c^2z^2 - \frac{m^2}{1 - z^2})w = 0,$$

given $m(= 0, 1, 2, \ldots)$ and c (a real number) where the double sign $\mp$ correspond, respectively, to the prolate or oblate case.

Fortunately, the assumption of the existence of $\boldsymbol{T}^{-1}$ will not present an adverse effect on the application of Theorem 1, to these examples, for we may consider, if necessary, the eigenvalue problem for $\boldsymbol{T} + \alpha\boldsymbol{I}$ with α taken appropriately so that $(\boldsymbol{T} + \alpha\boldsymbol{I})^{-1}$ may exist.

2 Theoretical Analysis

Throughout this section we assume as given the situation described by (1) through (5) in Section 1, where T^{-1} *is assumed to exist and* λ *given and simple.*

Lemma 1. *For all sufficiently large* n *(for all* $n \geq n_0$, *say),* $x^{(n)} \neq 0$.

Proof Suppose the contrary and let $x^{(n)} = 0$ for $n = n_1 < n_2 < \cdots \to \infty$. Let n denote any one of $n_1, n_2, \ldots$. Then $x^{(n+1)} \neq 0$, for otherwise all components of $\mathbf{x}$ would be zero due to the fact that none of the f's vanish, contradicting $\mathbf{x} \neq \mathbf{0}$. Substitition of $x^{(n)} = 0$ into $\mathbf{Tx} = \lambda \mathbf{x}$ gives, in particular,

$$(1) \qquad \begin{bmatrix} d_{n+1} & f_{n+2} & & \mathbf{0} \\ f_{n+2} & d_{n+2} & f_{n+3} & \\ & f_{n+3} & d_{n+3} & \ddots \\ \mathbf{0} & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} x^{(n+1)} \\ x^{(n+2)} \\ x^{(n+3)} \\ \vdots \end{bmatrix} = \lambda \begin{bmatrix} x^{(n+1)} \\ x^{(n+2)} \\ x^{(n+3)} \\ \vdots \end{bmatrix},$$

from which follows $|\lambda| \geq |d_{n+p}| - |f_{n+p}| - |f_{n+p+1}| \geq |d_{n+p}| - 2|\alpha|$, where p is some natural number such that $x^{(n+p)}$ is largest in modulus among $x^{(n+1)}, x^{(n+2)}, \ldots$. By letting $n = n_1, n_2, \ldots$ in turn one concludes that $|\lambda|$ would have to be greater than any positive number since $|d_k| \to \infty$ as $k \to \infty$. This is absurd since λ is a fixed eigenvalue of $\mathbf{T}$. ∎

Theorem 1. *For the given simple eigenvalue* λ *of* $\mathbf{T}$, *where the existence of* $\mathbf{T}^{-1}$ *is assumed, there exists a sequence* $\{\lambda_n\}$ *of an appropriate eigenvalue of* $\mathbf{T}_n$ *such that* $\lambda_n \to \lambda$ *and for any such sequence the error is given by*

$$(2) \qquad \lambda - \lambda_n = \frac{f_{n+1} x^{(n)} x^{(n+1)}}{\mathbf{x}^T \mathbf{x}} [1 + o(1)],$$

provided $\mathbf{x}^T \mathbf{x} \neq 0$ *and* $f_{n+1} x^{(n+1)} / x^{(n)} \to 0$ *as* $n \to \infty$.

Proof We factor $\boldsymbol{T}$ into

(3) $$\boldsymbol{T} = \boldsymbol{D}^{-1}(\boldsymbol{DSD} + \boldsymbol{I})\boldsymbol{D}^{-1},$$

where

(4) $$\boldsymbol{D} = \begin{bmatrix} 1/\sqrt{d_1} & & & \boldsymbol{0} \\ & 1/\sqrt{d_2} & & \\ & & 1/\sqrt{d_3} & \\ \boldsymbol{0} & & & \ddots \end{bmatrix}, \quad \boldsymbol{S} = \begin{bmatrix} 0 & f_2 & & \boldsymbol{0} \\ f_2 & 0 & f_3 & \\ & f_3 & 0 & \ddots \\ \boldsymbol{0} & & \ddots & \ddots \end{bmatrix}.$$

The matrix $\boldsymbol{DSD}$ is in $B(\ell^2)$, the space of all bounded linear operators mapping ℓ^2 into itself, and is compact since $\boldsymbol{S} \in B(\ell^2)$ and $\boldsymbol{D}$ compact by [2, p. 59]. Hence $\boldsymbol{T}^{-1}$ exists and is in $B(\ell^2)$ if and only if $\boldsymbol{DSD} + \boldsymbol{I}$ has an inverse, which is true if and only if -1 is not an eigenvalue of the compact operator $\boldsymbol{DSD}$. Hence the existence of $\boldsymbol{T}^{-1}$ guarantees the existence of $(\boldsymbol{DSD} + \boldsymbol{I})^{-1}$. Thus

(5) $$\boldsymbol{T}^{-1} = \boldsymbol{D}(\boldsymbol{DSD} + \boldsymbol{I})^{-1}\boldsymbol{D},$$

which is also compact, again by the compactness of $\boldsymbol{D}$. Therefore the eigenvalue problem (1) in Section 1, namely, $\boldsymbol{Tx} = \lambda\mathbf{x}$, is equivalent to

(6) $$\boldsymbol{A}\mathbf{x} = (1/\lambda)\mathbf{x}, \quad \boldsymbol{A} \equiv \boldsymbol{T}^{-1}.$$

We take the n–th approximation to $\boldsymbol{A}$ to be

(7) $$\boldsymbol{A}_n \equiv \boldsymbol{P}_n\boldsymbol{D}(\boldsymbol{P}_n\boldsymbol{DSDP}_n + \boldsymbol{I})^{-1}\boldsymbol{DP}_n, \quad n \geq n_0 \ (say),$$

where

(8) $$\boldsymbol{P}_n = \boldsymbol{P}_n^2 = \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}.$$

We will show that $\boldsymbol{A}_n$ is a well–defined compact operator for all sufficiently large n. To see this we note that for any compact operator $\boldsymbol{K} \in B(\ell^2)$

(9) $$\|\boldsymbol{P}_n\boldsymbol{K} - \boldsymbol{K}\| \to 0$$

by [11, p. 151, Lemma 3.7].

We also have

$$\|(P_nK - K)^*\| \to 0, \text{ (by [19, p.242])} \tag{10}$$

where '$*$' denotes the adjoint.

The '$*$' represents, in the present situation, the conjugate transpose. Hence

$$K^* = (DSD)^* = \overline{(DSD)^T} = \overline{DSD} \text{ and } (P_nK)^* = K^*P_n = \overline{DSD}\,P_n, \tag{11}$$

since D and S are symmetric (it is here that we need the symmetry of T).

Substituting these into (10), we find

$$\|KP_n - K\| \to 0. \tag{12}$$

It follows from (9), (12) and the fact that $\|P_n\| = 1 (n = 1, 2, \ldots)$ that

$$\|P_nKP_n - K\| \to 0, \quad \text{or} \quad \|P_nDSDP_n - DSD\| \to 0. \tag{13}$$

Since the existence of $(DSD + I)^{-1}$ has been assumed, (13) guarantees the existence of $(P_nDSDP_n + I)^{-1}$, and hence, of A_n, for all sufficiently large n.

Again, by a similar line of argument, we can prove

Lemma 2.

$$\|A_n - A\| \to 0. \tag{14}$$

We will next show the existence of T_n^{-1} for all large n. To see this, compute first

$$P_nDSDP_n + I = \left[\begin{array}{c|c} D_nT_nD_n & 0 \\ \hline 0 & I \end{array}\right], \tag{15}$$

where D_n denotes the $n \times n$ principal submatrix of D. Hence, the existence of $(P_nDSDP_n + I)^{-1}$ is equivalent to the existence of T_n^{-1} (since D_n^{-1} exists for all n). Thus, using (15) in the definition (7) of A_n, we find

Lemma 3.

$$A_n = \begin{bmatrix} T_n^{-1} & 0 \\ 0 & 0 \end{bmatrix} \quad n \geq n_0 \ (say). \tag{16}$$

This means that the eigenvalues of the approximate operator A_n are precisely the reciprocals of the n eigenvalues of T_n and zero.

