Outside-In Conditional Narrowing

Tetsuo Ida † ‡ and Satoshi Okui ††

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Abstract

We present outside-in conditional narrowing for orthogonal conditional term rewriting systems, and show the completeness of leftmost-outside-in conditional narrowing with respect to normalizable solutions. We consider orthogonal conditional term rewriting systems whose conditions consist of strict equality only. Completeness results are obtained for systems both with and without extra variables. The result bears practical significance since orthogonal conditional term rewriting systems can be viewed as a computation model for functional-logic programming languages and leftmost-outside-in conditional narrowing is the computing mechanism for the model.

†Inst. of Information Sciences and Electronics, Univ. of Tsukuba, Tsukuba, Ibaraki 305, Japan
‡ida@is.tsukuba.ac.jp
††okui@softlab.is.tsukuba.ac.jp

1 Introduction

Narrowing[13, 4] is an important computing mechanism for functional-logic programming languages. It comprises reduction of functional programming and term unification used in resolution of logic programming. A conditional term rewriting system with narrowing is a natural computation model for functional-logic programming languages. To design efficient narrowing while preserving a property of completeness is, therefore, a research not only of theoretical interest but also of practical importance.

In this paper we consider orthogonal conditional term rewriting systems since most of proposed functional-logic programming languages can be viewed as a syntactically sugared version of orthogonal conditional term rewriting systems.

Various narrowing methods have been proposed and their completeness has been studied. Among them innermost narrowing[5], outer narrowing[14] and basic narrowing [10, 12] have been well studied. These narrowing methods are originally presented for (unconditional) term rewriting systems. We consider outside-in conditional narrowing, which has close relationship with outside-in reduction. The outside-in reduction was presented by Huet and Lévy and its property was deeply analyzed[9]. We adapt this notion of outside-in to conditional narrowing. Using the lifting argument often used in the analysis of computations of logic programs and narrowing, we can relate outside-in reduction and outside-in conditional narrowing and show the completeness of outside-in conditional narrowing.

The organization of this paper is as follows. In Sect.2 and Sect.3 we give basic definitions used for our treatment of outside-in conditional narrowing in subsequent sections. Sect.3.1, in particular, summarizes the results of Huet and Lévy's work on reduction derivations of orthogonal term rewriting systems. Readers familiar with their work may skip Sect.2 and Sect.3.1 and refer to them only for resolving differences of notations between ours and Huet and Lévy's. Sect.4 is devoted to definitions on conditional narrowing. In Sect.5 we describe the main result of this paper. In Sect.6 we briefly discuss future research themes.

2 Preliminaries

Let $\mathcal{F}$ be a set of function symbols, and $\mathcal{V}$ a set of variables, satisfying $\mathcal{F} \cap \mathcal{V} = \emptyset$. Terms are defined as usual over a set of alphabet $\mathcal{F} \cup \mathcal{V}$. The set of terms is denoted by $\mathcal{T}(\mathcal{F}, \mathcal{V})$, or simply by $\mathcal{T}$. A term t is called *linear* when no variable occurs in t more than once. $\mathcal{V}(\mathcal{A})$ denotes a set of variables occurring in a syntactic object $\mathcal{A}$. For example, $\mathcal{A}$ is a term, a rewrite rule, an equation or a sequence of equations. The set $\mathcal{O}(t)$ of *positions* of a term t is a set of sequences of positive integers that address subterms of t. An empty sequence is denoted by ε. A position u in $\mathcal{O}(t)$ addresses a subterm $t|_u$, where $t|_u$ is defined as follows. Let $t \triangleq f(t_1, \ldots, t_n)$. $t|_u \triangleq t$ if $u = \varepsilon$ and $t|_u \triangleq t_i|_{u'}$ if $u = i.u'$. A position u in $\mathcal{O}(t)$ is called a *non-variable position* if $t|_u$ is not a variable. The set of non-variable positions of t is denoted by $\overline{\mathcal{O}}(t)$, and the set of variable positions of t by $\mathcal{O}_{\mathcal{V}}(t)$. A term obtained from t by replacing $t|_u$, where $u \in \mathcal{O}(t)$, by a term s is denoted by $t[s]_u$. A context, written as $C[\ ]$, is a term that has a single special constant $[]$, called a hole, as a subterm. $C[t]$ is a term whose hole is replaced by a term t, i.e., $C[\ ][t]_u$, where $C[\ ]|_u \equiv []$. A set of contexts is denoted by $\mathcal{T}_{[]}$.