Lemma 4. *There exists a sequence $\{1/\lambda_n\}$ of eigenvalues of $\boldsymbol{A}_n$, namely, of $\boldsymbol{T}_n^{-1}$, simple for all large n, and a corresponding sequence $\{\mathbf{x}_n\}$ of eigenvectors such that*

$$(17) \qquad \lambda_n \to \lambda \quad \text{and} \quad \mathbf{x}_n \to \mathbf{x}, \quad \text{where } \boldsymbol{A}_n\mathbf{x}_n = (1/\lambda_n)\mathbf{x}_n.$$

This is true from the known fact in [12, pp. 272–274, Theorem 18.1–3].

By (16) $\mathbf{x}_n$ has the form

$$(18) \qquad \mathbf{x}_n = \begin{bmatrix} \tilde{\mathbf{x}}_n \\ \boldsymbol{0} \end{bmatrix}$$

and the relation $\boldsymbol{A}_n\mathbf{x}_n = (1/\lambda_n)\mathbf{x}_n$ translates to

$$(19) \qquad \boldsymbol{T}_n^{-1}\tilde{\mathbf{x}}_n = (1/\lambda_n)\tilde{\mathbf{x}}_n, \quad \text{or} \quad \boldsymbol{T}_n\tilde{\mathbf{x}}_n = \lambda_n\tilde{\mathbf{x}}_n.$$

The argument up to this point thus proves the first half of Theorem 1.

We will now proceed to the proof of the last half, namely, the error expression (2). To this end we begin by decomposing $\lambda - \lambda_n$ into

$$(20) \qquad \lambda - \lambda_n = (\lambda - \mu_n) + (\mu_n - \lambda_n),$$

where μ_n denotes the *Generalized Rayleigh Quotient*

$$(21) \qquad \mu_n = \mathbf{v}_n^T\boldsymbol{T}_n\mathbf{v}_n/(\mathbf{v}_n^T\mathbf{v}_n),$$

with $\mathbf{v}_n$ denoting the n–vector consisting of the first n components of $\mathbf{x}$, the exact eigenvector of $\boldsymbol{T}$ corresponding to the eigenvalue λ (see (1) in Section 1). Note that $\mathbf{v}_n^T\mathbf{v}_n \to \mathbf{x}^T\mathbf{x} \neq 0$ (by assumption), hence $\mathbf{v}_n^T\mathbf{v}_n \neq 0$ for all large n.

We will show

$$(22) \qquad \lambda - \mu_n = f_{n+1}x^{(n)}x^{(n+1)}/(\mathbf{v}_n^T\mathbf{v}_n)$$

and

$$(23) \qquad |\mu_n - \lambda_n| \leq \text{const} \cdot \left|f_{n+1}x^{(n+1)}\right|^2.$$

Using in (20) these two relations and the stated assumption $f_{n+1}x^{(n+1)}/x^{(n)} \to 0$, we would obtain the error expression (2), completing the proof of Theorem 1.

The derivation of the expression (22) for $\lambda - \mu_n$ is straightforward. Indeed, the substitution of the definition (21) of μ_n into $\lambda - \mu_n$ gives

$$\lambda - \mu_n = \mathbf{v}_n^T(\lambda \boldsymbol{I}_n - \boldsymbol{T}_n)\mathbf{v}_n/(\mathbf{v}_n^T\mathbf{v}_n). \tag{24}$$

Expanding $\boldsymbol{T}\mathbf{x} = \lambda\mathbf{x}$, or

$$\left[\begin{array}{c|c} \boldsymbol{T}_n & \begin{matrix} & \boldsymbol{0} \\ f_{n+1} & \end{matrix} \\ \hline \begin{matrix} & f_{n+1} \\ \boldsymbol{0} & \end{matrix} & \begin{matrix} \ddots & \ddots \\ \ddots & \ddots \end{matrix} \end{array}\right] \left[\begin{array}{c} \mathbf{v}_n \\ \hline \mathbf{w}_n \end{array}\right] = \lambda \left[\begin{array}{c} \mathbf{v}_n \\ \hline \mathbf{w}_n \end{array}\right], \quad \text{where} \quad \left[\begin{array}{c} \mathbf{v}_n \\ \hline \mathbf{w}_n \end{array}\right] = \mathbf{x}, \tag{25}$$

one finds

$$(\boldsymbol{T}_n - \lambda \boldsymbol{I}_n)\mathbf{v}_n = [0, \ \ldots, \ 0 \ , -f_{n+1}x^{(n+1)}]^T. \tag{26}$$

Substitution of this into (24) gives (22).

It remains to prove (23), which requires more steps as we will show. First,

$$\mu_n - \lambda_n = \mathbf{v}_n^T(\boldsymbol{T}_n - \lambda_n \boldsymbol{I}_n)\mathbf{v}_n/(\mathbf{v}_n^T\mathbf{v}_n). \tag{27}$$

For brevity, let

$$\mathbf{z}_n \equiv (\boldsymbol{T}_n - \lambda_n \boldsymbol{I}_n)\mathbf{v}_n. \tag{28}$$

Then (26) reads

$$\mu_n - \lambda_n = \mathbf{v}_n^T\mathbf{z}_n/(\mathbf{v}_n^T\mathbf{v}_n) \quad \text{with} \quad \tilde{\mathbf{x}}_n^T\mathbf{z}_n = 0, \tag{29}$$

the latter being true since $\boldsymbol{T}_n\tilde{\mathbf{x}}_n = \lambda_n\tilde{\mathbf{x}}_n$ by (19) and $\boldsymbol{T}_n$ is *symmetric.*

Let $\tilde{X}_n$ denote the subspace of C^n (= the space of all complex column vectors of order n), consisting of all those $\boldsymbol{y}$ which satisfy $\tilde{\mathbf{x}}_n^T\boldsymbol{y} = 0$, or

$$\tilde{X}_n = \{\boldsymbol{y} \in C^n : \tilde{\mathbf{x}}_n^T\boldsymbol{y} = 0\}. \tag{30}$$

By (29) $\mathbf{z}_n \in \tilde{X}_n$.

Lemma 5. *$T_n - \lambda_n I_n$ maps C^n into $\tilde{X}_n$ and is non-singular when restricted to $\tilde{X}_n$ for all large n.*

The first half follows again from $T_n\tilde{x}_n = \lambda_n\tilde{x}_n$ and the symmetry of T_n. To verify the last half, one observes that $(T_n - \lambda_n I_n)y = 0$ with $\tilde{x}_n^T y = 0$ ($y \in C^n$) implies $y = 0$, due to the fact that λ_n^{-1} is a simple eigenvalue of A_n, or equivalently λ_n is a simple eigenvalue of T_n by Lemma 3, and $\tilde{x}_n^T\tilde{x}_n = x_n^T x_n$ (by (18)) $\to x^T x \neq 0$ (by assumption).