Partial order $\preceq$ over positions is defined as follows. For $u, v \in \mathcal{O}(t)$, $v \preceq u$ if v is a prefix of u, i.e., $\exists w$ such that $vw = u$. $v \prec u$ if $v \preceq u$ and $v \neq u$. Positions u and v are called *disjoint*, written as $v \mid u$, if $\neg(v \preceq u$ or $u \preceq v)$. u is said to be to the left of v if u is written as $w_1 i w_2$ and v is written as $v = w_1 j w_3$ for $i < j$. u is the leftmost in $\mathcal{O}(t)$ if for any $v \in \mathcal{O}(t)$ $\neg(v$ is to the left of $u)$.

A *substitution* is a mapping from $\mathcal{V}$ to $\mathcal{T}$. The *domain* and *codomain* of a substitution θ are respectively defined as $\mathcal{D}(\theta) = \{x \mid \theta x \not\equiv x, x \in \mathcal{V}\}$ and $\mathcal{C}od(\theta) = \{\theta x \mid x \in \mathcal{D}(\theta)\}$. We identify a substitution θ with the set $\{x \mapsto \theta x \mid x \in \mathcal{D}(\theta)\}$. An *empty substitution* is defined as the empty set $\emptyset$. A substitution is extended to an endomorphism over $\mathcal{T}$ as usual. Let $V \subseteq \mathcal{V}$. The restriction of θ to V is denoted by $\theta \restriction_V$. The *composition* of θ_2 and θ_1, (first apply θ_1, then θ_2) is denoted by $\theta_2\theta_1$. We write $\theta_1 = \theta_2[V]$ when $\theta_1 \restriction_V = \theta_2 \restriction_V$ holds. When $\sigma\theta_1 = \theta_2$ for some substitution σ, we write $\theta_1 \preceq \theta_2$. When $\sigma\theta_1 = \theta_2[V]$ for some substitution σ, we write $\theta_1 \preceq \theta_2[V]$. The set of substitutions is denoted by Θ.

3 Derivations in orthogonal term rewriting systems

3.1 Reduction derivation

A rewrite rule is a directed equation of terms, written as $l \rightarrow r$ satisfying $l \notin \mathcal{V}$ and $\mathcal{V}(r) \subseteq \mathcal{V}(l)$. A term rewriting system (abbreviated as TRS) is a set of rewrite rules.

Definition 3.1 A TRS $\mathcal{R}$ is called *orthogonal* if the following conditions are satisfied:

- For any rule $l \rightarrow r$ in $\mathcal{R}$, l is linear.
- For any two rewrite rules $l \rightarrow r$ and $l' \rightarrow r'$ of variants of rewrite rules in $\mathcal{R}$ there exists no unifier of l and a subterm $l'|_u$, where $u \in \overline{\mathcal{O}}(l')$, except in the case where $l \rightarrow r$ and $l' \rightarrow r'$ are the same rewrite rules (modulo renaming) and $u = \varepsilon$.

Definition 3.2 A TRS $\mathcal{R}$ is called *constructor-based* if $\mathcal{F}$ is partitioned into disjoint sets $\mathcal{F}_\mathcal{D}$ and $\mathcal{F}_\mathcal{C}$ such that the left-hand side $f(t_1, \ldots t_n)$ of every rewrite rule of $\mathcal{R}$ satisfies $f \in \mathcal{F}_\mathcal{D}$ and $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}_\mathcal{C}, \mathcal{V})$.

A function symbol in $\mathcal{F}_\mathcal{C}$ is called a *constructor symbol.* A term in $\mathcal{T}(\mathcal{F}_\mathcal{C}, \mathcal{V})$ is called a *data term.*

Definition 3.3 Let $\mathcal{R}$ be a TRS. The single-step reduction $\rightarrow_\mathcal{R}$ is defined as follows. Suppose s and t are terms. $s \rightarrow_\mathcal{R} t$ if there exist a position $u \in \mathcal{O}(s)$, a new variant $l \rightarrow r$ of a rewrite rule in $\mathcal{R}$ and a substitution σ such that

- $s|_u \equiv \sigma l$,
- $t \equiv s[\sigma r]_u$.

The subterm $s|_u$ is called a *redex* and u is called a *redex position.* The set of redex positions in s is denoted by Red(s).

The reflexive and transitive closure of $\rightarrow_{\mathcal{R}}$ is denoted by $\twoheadrightarrow_{\mathcal{R}}$. A term that does not contain a redex is called a ($\rightarrow_{\mathcal{R}}$)-*normal form*. The subscript $\mathcal{R}$ in $\rightarrow_{\mathcal{R}}$ is often omitted if $\mathcal{R}$ is clear from the context. This convention is applied to other relations associated with $\mathcal{R}$, which will be introduced later.

We give several definitions and important results pertaining to orthogonal TRSs, which were obtained by Huet and Lévy. An elementary reduction derivation $s \rightarrow_u t$ is a process of rewriting a term s at u into t by a single-step reduction. The redex $s|_u$ is said to be contracted in this reduction derivation.[1] Note that in an orthogonal TRS, a pair (s, u) unambiguously determines the rewrite rule $l \rightarrow r$ and terms $\sigma l, \sigma r$, and hence the term t. A reduction derivation is a sequence $A \triangleq A_1 A_2 \cdots A_n$ of elementary reduction derivations $A_i : s_{i-1} \rightarrow_{u_i} s_i$. An empty reduction derivation is denoted by 0. We use notation $A[i,j]$ also as an abbreviation of $A_i \cdots A_j$.

Let $u \in \mathrm{Red}(s)$, and $l \rightarrow r$ be a rewrite rule used to contract the redex $s|_u$. We define $\mathrm{Pattern}(s, u) \triangleq \{uv \in \mathcal{O}(s) \mid v \in \overline{\mathcal{O}}(l)\}$.

Definition 3.4 Let $A : s \rightarrow_u t$ be an elementary reduction derivation and $v \in \mathcal{O}(s)$. Suppose $l \rightarrow r$ is used to contract the redex $s|_u$. The set $v \backslash A$ of descendants of v by A is defined as follows:

$$v \backslash A = \begin{cases} \{v\} & \text{if } v|u \text{ or } v \prec u \\ \{uw_1v_1 \mid r|_{w_1} \equiv l|_w\} & \text{if } v = uwv_1 \text{ and } w \in \mathcal{O}_{\mathcal{V}}(l) \\ \emptyset & \text{otherwise.} \end{cases}$$

The notion of descendant is extended to non-elementary reduction derivations in the following way:

$$v \backslash 0 = \{v\},$$
$$v \backslash (AB) = \cup_{w \in v \backslash A} w \backslash B.$$

For a set $U = \{u_1, \ldots, u_n\}, n \geq 0$, of mutually disjoint redex positions in s, we can define a reduction derivation in which n redexes are simultaneously contracted. We call this derivation an *elementary multiderivation* defined by $A = (s, U)$. The multiderivation is denoted by $s \rightarrow_U s[\sigma_1 r_1]_{u_1} \cdots [\sigma_n r_n]_{u_n}$, where $l_i \rightarrow r_i$ and σ_i is a rewrite rule and a substitution, respectively, that are used to contract the redex $s|_{u_i}$ for $i = 1, \ldots n$. $U \backslash A$ is defined as $\cup_{u \in U} u \backslash A$. Note that in this definition of a multiderivation, U may be $\emptyset$. Thus $s \rightarrow_\emptyset s$ is possible.[2] Since an elementary derivation is an elementary multiderivation, derivation refers to multiderivation hereafter. The length $|A|$ of the derivation A is defined as follows.[3]

$$\begin{aligned} |0| &= 0 \\ |A_1 A| &= 1 + |A| \end{aligned}$$

[1]We even say that u is contracted if s is clear from the context.

[2]This should not be confused with an empty derivation 0.

[3]Later we introduce derivations of different kinds. We assume that the notational convention given here and the definition of length are carried over to those derivations.

where A_1 is an elementary derivation. Note that the length of a derivation $s \to_\emptyset s$ is 1.

We denote by $D(s)$ the set of reduction derivations starting from s, and by $F(A)$ the final term of the reduction derivation A, i.e., $F(A) = s_k$ for $A : s_0 \to_{U_1} s_1 \to_{U_2} \cdots \to_{U_k} s_k$. Let $A, B \in D(s)$, where $|B| = 1$ and B contracts all the redexes in the set $U \subseteq \mathrm{Red}(s)$. We define the *residual* $B \backslash A$ of B by A as an elementary reduction derivation in $D(F(A))$ that contracts the redexes in the set $U \backslash A$. This definition is generalized to reduction derivations of arbitrary length (by the parallel moves lemma[9]). Reduction derivations A and B are called *equivalent* (notation $A \equiv B$) if $A, B \in D(s)$ and for all $C \in D(s)$, $C \backslash A = C \backslash B$. The relation $\equiv$ is an equivalence relation on the set of reduction derivations, and hence it determines the quotient set. An equivalent class is denoted by $[A]_\equiv$ with A a representative of the class.