We denote by $(T_n - \lambda_n I_n)^{-1}_{\tilde{X}_n}$ the inverse of $T_n - \lambda_n I_n$ with its domain restricted to $\tilde{X}_n$. We now return to (29) and compute

$$
\begin{aligned}
(31)\qquad \mu_n - \lambda_n &= v_n^T z_n/(v_n^T v_n) \\
&= v_n^T[(T_n - \lambda_n I_n)\{(T_n - \lambda_n I_n)^{-1}_{\tilde{X}_n} z_n\}]/(v_n^T v_n) \\
&= z_n^T(T_n - \lambda_n I_n)^{-1}_{\tilde{X}_n} z_n/(v_n^T v_n).
\end{aligned}
$$

Hence

$$
\begin{aligned}
(32)\,|\mu_n - \lambda_n| &\leq \|z_n\| \left\|(T_n - \lambda_n I_n)^{-1}_{\tilde{X}_n} z_n\right\| / \left|v_n^T v_n\right| \quad \text{(by the Cauchy} - \text{Schwarz inequality)} \\
&\leq \left\|(T_n - \lambda_n I_n)^{-1}\right\|_{\tilde{X}_n} \|z_n\|^2 / \left|v_n^T v_n\right|.
\end{aligned}
$$

Lemma 6. $\|(T_n - \lambda_n I_n)^{-1}\|_{\tilde{X}_n} \leq \beta$ *(say, a constant) for all large n.*

To prove this, we compute

$$
(33)\qquad (T_n - \lambda_n I_n)^{-1}_{\tilde{X}_n} = -\lambda_n^{-1} T_n^{-1}(T_n^{-1} - \lambda_n^{-1} I_n)^{-1}_{\tilde{X}_n}
$$

We know that $\lambda_n \to \lambda$ and $\|T_n^{-1}\| = \|A_n\|$ (by Lemma 3) $\to \|A\|$ (by Lemma 2). Hence, it suffices now to show that $\|(T_n^{-1} - \lambda_n^{-1} T_n)^{-1}\|_{\tilde{X}_n}$ is bounded for all large n.

To show this, we need subspaces X and X_n defined by

$$
\begin{aligned}
(34)\qquad X &= \{y \in \ell^2 : x^T y = 0\} \text{ and} \\
(35)\qquad X_n &= \{y \in \ell^2 : x_n^T y = 0\} = \{\begin{bmatrix} u \\ v \end{bmatrix} \in \ell^2 : u \in \tilde{X}_n, v \in \ell^2\} \quad \text{(by (18))}
\end{aligned}
$$

Using Lemma 3, we find

$$
(36)\qquad (A_n - \lambda_n^{-1} I)^{-1}_{X_n} = \begin{bmatrix} (T_n^{-1} - \lambda_n^{-1} I_n)^{-1}_{\tilde{X}_n} & 0 \\ 0 & -\lambda_n I \end{bmatrix},
$$

where the meaning, and the argument for showing the existence, of $(\boldsymbol{A}_n - \lambda_n^{-1}\boldsymbol{I})_{X_n}^{-1}$ is similar for $(\boldsymbol{T}_n - \lambda_n \boldsymbol{I}_n)_{\tilde{X}_n}$ (see the proof for Lemma 5). It easily follows from (36) that

$$\left\|(\boldsymbol{T}_n^{-1} - \lambda_n^{-1}\boldsymbol{I}_n)^{-1}\right\|_{\tilde{X}_n} \le \left\|(\boldsymbol{A}_n - \lambda_n^{-1}\boldsymbol{I})^{-1}\right\|_{X_n} \tag{37}$$

But by [10, Theorem 1.2],

$$\left\|(\boldsymbol{A}_n - \lambda_n^{-1}\boldsymbol{I})^{-1}\right\|_{X_n} \to \left\|(\boldsymbol{A} - \lambda^{-1}\boldsymbol{I})^{-1}\right\|_{X} \quad (n \to \infty) \tag{38}$$

This completes the proof of Lemma 6.

Lemma 7. $\|z_n\| = \left|f_{n+1}x^{(n+1)}\right| \cdot [1 + o(1)]$.

To prove this, we rewrite $\mathbf{z}_n$ into the form

$$\begin{aligned} \mathbf{z}_n &= (\boldsymbol{T}_n - \lambda_n \boldsymbol{I}_n)\mathbf{v}_n \text{ (by definition (28))} = (\boldsymbol{T}_n - \lambda \boldsymbol{I}_n)\mathbf{v}_n + (\lambda - \lambda_n)\mathbf{v}_n \\ &= [0, \ldots, 0, -f_{n+1}x^{(n+1)}]^T + (\lambda - \lambda_n)\mathbf{v}_n \text{ (by (26))}, \end{aligned} \tag{39}$$

whence

$$\|\mathbf{z}_n\| \le \left|f_{n+1}x^{(n+1)}\right| + |\lambda - \lambda_n| \cdot \|\mathbf{x}\| \quad \text{(by(25))}. \tag{40}$$

Hence

$$\|\mathbf{z}_n\| \to 0. \tag{41}$$

We now decompose $\mathbf{z}_n$ further into the form

$$\mathbf{z}_n = (\boldsymbol{T}_n - \lambda \boldsymbol{I}_n)\mathbf{v}_n + (\lambda - \mu_n)\mathbf{v}_n + (\mu_n - \lambda_n)\mathbf{v}_n \tag{42}$$

We evaluate each term on the right hand side of (42). For the first term

$$\|(\boldsymbol{T}_n - \lambda \boldsymbol{I}_n)\mathbf{v}_n\| = \left|f_{n+1}x^{(n+1)}\right| \quad \text{(by (26))}. \tag{43}$$

For the second term we have from (22)

$$\begin{aligned} \|(\lambda - \mu_n)\mathbf{v}_n\| &= \left|f_{n+1}x^{(n)}x^{(n+1)}\right| \|\mathbf{v}_n\| / \left|\mathbf{v}_n^T\mathbf{v}_n\right| \\ &= \left|x^{(n)}\right| \|(\boldsymbol{T}_n - \lambda \boldsymbol{I}_n)\mathbf{v}_n\| \, \|\mathbf{v}_n\| / \left|\mathbf{v}_n^T\mathbf{v}_n\right| \quad \text{(by (43))}. \end{aligned} \tag{44}$$

For the third term

(45) $\|(\mu_n - \lambda_n)\mathbf{v}_n\| \leq \text{const.} \|\mathbf{z}_n\|^2$ (by(32), Lemma 6 and the fact $\mathbf{v}_n^T\mathbf{v}_n \to \mathbf{x}^T\mathbf{x}$).

Using (43), (44) and (45) together with the fact $x^{(n)} \to 0$, $\|\mathbf{v}_n\| \to \|\mathbf{x}\|$, $\mathbf{v}_n^T\mathbf{v}_n \to \mathbf{x}^T\mathbf{x}$ and $\|\mathbf{z}_n\| \to 0$ in (42) we obtain

(46) $$\|\mathbf{z}_n\| = \|(\boldsymbol{T}_n - \lambda \boldsymbol{I}_n)\mathbf{v}_n\| [1 + o(1)] = \left|f_{n+1}x^{(n+1)}\right| [1 + o(1)].$$

This proves Lemma 7.

Using Lemmas 6 and 7 in (32) we finally have (23), i.e.,

(47) $$|\mu_n - \lambda_n| \leq \text{const.} \left|f_{n+1}x^{(n+1)}\right|^2 . \blacksquare$$

3 Application to $J_\nu(z) = 0$

We consider solving

$$J_\nu(z) = 0 \tag{1}$$

for ν, where $0 \neq z$ is given and generally complex. For the solution of (1) for z with a given ν see [10].