Definition 3.5 Let $A : s_0 \to_{U_1} s_1 \to_{U_2} \cdots \to_{U_k} s_k$ be a reduction derivation. A set $\mathrm{Red}(A)$ of initial redex positions contributing to A is defined as follows:[4]

$$\mathrm{Red}(A) = \{u \in \mathrm{Red}(s_0) \mid \exists i \leqslant k \; U_i \cap (u \backslash A[1, i-1]) \neq \emptyset\}.$$

Definition 3.6 Let $A : s_0 \to_{U_1} s_1 \to_{U_2} \cdots \to_{U_k} s_k$ be a reduction derivation and $u \in \mathcal{O}(s_0)$. The derivation A preserves u if A does not contract a redex above u, i.e., $\forall i \leqslant k \; \neg(\exists v \in U_i, v \prec u)$.

Definition 3.7 Let A be a reduction derivation starting from s. A position u in s is *external* for A if

- A preserves u, or
- $A = A_1 A_2 A_3$ and there exists $v \prec u$ such that
  - A_1 preserves u,
  - $A_2 : t \to_U t'$, with $v \in U$ and $u \in \mathrm{Pattern}(t, v)$,
  - v is external for A_3.

The set of external positions for A is denoted by $\mathcal{X}(A)$. The set $\mathcal{E}(A)$ of external redex positions for A is defined as $\mathcal{E}(A) = \mathcal{X}(A) \cap \mathrm{Red}(A)$.

Definition 3.8 Let $\mathcal{R}$ be an orthogonal TRS. The reduction derivation A is called *outside-in* if either $A = 0$ or $A = A_1 A_2$, where A_1 is the elementary reduction derivation contracting a redex position u in $\mathcal{E}(A)$ and A_2 is outside-in.

For a given reduction derivation A, its outside-in reduction derivation is not unique since $\mathcal{E}(A)$ contains more than one positions in general. The choice of the position may be fixed. Namely we have the following.

Definition 3.9 A reduction derivation A is called *standard* if either $A = 0$ or $A = A_1 A_2$, where A_1 is the elementary reduction derivation contracting the leftmost redex position in $\mathcal{E}(A)$ and A_2 is standard.

[4]Note the overloaded use of the symbol Red.

Note that since $\mathcal{E}(A)$ consists of pairwise disjoint positions, the leftmost position is uniquely determined in $\mathcal{E}(A)$.

Huet and Lévy obtained the following result on the standard reduction derivation.

Theorem 3.1 (Standardization Theorem [9]) Let $\mathcal{R}$ be an orthogonal TRS. For any reduction derivation A, $[A]_\equiv$ contains a unique standard reduction derivation.

3.2 Narrowing derivation

Definition 3.10 Let $\mathcal{R}$ be a TRS. The narrowing relation $\rightsquigarrow$ is defined as follows. Suppose s and t are terms. $s \rightsquigarrow t$ if there exists a non-variable position $u \in \overline{\mathcal{O}}(s)$, a new variant $l \rightarrow r$ of a rewrite rule in $\mathcal{R}$ and a substitution σ such that

- σ is a most general unifier of $s|_u$ and l,
- $t \equiv \sigma(s[r]_u)$.

The subterm $s|_u$ is called a *narex* (*na*rrowable *ex*pression), and u the narex position. The reflexive and transitive closure of $\rightsquigarrow$ is denoted by $\rightsquigarrow^*$.

An elementary narrowing derivation $s \rightsquigarrow_{[U,\sigma]} t$ is the process of rewriting a term s into a term t by the narrowing relation $s \rightsquigarrow t$. In order to be compatible with the notion of multiderivation of reduction we define an elementary narrowing derivation as follows:

$$s \rightsquigarrow_{[U,\sigma]} t \text{ is } \begin{cases} s \rightsquigarrow t \text{ with } \sigma \text{ and } u \in \overline{\mathcal{O}}(s) & \text{if } U = \{u\}, \\ t \equiv \sigma s & \text{if } U = \emptyset. \end{cases}$$

Note that for an orthogonal TRS a triple (s, U, σ) unambiguously determines the term t and the rewrite rule $l \rightarrow r$ to be applied, and hence the elementary narrowing derivation. A (non-elementary) narrowing derivation is defined as in a reduction derivation. A narrowing derivation $s_0 \rightsquigarrow_{[U_1,\sigma_1]} s_1 \rightsquigarrow_{[U_2,\sigma_2]} \cdots \rightsquigarrow_{[U_k,\sigma_k]} s_k$ is abbreviated as $s_0 \rightsquigarrow^*_\sigma s_k$ where $\sigma = \sigma_k \cdots \sigma_1$ if we are only interested in the substitution σ. For a narrowing derivation $s \rightsquigarrow_{[U,\sigma]} t$, there exists a corresponding reduction derivation $\sigma s \rightarrow_U t$. Hence, we define leftmost-outside-in narrowing derivation by means of reduction derivation.