The fact that the $x^{(k)} = J_{\nu+k}(z)$ are the minimal solution of the second order difference equation

$$(z/2)x^{(k)} - (\nu + k + 1)x^{(k+1)} + (z/2)x^{(k+2)} = 0, \; k = 0, \; 1, \; 2, \; \ldots, \tag{2}$$

(see [9, Theorem 2.3]) implies that $J_\nu(z) = 0$ if and only if for some nonzero $\mathbf{x} \in \ell^2$

$$\boldsymbol{T}\mathbf{x} = \nu \mathbf{x}, \tag{3}$$

where

$$\boldsymbol{T} = \begin{bmatrix} -1 & z/2 & & \boldsymbol{0} \\ z/2 & -2 & z/2 & \\ & z/2 & -3 & \ddots \\ \boldsymbol{0} & & \ddots & \ddots \end{bmatrix}. \tag{4}$$

In fact,

$$\mathbf{x} = [x^{(1)}, \; x^{(2)}, \; \ldots]^T = [J_{\nu+1}(z), \; J_{\nu+2}(z), \; \ldots]^T \neq \boldsymbol{0}. \tag{5}$$

The matrix $\boldsymbol{T}$ in (4) satisfies the condition in Theorem 1 in Section 2, except possibly that $\boldsymbol{T}^{-1}$ may not always exist depending on the value of z. In fact, $\boldsymbol{T}^{-1}$ exists if and only if $J_0(z) \neq 0$.

However, $\boldsymbol{T} + \alpha \boldsymbol{I}$ with $\alpha = -|z|$ has an inverse, since none of the Gerschgorin disks for $\boldsymbol{T} + \alpha \boldsymbol{I}$ contains zero. By applying Theorem 1 to $\boldsymbol{T} + \alpha \boldsymbol{I}$, we see immediately that for any simple eigenvalue ν of $\boldsymbol{T}$, there exists a sequence $\{\nu_n\}$ of an appropriate eigenvalue of $\boldsymbol{T}_n$ such that $\nu_n \to \nu$ and

$$\begin{aligned} \nu - \nu_n &= f_{n+1} x^{(n)} x^{(n+1)} [1 + o(1)] / \mathbf{x}^T \mathbf{x}, \\ &= (z/2) J_{\nu+n}(z) J_{\nu+n+1}(z) [1 + o(1)] / \sum_{k=1}^{\infty} J_{\nu+k}^2(z). \end{aligned} \tag{6}$$

Hence

$$(7) \qquad (\nu - \nu_{n+1})/(\nu - \nu_n) = J_{\nu+n+2}(z)[1 + o(1)]/J_{\nu+n}(z) = z^2[1 + o(1)]/4(\nu + n)^2,$$

which shows that the relative error is diminished approximately by the factor of $z^2/\{4(\nu+n)^2\}$ with the unit increase of the value of n.

Dougall [6] appears to be among the firsts to be concerned with the localization of the zeros ν in the case z is given and pure imaginary. Coulomb [5] gives a more systematic study on the zeros ν apparently without the knowledge of Dougall's work. More recently, Flajolet and Schott [8] encounter the need of solving $J_\nu(2) = 0$ in studying a class of combinatorial problems called *non-overlapping partitions.* They numerically compute the zeros of the Lommel polynomial $R_{n,\,\nu}(2)$ $(n = 1,\ 2,\ \ldots)$ [20, p.294] as an approximation to the zeros of $J_\nu(2)$. It turns out this is precisely equivalent to solving the eigenvalue problem for $\boldsymbol{T}_n$, the $n \times n$ principal submatrix of $\boldsymbol{T}$. Feinsilver and Schott [7] give an estimate for a quantity $|\nu - \nu_n|$, $(\nu,\ \nu_n$ in our notation), which appears weaker than our estimate (6).

For an important special case where z is real and nonzero, every eigenvalue of $\boldsymbol{T}$ may be shown to be real and simple. Figure 1 gives the complete family of curves representing the $z - \nu$ relation satisfying $J_\nu(z) = 0$ with z restricted to reals (in [20, p.510, Figure 33] a similar and less complete plot is given). The approximate values of ν corresponding to a given z are computed as the eigenvalues of $\boldsymbol{T}_n$ for a sufficiently large n through the use of the standard subroutines such as those in the EISPACK package[17] or in the LAPACK package[3].

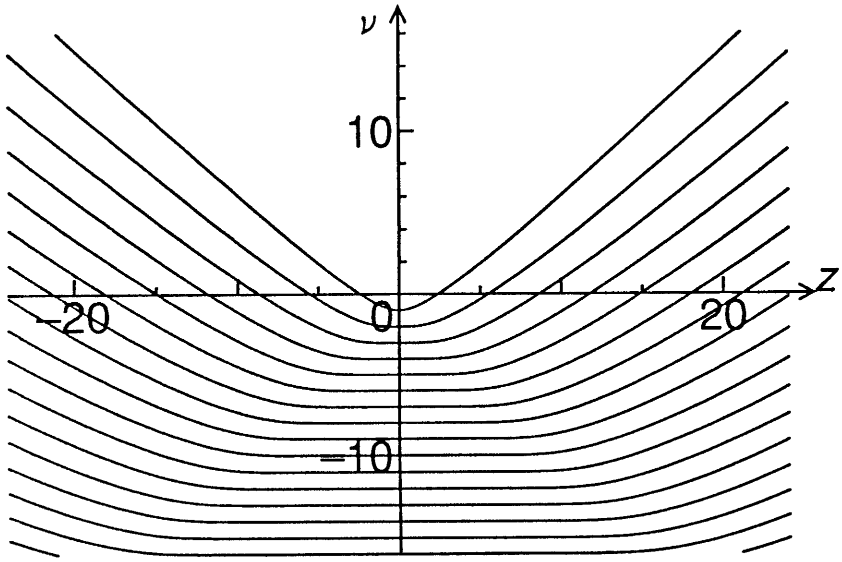

Figure 1. The real z - real ν relation

We consider the case where z is pure imaginary. Dougall [6] proves no zeros ν exists for which $Re(\nu) \geq 0$. In Figure 2 a plot of zeros of $J_\nu(i6)$ are given, where $\nu^{(k)}$ denotes the zero of $J_\nu(i6)$ whose real part is k-th largest. The first 8 zeros are complex (i.e., non-real) and the rest are negative reals close to negative integers.

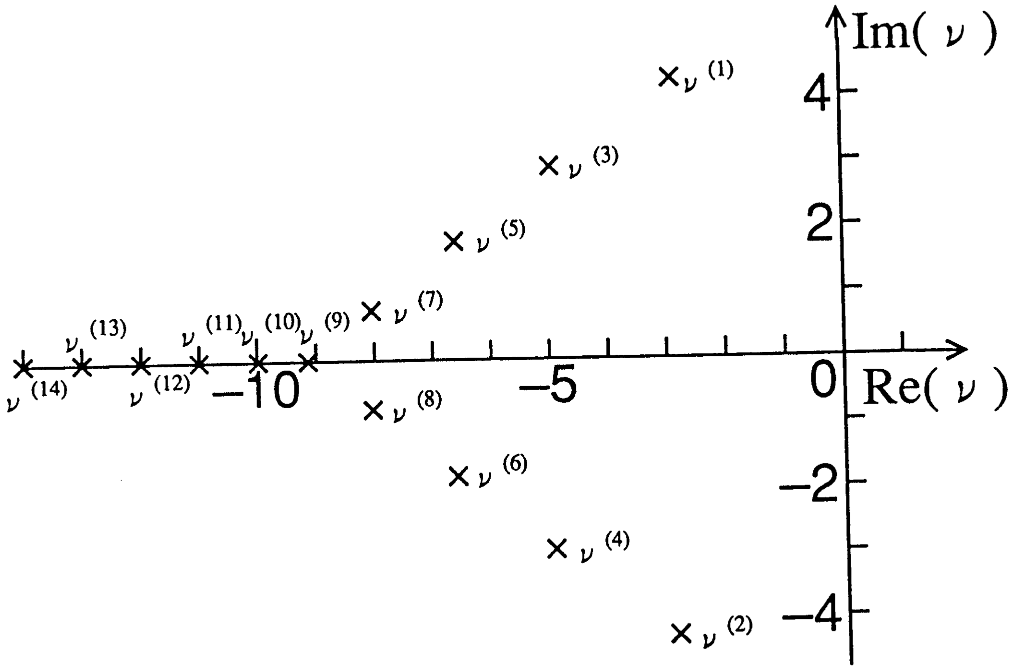

Figure 2. A plot of the first 14 zeros of $J_\nu(i6)$

Their values correct to 20 decimals are given in Table 1 computed through the procedure indicated in Theorem 1 where the values of n large enough to give the prescribed accuracy are numerically found.