Definition 3.11 A narrowing derivation

$$A : s_0 \rightsquigarrow_{[U_1,\sigma_1]} s_1 \rightsquigarrow_{[U_2,\sigma_2]} \cdots \rightsquigarrow_{[U_k,\sigma_k]} s_k$$

is called *leftmost-outside-in* if the corresponding reduction derivation $A' : \mu_1 s_0 \rightarrow_{U_1} \mu_2 s_1 \rightarrow_{U_2} \ldots \rightarrow_{U_k} s_k$, where $\mu_i = \sigma_k \sigma_{k-1} \ldots \sigma_i$, is standard.

Leftmost-outside-in narrowing derivation is abbreviated as LOI narrowing derivation.

Example 3.1 Let a TRS $\mathcal{R}$ be

$$\mathcal{R} = \begin{cases} f(g(d)) \rightarrow a \\ g(c) \rightarrow g(d) \\ h(x) \rightarrow x. \end{cases}$$

There are 5 narrowing derivations issuing from $h(f(g(z)))$ and ending at a. Among them the following two narrowing derivations are LOI.

$$A_1 : h(f(g(z))) \leadsto_{[\{\varepsilon\},\{x \mapsto f(g(z))\}]} f(g(z)) \leadsto_{[\{\varepsilon\},\{z \mapsto d\}]} a$$
$$A_2 : h(f(g(z))) \leadsto_{[\{\varepsilon\},\{x \mapsto f(g(z))\}]} f(g(z)) \leadsto_{[\{1\},\{z \mapsto c\}]} f(g(d)) \leadsto_{[\{\varepsilon\},\emptyset]} a$$

Others, e.g., $A_3 : h(f(g(z))) \leadsto_{[\{1\},\{z \mapsto d\}]} h(a) \leadsto_{[\{\varepsilon\},\{x \mapsto a\}]} a$, are not LOI.

Closely related to the LOI narrowing derivation is You's outer narrowing [14]. Intuitively, an outer narrowing derivation is a derivation in which no later narrowing steps at higher positions can be performed earlier in the derivation.[5] In Example 3.1 the narrowing derivation A_1 is outer, whereas A_2 is not. The derivation A_2 is not outer since the narex at the position ε can be contracted earlier.

4 Conditional term rewriting systems

4.1 Classes of conditional term rewriting systems

A conditional rewrite rule is a rewrite rule with a condition, written as $l \to r \Leftarrow Q$, where Q is a condition. A conditional term rewriting system (abbreviated as CTRS) is a set of conditional rewrite rules. In this paper we are concerned with a condition that is specified by a sequence of equations $s_1 = t_1, \ldots, s_n = t_n$. The order of the equations is insignificant. Syntactically, equations are treated as a special term, written in infix form, whose root symbol is $=$. The symbol $=$ is allowed only to be at the root of a term.

CTRSs are are called 1-CTRS if $\mathcal{V}(r) \cup \mathcal{V}(Q) \subseteq \mathcal{V}(l)$, 2-CTRS if $\mathcal{V}(r) \subseteq \mathcal{V}(l)$, and 3-CTRS if $\mathcal{V}(r) \subseteq \mathcal{V}(l) \cup \mathcal{V}(Q)$ for every rewrite rule $R : l \to r \Leftarrow Q \in \mathcal{R}$ as in [12].

Let $\mathrm{Ext}(R) = \mathcal{V}(Q) - \mathcal{V}(l)$. $\mathrm{Ext}(R)$ denotes a set of variables in Q without occurring in l. Variables in $\mathrm{Ext}(R)$ are called *extra variables*. In this paper we treat only 1- and 2-CTRSs.

We further restrict ourselves to a class of CTRSs whose conditional part consists of equations of the form $f_{\equiv}(s,t) = \mathrm{true}$, where true is a distinguished symbol in $\mathcal{F}_C$. The function symbol $f_{\equiv}$ defines the so-called strict equality; $f_{\equiv}(s,t) \twoheadrightarrow_{\mathcal{R}}$ true if and only if normal forms of s and t are syntactically equal closed data terms. We then write $s \equiv t$[6] instead of $f_{\equiv}(s,t) = \mathrm{true}$.

Definition 4.1 A CTRS $\mathcal{R}$ is called s-CTRS (s for strict) if the condition of every rewrite rule in $\mathcal{R}$ is of the form $s_1 \equiv t_1, \ldots, s_n \equiv t_n$.

The restriction of CTRSs to s-CTRSs is essential in the following discussions. One would like to have the ordinary equality that is realized by a rewrite rule $x = x \to \mathrm{true}$. It cannot be incorporated into our CTRSs , however, since it is not a linear rule and hence the notion of outside-in derivation is not well-defined. Properties of s-CTRSs are first investigated

[5] Refer to Definition 3.10 in You [14] for the precise definition.