Table 1. The first 14 zeros of $J_\nu(i6)$ correct to 20 decimals

k	$\nu^{(k)}$		
1	$-2.86497\ 19531\ 79587\ 81162 \cdots$	$+$	$i\ 4.28177\ 75584\ 62667\ 05260 \cdots$
2	$-2.86497\ 19531\ 79587\ 81162 \cdots$	$-$	$i\ 4.28177\ 75584\ 62667\ 05260 \cdots$
3	$-4.93981\ 14391\ 13353\ 78403 \cdots$	$+$	$i\ 2.94065\ 36043\ 64872\ 32355 \cdots$
4	$-4.93981\ 14391\ 13353\ 78403 \cdots$	$-$	$i\ 2.94065\ 36043\ 64872\ 32355 \cdots$
5	$-6.59799\ 73051\ 78632\ 80040 \cdots$	$+$	$i\ 1.80238\ 99225\ 15421\ 84748 \cdots$
6	$-6.59799\ 73051\ 78632\ 80040 \cdots$	$-$	$i\ 1.80238\ 99225\ 15421\ 84748 \cdots$
7	$-8.03737\ 27373\ 31138\ 76955 \cdots$	$+$	$i\ 0.76530\ 46391\ 84327\ 65676 \cdots$
8	$-8.03737\ 27373\ 31138\ 76955 \cdots$	$-$	$i\ 0.76530\ 46391\ 84327\ 65676 \cdots$
9	$-9.13439\ 42110\ 61678\ 85098 \cdots$		
10	$-9.98430\ 00231\ 78234\ 52853 \cdots$		
11	$-11.00105\ 96223\ 84819\ 62060 \cdots$		
12	$-11.99993\ 61086\ 80997\ 29165 \cdots$		
13	$-13.00000\ 33084\ 24652\ 83023 \cdots$		
14	$-13.99999\ 98509\ 76760\ 64235 \cdots$		

In Table 2 the actual relative errors $(\nu - \nu_n)/\nu$ are checked against the theoretical estimates (see (6))

$$E_n(\nu) \equiv (z/2)J_{\nu+n}(z)J_{\nu+n+1}(z)/(\nu \sum_{k=1}^{\infty} J^2_{\nu+k}(z)) \quad (z = i6) \tag{8}$$

for a selected set of values of n. In the table, $\nu_n^{(k)}$ denotes the approximation to $\nu^{(k)}$ computed from the $n \times n$ matrix $\boldsymbol{T}_n$. The corresponding values, namely, $(\nu^{(k)} - \nu_n^{(k)})/\nu^{(k)}$ and $E_n(\nu^{(k)})$, may be seen to be in agreement, approximately 1 digit except for a few low values of n.

Table 2. Actual relative errors and theoretical estimates

n	$(\nu^{(1)} - \nu_n^{(1)})/\nu^{(1)}$		$E_n(\nu^{(1)})$		$(\nu^{(3)} - \nu_n^{(3)})/\nu^{(3)}$		$E_n(\nu^{(3)})$	
	real	imaginary	real	imaginary	real	imaginary	real	imaginary
4	4.25e-01	-1.99e-01	-9.83e-02	1.32e-01	7.95e-01	-5.05e-01	6.06e-01	-2.31e-00
6	1.52e-02	-9.21e-03	1.27e-02	-7.29e-03	3.53e-01	-1.03e-01	-1.28e-00	6.02e-01
8	-4.75e-04	-1.40e-04	-4.14e-04	-1.82e-04	5.50e-02	5.72e-02	1.29e-01	1.31e-01
10	-5.40e-07	6.33e-06	-9.31e-07	5.86e-06	6.63e-03	-5.78e-03	5.49e-03	-5.26e-03
12	3.93e-08	5.31e-09	3.70e-08	6.75e-09	-7.01e-05	-1.16e-04	-5.75e-05	-1.10e-04
14	4.97e-11	-1.23e-10	5.12e-11	-1.16e-10	-9.89e-07	3.73e-08	-9.26e-07	1.55e-09
16	-1.79e-13	-1.82e-13	-1.69e-13	-1.79e-13	-2.15e-09	3.07e-09	-2.11e-09	2.87e-09
18	-2.91e-16	7.11e-17	-2.84e-16	6.48e-17	1.59e-12	7.81e-12	1.41e-12	7.54e-12
20	-8.85e-20	2.08e-19	-8.88e-20	2.03e-19	8.11e-15	6.17e-15	7.80e-15	6.07e-15
22	4.89e-23	1.03e-22	4.71e-23	1.02e-22	8.26e-18	3.74e-20	8.05e-18	1.02e-19
24					3.70e-21	-2.43e-21	3.64e-21	-2.36e-21
26					7.17e-25	-1.47e-24	7.12e-25	-1.44e-24

If z is not real $\boldsymbol{T}$ may have multiple eigenvalues. In fact, consider in particular the case where z is pure imaginary. *A theorem of Hurwitz* asserts that if $\nu > -1$ the zeros z of $J_\nu(z)$ are all real and if $\nu \in (-p-1, \ -p)$ for a natural number p, $J_\nu(z)$ has exactly $2p$ complex zeros z, of which two are pure imaginary if p is odd, and the rest real[20, p.483]. Hence, for $\nu \in (-2,-1) \cup (-4,-3) \cup (-6,-5) \cup \cdots$ there exists exactly a pair of pure imaginary zeros z of $J_\nu(z)$. In Figure 3 such $z - \nu$ relation is plotted. To obtain this relation we compute the real values of ν for a given set of pure imaginary z instead of solving $J_\nu(z) = 0$ for z for a given set of values of ν. The reason is that the former is a well-conditioned problem while the latter is ill-conditioned, as can be seen from Figure 3.

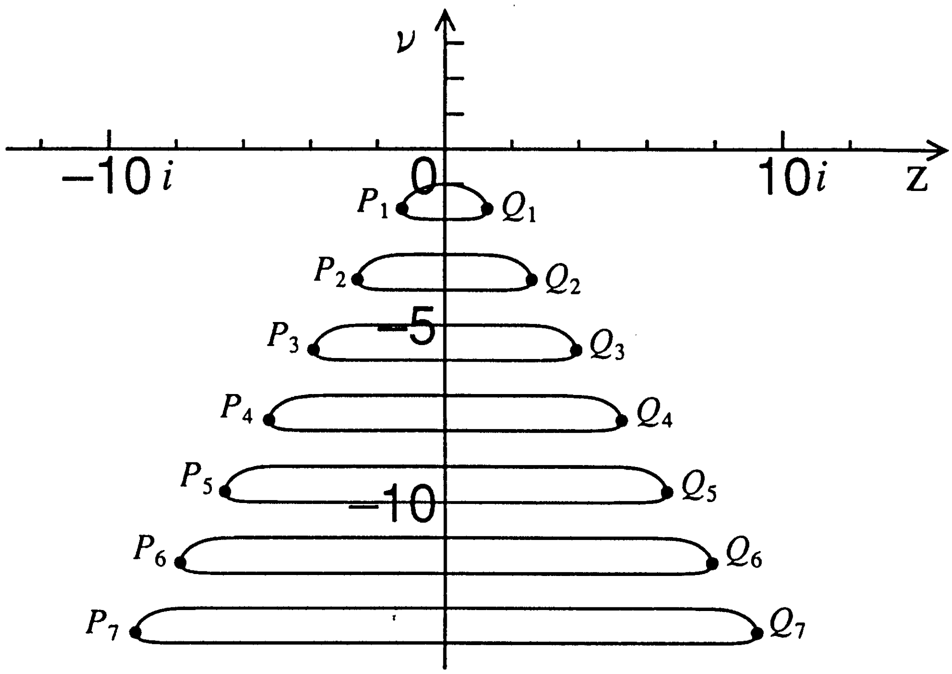

Figure 3. The pure imaginary z - real ν relation

Figure 3 also indicates the presence of double eigenvalues of $\boldsymbol{T}$, which are represented by the leftmost and rightmost extreme points (such as P_1, Q_1, P_2, Q_2, ...) of each closed curve. The first several double eigenvalues correct to 10 decimals are given in Table 3.