[6] The symbol $\equiv$ is used not only as the strict equality but also as a meta symbol; the distinction should be clear from the context.

by Giovannetti *et al.* in conjunction with the design of logic plus functional language K-LEAF[7, 6]. An s-CTRS of class 1 and of class 2 is denoted by 1s-CTRS and 2s-CTRS respectively. Note that 1s- and 2s-CTRSs are special cases of $\mathrm{III_n}$ CTRSs [1].

Given a CTRS $\mathcal{R}$, we define an associated unconditional TRS, written as $\mathcal{U}(\mathcal{R})$, by removing the condition of each rewrite rule in $\mathcal{R}$. A CTRS $\mathcal{R}$ is called orthogonal if $\mathcal{U}(\mathcal{R})$ is orthogonal, and constructor-based if $\mathcal{U}(\mathcal{R})$ is.

4.2 Reduction in CTRSs

In the literature there are several different ways to define the notion of conditional reduction and narrowing. We basically follow Bockmayr's formulation[2]. For technical reasons we assign pairwise distinct labels to the equations in the condition of a rewrite rule whenever it is used for reduction and narrowing. This enables us to keep track of each equation during the reduction and narrowing, and furthermore to treat the condition of a rewrite rule as a set of equations.

We define a goal as follows.

Definition 4.2 Let $\mathcal{I}$ be a countably infinite index set, and $=^?_\ell$, where $\ell \in \mathcal{I}$, be (infinitely many) equality symbols. A *labeled equation* is of the form $s =^?_\ell t$, where ℓ is called a label of the (labeled) equation. A set of labeled equations whose labels are pairwise distinct is called a *goal*.

We denote by $\ell{:}S$ the labeled equation $s =^?_\ell t$ in a goal S also. The set of labels in a goal S is denoted by Label(S). Let Q be the condition $s_1 = t_1, \ldots, s_n = t_n$ of a rewrite rule. Then $\overline{Q}$ denotes a goal $\{s_1 =^?_{\ell_1} t_1, \ldots, s_n =^?_{\ell_n} t_n\}$, where labels $\ell_1, \ldots, \ell_n$ are pairwise distinct. As a shorthand notation a labeled equation $f_\equiv(s,t) =^?_\ell$ true is written as $s \equiv^?_\ell t$.

We now discuss the notion of reductions in s-CTRSs. We will follow the formulation of Middeldorp and Hamoen [12] to define notions of conditional narrowing, reductions and intermediate reduction associated with CTRSs.

In the sequel we assume that s-CTRSs contain a set of rewrite rules:

$$\{f_\equiv(c(x_1,\ldots,x_n), c(y_1,\ldots,y_n)) \rightarrow \text{true} \Leftarrow f_\equiv(x_1,y_1) = \text{true}, \ldots, f_\equiv(x_n,y_n) = \text{true}$$

$$\mid c \in \mathcal{F}_C \text{ and the arity of } c \text{ is } n\}.$$

Definition 4.3 Let $\mathcal{R}$ be an s-CTRS. We define inductively TRSs $\mathcal{R}_n$ associated with $\mathcal{R}$ as follows:

$\mathcal{R}_0 = \emptyset$
$\mathcal{R}_{n+1} = \{\ \sigma l \rightarrow \sigma r \mid l \rightarrow r \Leftarrow Q \in \mathcal{R}, \sigma \in \Theta, \text{ such that for every } e \in \overline{Q}\ \ \sigma e \twoheadrightarrow_{\mathcal{R}_n} \text{true} =^?_\ell \text{true}\}$.

Relation $\rightarrow_\mathcal{R}$ is defined as $\cup_n \rightarrow_{\mathcal{R}_n}$. We sometimes abbreviate $s \rightarrow_{\mathcal{R}_n} t$ as $s \xrightarrow{n} t$. We call the reduction $s \rightarrow_\mathcal{R} t$ *n-level* if $s \rightarrow_{\mathcal{R}_n} t$.

The reduction relation over terms is extended in the obvious way to the one over goals. Let $\top$ generically represents a set consisting only of finite number of labeled equations of the form true $=^?_\ell$ true.

A goal S is called *solvable* in a CTRS $\mathcal{R}$ if there exists a substitution σ such that $\sigma S \twoheadrightarrow_{\mathcal{R}} \top$. The substitution σ is called a *solution* of the goal S. We write $\mathcal{R} \vdash S$ if $S \twoheadrightarrow_{\mathcal{R}} \top$. The least n such that $\mathcal{R}_n \vdash S$ is called the *level* of the goal S.