Table 3. Examples of pure imaginary z giving double eigenvalues ν

	z	ν
Q_1, P_1	$\pm i$ 1.26786 89031 $\cdots$	-1.69752 36772 $\cdots$
Q_2, P_2	$\pm i$ 2.58947 93891 $\cdots$	-3.70245 24295 $\cdots$
Q_3, P_3	$\pm i$ 3.91357 50289 $\cdots$	-5.70416 30259 $\cdots$
Q_4, P_4	$\pm i$ 5.23835 03301 $\cdots$	-7.70502 50615 $\cdots$
Q_5, P_5	$\pm i$ 6.56340 57743 $\cdots$	-9.70554 33265 $\cdots$
Q_6, P_6	$\pm i$ 7.88860 33324 $\cdots$	-11.70588 89803 $\cdots$
Q_7, P_7	$\pm i$ 9.21388 27795 $\cdots$	-13.70613 58495 $\cdots$
Q_8, P_8	$\pm i$ 10.53921 36827 $\cdots$	-15.70632 09487 $\cdots$
Q_9, P_9	$\pm i$ 11.86457 90177 $\cdots$	-17.70646 48665 $\cdots$

Remark. From the numerical evidence (see Figure 2 and Table 1, for example), one might conjecture that if z is pure imaginary the zeros ν consist of a finite number of non-reals and an infinity of reals, a situation somewhat similar to the theorem of Hurwitz stated above.

4 Application to the Mathieu equation

We consider the eigenvalue problem of the Mathieu equation

$$w'' + (\lambda - 2q\cos 2z)w = 0 \tag{1}$$

where the *parameter* q is given, complex and nonzero. We will concern ourselves with the problem of finding the *eigenvalues* λ so that (1) admits *eigenfunctions* that are π- or 2π- periodic and even or odd. Thus written in Fourier series, they may be represented by an even–cosine, or odd–cosine, or odd–sine, or even–sine Fourier series. They are commonly referred to simply as *Mathieu functions*. For the standard reference on the Mathieu equation see, for example, [14] or [15].

The method of this paper is best illustrated by an example. Thus we consider computing the eigenvalues corresponding to the Mathieu functions that are represented by an even–sine series, namely, $se_{2k}(z,\ q)$, $k = 1,\ 2,\ 3,\ \ldots,$

$$se_{2k}(z,\ q) = B_2\sin 2z + B_4\sin 4z + B_6\sin 6z + \cdots. \tag{2}$$

The following fact is well-known: the $x^{(k)} = B_{2k}$ represent the minimal solution of the linear second order difference equation

$$(4-\lambda)x^{(1)} + qx^{(2)} = 0 \tag{3}$$

$$qx^{(k-1)} + (4k^2-\lambda)x^{(k)} + qx^{(k+1)} = 0, \quad k = 2,\ 3,\ 4,\ \ldots, \tag{4}$$

so that

$$x^{(k)}/x^{(k-1)} = B_{2k}/B_{2k-2} = q[1+o(1)]/(\lambda - 4k^2) \quad \text{(by [9, Theorem 2.3])}. \tag{5}$$

It follows that λ is an eigenvalue of the indicated type if and only if for some nonzero $\mathbf{x} \in \ell^2$

$$\boldsymbol{T}\mathbf{x} = \lambda\mathbf{x}, \tag{6}$$

where

$$\boldsymbol{T} = \begin{bmatrix} 2^2 & q & & \boldsymbol{0} \\ q & 4^2 & q & \\ & q & 6^2 & \ddots \\ \boldsymbol{0} & & \ddots & \ddots \end{bmatrix}. \tag{7}$$

Indeed,

$$\mathbf{x} = [x^{(1)},\ x^{(2)},\ \ldots]^T = [B_2,\ B_4,\ \ldots]^T \neq \boldsymbol{0}. \tag{8}$$

Theorem 1 again applies (we may consider $\boldsymbol{T} + 2\,|q|\,\boldsymbol{I}$, if necessary, whose inverse exists for any q). Hence, for any simple eigenvalue λ, there is a sequence $\{\lambda_n\}$ of appropriate eigenvalues of $\boldsymbol{T}_n$ such that $\lambda_n \to \lambda$ and

$$\lambda - \lambda_n = qB_{2n}B_{2n+2}[1 + o(1)] / \sum_{k=1}^{\infty} B_{2k}^2. \tag{9}$$

Again,

$$(\lambda - \lambda_n)/(\lambda - \lambda_{n-1}) = B_{2n+2}[1 + o(1)]/B_{2n-2} = q^2[1 + o(1)]/(\lambda - 4n^2)^2. \tag{10}$$

As an example we take the case $q = i50$. The first 12 eigenvalues correct to 20 decimals are tabulated in Table 1 where $\lambda^{(k)}$ denotes the eigenvalue whose real part is k-th smallest. A plot of these eigenvalues is given in Figure 1.

Table 1. The first 12 eigenvalues correct to 20 decimals for the case $q = i50$

k	$\lambda^{(k)}$		
1	28.72229 11370 32355 49601 ···	+	i 69.96801 99265 72528 65758 ···
2	28.72229 11370 32355 49601 ···	−	i 69.96801 99265 72528 65758 ···
3	63.39929 14509 62812 29031 ···	+	i 29.60852 31269 66005 20473 ···
4	63.39929 14509 62812 29031 ···	−	i 29.60852 31269 66005 20473 ···
5	92.06491 93049 30234 38145 ···		
6	135.51494 61243 03635 01229 ···		
7	189.71757 07735 82690 73268 ···		
8	251.15610 66985 33519 76394 ···		
9	320.15885 67485 25160 00697 ···		
10	396.88252 79468 21650 31833 ···		
11	481.42068 67817 00903 23046 ···		
12	573.83123 64944 14797 70809 ···		

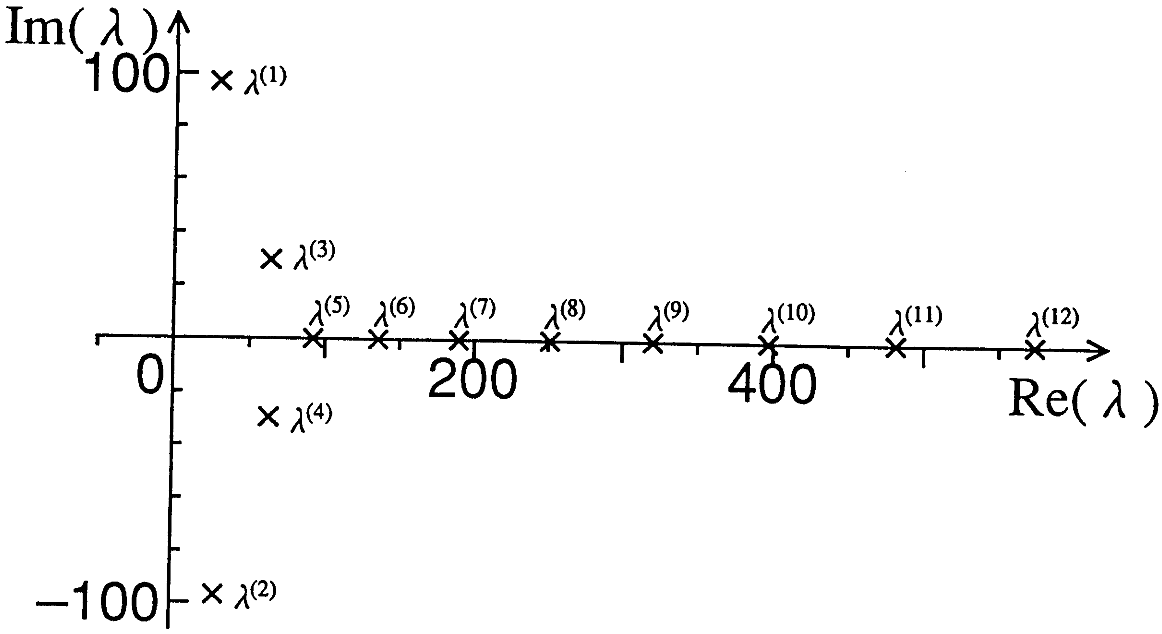

Figure 1. A plot of the first 12 eigenvalues for the case $q = i50$

In Table 2 the actual relative errors are compared with the corresponding theoretical estimates