Definition 4.4 (1) A CTRS $\mathcal{R}$ is called *level-confluent* if each $\mathcal{R}_n (n \geqslant 0)$ is confluent.

(2) A TRS is called *complete*[7] if it is strongly normalizing and confluent, and *semi-complete* if it is weakly normalizing and confluent.

(3) A CTRS $\mathcal{R}$ is called *level-complete* if each $\mathcal{R}_n (n \geqslant 0)$ is complete, and *level-semi-complete* if each $\mathcal{R}_n (n \geqslant 0)$ is semi-complete.

The following lemma [1, Theorem 3.5] is a version of a parallel moves lemma for orthogonal 2s-CTRSs. This lemma is used to prove the confluence of 2s-CTRSs, as well as to prove the standardization theorem for orthogonal 2s-CTRSs.

Lemma 4.1 Let $\mathcal{R}$ be an orthogonal 2s-CTRS. If $A_1 : s \overset{n}{\rightarrow}_{U_1} s_1$ and $A_2 : s \overset{m}{\rightarrow}_{U_2} s_2$ then there exists s' such that $s_1 \overset{m}{\rightarrow}_{U_1 \backslash A_2} s'$ and $s_2 \overset{n}{\rightarrow}_{U_2 \backslash A_1} s'$.

The following theorem is an immediate consequence of Lemma 4.1.

Theorem 4.1 An orthogonal 2s-CTRS is level-confluent.

Since level-confluence implies confluence, orthogonal 2s-CTRSs are confluent.

We next define the standard reduction derivation for an orthogonal s-CTRS $\mathcal{R}$ with respect to $\rightarrow_{\mathcal{R}}$. In Sect.3.1 we gave the definition of standard reduction derivation with respect to $\rightarrow_{\mathcal{R}}$ for an orthogonal TRS $\mathcal{R}$. Since the TRSs $\mathcal{R}_1, \mathcal{R}_2, \ldots$ induced from an s-CTRS $\mathcal{R}$ are not orthogonal in general, we cannot apply the definition of the standard derivation for orthogonal TRSs to CTRSs directly. Instead, we define the notion of the standard derivation via $\mathcal{U}(\mathcal{R})$. The relation $\rightarrow_{\mathcal{R}}$ is included in the relation $\rightarrow_{\mathcal{U}(\mathcal{R})}$. Hence, a reduction derivation with respect to $\rightarrow_{\mathcal{R}}$ is well-defined as a reduction derivation with respect to $\rightarrow_{\mathcal{U}(\mathcal{R})}$. With this observation we state the definition of standard reduction derivation for orthogonal s-CTRSs as follows.

Definition 4.5 Let $\mathcal{R}$ be an orthogonal s-CTRS. A reduction derivation with respect to $\rightarrow_{\mathcal{R}}$ is called standard if it is standard with respect to $\rightarrow_{\mathcal{U}(\mathcal{R})}$.

The equivalence relation $\equiv$ on the set of reduction derivations with respect to $\rightarrow_{\mathcal{U}(\mathcal{R})}$ is carried over to the case of $\rightarrow_{\mathcal{R}}$.

Lemma 4.2 Let $\mathcal{R}$ be an orthogonal 2s-CTRS. For any reduction derivation A, $[A]_{\equiv}$ contains a unique standard reduction derivation.

Proof: By the parallel moves lemma for 2s-CTRSs (Lemma 4.1). The proof of Theorem 3.1 in [9] is applicable to 2s-CTRSs.∎

We next extend the notion of standard reduction over terms to the one over goals.

[7]This should not be confused with completeness of narrowing.

Definition 4.6 Let $\mathcal{R}$ be an orthogonal 2s-CTRS, and $S_0, \dots, S_k$ be goals. A reduction derivation $S_0 \to S_1 \to \cdots \to S_k$ is standard if $\ell{:}S_0 \to \ell{:}S_1 \to \cdots \to \ell{:}S_k$ is standard for all $\ell \in \text{Label}(S_0)$.

The reduction derivation with respect to $\to_{\mathcal{R}}$ does not entirely capture the process of rewriting since rewriting of the goals originating in the conditions of the rewrite rules used during the derivation is not recorded in the derivation. To the rewriting process, we introduce a notion of intermediate reduction which is originally due to Bockmayr[2].[8] The intermediate reduction is the right notion when we study the correspondence between the reduction and narrowing in the setting of CTRS as we will see in Sect.5.1 and Sect.5.2.