$$E_n(\lambda) \equiv qB_{2n}B_{2n+2}/(\lambda \sum_{k=1}^{\infty} B_{2k}^2) \quad (q = i50) \tag{11}$$

for a selected set of values of n. In this table, $\lambda_n^{(k)}$ denotes the approximation to $\lambda^{(k)}$ computed from the $n \times n$ matrix $\boldsymbol{T}_n$. They match up at least to a few digits except for low values of n.

Table 2. Actual relative errors and theoretical estimates

n	$(\lambda^{(1)} - \lambda_n^{(1)})/\lambda^{(1)}$		$E_n(\lambda^{(1)})$		$(\lambda^{(3)} - \lambda_n^{(3)})/\lambda^{(3)}$		$E_n(\lambda^{(3)})$	
	real	imaginary	real	imaginary	real	imaginary	real	imaginary
4	-4.19e-02	-4.10e-02	-6.53e-02	-1.94e-02	3.82e-01	-2.10e-01	1.51e+00	-5.64e-01
6	6.99e-04	-8.32e-04	6.18e-04	-8.19e-04	8.75e-02	2.47e-02	7.75e-02	6.89e-02
8	1.82e-06	4.48e-07	1.78e-06	4.09e-07	1.04e-04	3.97e-04	1.03e-04	3.79e-04
10	3.11e-10	4.51e-10	3.09e-10	4.44e-10	-1.57e-08	1.83e-07	-1.49e-08	1.80e-07
12	3.56e-15	3.61e-14	3.59e-15	3.58e-14	-5.00e-12	1.54e-11	-4.94e-12	1.53e-11
14	-1.82e-19	6.55e-19	-1.81e-19	6.53e-19	-1.71e-16	3.29e-16	-1.70e-16	3.28e-16
16	-2.24e-24	3.64e-24	-2.24e-24	3.64e-24	-1.53e-21	2.24e-21	-1.53e-21	2.23e-21

In Figure 2 is shown the pure imaginary q – real λ relation, where the presence of double eigenvalues such as P_1, Q_1, P_2, Q_2, ... are evident whose values correct to 10 decimals are tabulated in Table 3 together with the corresponding values of q. The reader might recall a similar situation in Figure 3 in Section 3.

Table 3. Examples of pure imaginary q giving double eigenvalues λ

	q	λ
$Q_1,\ P_1$	$\pm i$ 6.92895 47587 $\cdots$	11.19047 35991 $\cdots$
$Q_2,\ P_2$	$\pm i$ 30.09677 28375 $\cdots$	50.47501 61557 $\cdots$
$Q_3,\ P_3$	$\pm i$ 69.59879 32768 $\cdots$	117.86892 41608 $\cdots$
$Q_4,\ P_4$	$\pm i$ 125.43541 13143 $\cdots$	213.37256 86374 $\cdots$
$Q_5,\ P_5$	$\pm i$ 197.60667 86924 $\cdots$	336.98604 39502 $\cdots$
$Q_6,\ P_6$	$\pm i$ 286.11260 87616 $\cdots$	488.70938 44758 $\cdots$
$Q_7,\ P_7$	$\pm i$ 390.95320 62955 $\cdots$	668.54260 56541 $\cdots$

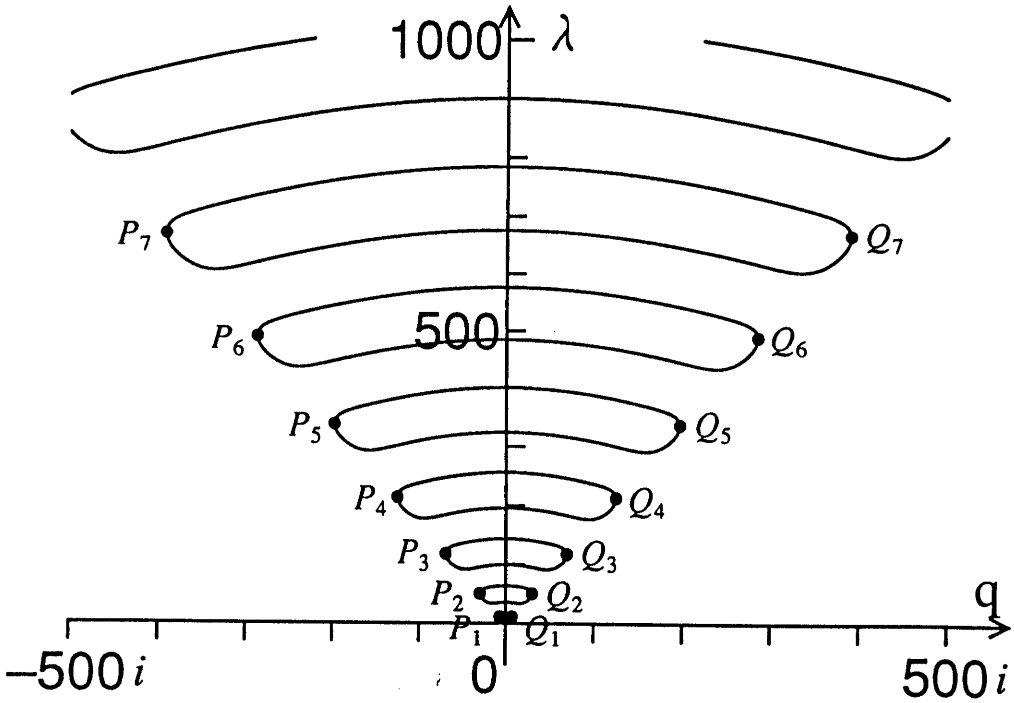

Figure 2. The pure imaginary q – real λ relation

Remark. One suspects that if q is pure imaginary, only a finite number of eigenvalues are non-real and the rest real, a similar situation noted in the Remark in Section 3.

5 Application to the spheroidal wave equation

We will be brief in this section. For the general background of the spheroidal wave equation, see, for example, [15] or [18]. For definiteness we consider finding the eigenvalues λ corresponding to *the prolate angular spheroidal functions* $w(z)$, namely, those values of λ for which the differential equation

$$\{(1-z^2)w'\}' + \{\lambda - c^2z^2 - m^2/(1-z^2)\}w = 0, \tag{1}$$

where $m = 0,\ 1,\ 2,\ \ldots$ and c is a given nonzero real number, admits solutions $w(z)$ which are analytic on $(-1,\ 1)$ and finite at $z = \pm 1$. It is well-known that λ is such an eigenvalue if and only if for some nonzero $\boldsymbol{x}$ or $\boldsymbol{y} \in \ell^2$

$$\mathbf{T}\mathbf{x} = \lambda\mathbf{x} \quad \text{or} \quad \boldsymbol{U}\boldsymbol{y} = \lambda\boldsymbol{y}, \tag{2}$$

where

$$\boldsymbol{T} = \begin{bmatrix} \beta_0 & \alpha_0 & & \boldsymbol{0} \\ \gamma_2 & \beta_2 & \alpha_2 & \\ & \gamma_4 & \beta_4 & \ddots \\ \boldsymbol{0} & & \ddots & \ddots \end{bmatrix}, \quad \boldsymbol{U} = \begin{bmatrix} \beta_1 & \alpha_1 & & \boldsymbol{0} \\ \gamma_3 & \beta_3 & \alpha_3 & \\ & \gamma_5 & \beta_5 & \ddots \\ \boldsymbol{0} & & \ddots & \ddots \end{bmatrix}, \tag{3}$$

$$\begin{aligned}
(4)\quad \alpha_k &= \frac{(2m+k+2)(2m+k+1)c^2}{(2m+2k+3)(2m+2k+5)} = \frac{c^2}{4}[1+o(1)] \neq 0,\\
(5)\quad \beta_k &= (m+k)(m+k+1) + \frac{2(m+k)(m+k+1)-2m^2-1}{(2m+2k-1)(2m+2k+3)}c^2 = k^2[1+o(1)] \neq 0,\\
(6)\quad \gamma_k &= \frac{k(k-1)c^2}{(2m+2k-3)(2m+2k-1)} = \frac{c^2}{4}[1+o(1)] \neq 0, \quad \text{(see [1, Formula 21.7.3])}.
\end{aligned}$$

The matrices $\boldsymbol{T}$ and $\boldsymbol{U}$ are real and may be symmetrized with a diagonal similarity transformation. Indeed, $\boldsymbol{T}$ and $\boldsymbol{U}$ are, respectively, similar to $\tilde{\boldsymbol{T}}$ and $\tilde{\boldsymbol{U}}$ defined by

$$(\boldsymbol{T}) = \begin{bmatrix} \beta_0 & \sqrt{\alpha_0}\sqrt{\gamma_2} & & \boldsymbol{0} \\ \sqrt{\alpha_0}\sqrt{\gamma_2} & \beta_2 & \sqrt{\alpha_2}\sqrt{\gamma_4} & \\ & \sqrt{\alpha_2}\sqrt{\gamma_4} & \beta_4 & \ddots \\ \boldsymbol{0} & & \ddots & \ddots \end{bmatrix}, \quad \boldsymbol{U} = \begin{bmatrix} \beta_1 & \sqrt{\alpha_1}\sqrt{\gamma_3} & & \boldsymbol{0} \\ \sqrt{\alpha_1}\sqrt{\gamma_3} & \beta_3 & \sqrt{\alpha_3}\sqrt{\gamma_5} & \\ & \sqrt{\alpha_3}\sqrt{\gamma_5} & \beta_5 & \ddots \\ \boldsymbol{0} & & \ddots & \ddots \end{bmatrix},$$

Letting

$$\mathbf{x} = [x^{(1)},\ x^{(2)},\ \ldots]^T \quad \text{and} \quad \boldsymbol{y} = [y^{(1)},\ y^{(2)},\ \ldots]^T, \tag{8}$$

we may show that

$$\frac{x^{(n+1)}}{x^{(n)}} = \frac{\gamma_{2n}}{\lambda - \beta_{2n}}[1 + o(1)] = \frac{-c^2}{16n^2}[1 + o(1)] \to 0 \tag{9}$$

$$\frac{y^{(n+1)}}{y^{(n)}} = \frac{\gamma_{2n+1}}{\lambda - \beta_{2n+1}}[1 + o(1)] = \frac{-c^2}{16n^2}[1 + o(1)] \to 0 \tag{10}$$

Theorem 1 once again applies to the eigenvalue problems (2) after symmetrizing them. The eigenvalues of $\boldsymbol{T}$ are usually denoted by

$$\lambda = \lambda_{m,m} < \lambda_{m,m+2} < \lambda_{m,m+4} < \cdots, \tag{11}$$

and those of $\boldsymbol{U}$ are by

$$\lambda = \lambda_{m,m+1} < \lambda_{m,m+3} < \lambda_{m,m+5} < \cdots. \tag{12}$$

The eigenvalue problem for the oblate case may be studied exactly in parallel with the prolate case except that c^2 in the prolate case is to be replaced by $-c^2$ (i.e., $c^2 \to -c^2$ in (1), (4), (5), (6)).

We may prove after some computation that

$$\frac{\lambda - \lambda^{(n+1)}}{\lambda - \lambda^{(n)}} = \frac{\alpha_{2n}}{\gamma_{2n}} \cdot \frac{A_{2n+2}}{A_{2n-2}}[1 + o(1)] = \frac{\alpha_{2n} \cdot \gamma_{2n+2}}{(\lambda - \beta_{2n})^2}[1 + o(1)] = \left(\frac{c}{4n}\right)^4 [1 + o(1)]. \tag{13}$$

where $\lambda^{(n)}$ denotes an appropriate eigenvalue of $\boldsymbol{T}_n$ or $\boldsymbol{U}_n$, an $n \times n$ principal submatrix of $\boldsymbol{T}$ or $\boldsymbol{U}$ (of course, λ means one of $\lambda_{m,m}, \lambda_{m,m+1}, \ldots$).

In Figure 1, we give a plot of $\lambda_{m,n}$ as a function of c^2 (the prolate case) or $-c^2$ (the oblate case) where $m = 0$ and $n = 0, 1, 2, \ldots, 7$.

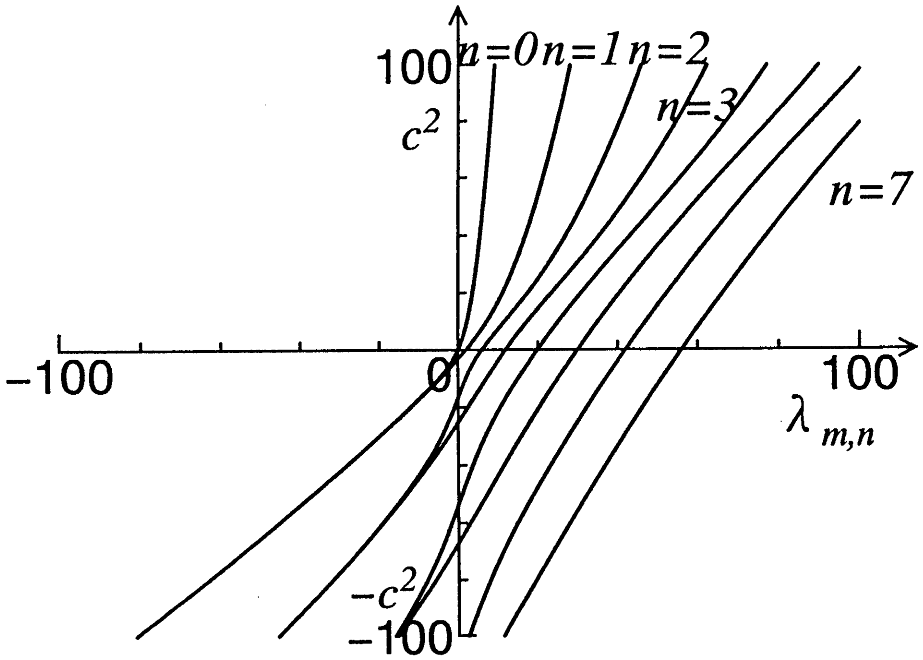

Figure 1. A plot of $\lambda_{m,n}$ as a function of c^2 (the prolate case) or $-c^2$ (the oblate case) where $m = 0$ and $n = 0, 1, 2, \ldots, 7$

We might add that the actual relative errors $(\lambda - \lambda_n)/\lambda$ and their theoretical estimates given in Theorem 1 are in good agreement, mainly because $\boldsymbol{T}$ and $\boldsymbol{U}$ are real and similar to a real symmetric matrix. We omit the details.

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