Definition 4.7 Let $\mathcal{R}$ be a CTRS. The single-step intermediate reduction $\rightarrowtail$ over goals is defined as follows. Suppose S and T are goals. $S \rightarrowtail T$ if there exist an equation $e \in S$, a position $u \in \mathcal{O}(e)$, a new variant $l \to r \Leftarrow Q$ of a rewrite rule in $\mathcal{R}$, and a substitution σ such that

- $e|_u \equiv \sigma l$,
- $T = (S - \{e\}) \cup \{e[\sigma r]_u\} \cup \sigma\overline{Q}$,
- $\mathcal{R} \vdash \sigma\overline{Q}$.

We implicitly assume that whenever $\overline{Q}$ is taken, $\text{Label}(S) \cap \text{Label}(\overline{Q}) = \emptyset$. The same assumption is made in Definitions 4.11 and 5.4.

The reflexive and transitive closure of $\rightarrowtail$ is denoted by $\twoheadrightarrow$. An intermediate reduction is called an *i-reduction* for short, hereafter. For orthogonal CTRSs an elementary i-reduction derivation is defined similarly to the elementary reduction derivation. An elementary i-reduction derivation is a 4-tuple (S, e, u, σ), and is denoted by $S \rightarrowtail_{[e,u,\sigma]} T$. To make explicit the rewrite rule $l \to r \Leftarrow Q$ employed in the derivation we write also $S \rightarrowtail_{[e,u,l \to r \Leftarrow Q,\sigma]} T$.

Definition 4.8 Let $S \rightarrowtail_{[e,u,\sigma]} T$ be an elementary i-reduction derivation and $\ell \in \text{Label}(S)$. The *successor* of $\ell{:}S$ is defined as $\ell{:}T$.

Definition 4.9 Let $A : S_0 \rightarrowtail S_1 \rightarrowtail \cdots \rightarrowtail S_k$ be an i-reduction derivation.

1. Every equation $\ell{:}S_0 \in S_0$ is initial in A.
2. An equation $\ell{:}S_i, 1 \leqslant i \leqslant k$, is initial if $\ell \notin \text{Label}(S_{i-1})$.

An initial equation is either an equation in an initial goal or an equation that originates in the condition of a rewrite rule.

Given an i-reduction derivation $A : S_0 \rightarrowtail S_1 \rightarrowtail \cdots \rightarrowtail S_k$, for every equation e in S_k we can extract a reduction derivation $e_0 \to_{U_1} e_1 \to_{U_2} \cdots \to_{U_l} e_l (\equiv e)$, where e_0 is an initial equation, e_i is a successor of e_{i-1}, and U_i is either $\emptyset$ or a singleton set for $i = 1, \dots, l$.

We call the reduction derivation $e_0 \to_{U_1} e_1 \to_{U_2} \cdots \to_{U_n} e_n$ a *trace* of the i-reduction derivation A. We denote by $\mathcal{TR}(A)$ the set consisting of these traces.

[8] Bockmayr called the intermediate reduction *Reduktionssrelation ohne Auswertung der Prämisse* (reduction relation without evaluating conditions). Although his terminology conveys exactly the meaning of what is involved, we prefer a more concise terminology for it.

It is meaningful to talk about a standard derivation of i-reduction as the following Lemma 4.3 shows.

Definition 4.10 Let $\mathcal{R}$ be an orthogonal s-CTRS. An i-reduction derivation A for $\mathcal{R}$ is called standard if every trace $B \in \mathcal{TR}(A)$ is standard.

Lemma 4.3 Let $\mathcal{R}$ be an orthogonal s-CTRS and S be a goal. If there exists a reduction derivation $S \twoheadrightarrow_{\mathcal{R}} \top$ then there exists a standard i-reduction derivation $S \rightarrowtail\!\!\!\twoheadrightarrow_{\mathcal{R}} \top$.

Proof: We prove by induction on the level n of the goal S. For $n = 0$ it is trivial. Suppose that the result holds for $n - 1 > 0$. By assumption and by Lemma 4.2, for every equation $\ell{:}S \in S$ there exists a standard reduction derivation

$$A^{\ell} : \ell{:}S \xrightarrow{n}_{[\sigma_1, l_1 \to r_1 \Leftarrow Q_1]} \ell{:}S_1 \xrightarrow{n}_{[\sigma_2, l_2 \to r_2 \Leftarrow Q_2]} \ell{:}S_2 \xrightarrow{n} \cdots \xrightarrow{n}_{[\sigma_k, l_k \to r_k \Leftarrow Q_k]} \ell{:}S_k (\equiv \text{true} =^?_{\ell} \text{true}).$$

In the above derivation, with $\xrightarrow{n}$ $(\subseteq \to_{\mathcal{U}(\mathcal{R})})$ we associate a pair $[\sigma, l \to r \Leftarrow Q]$, where σ and $l \to r \Leftarrow Q$ are a substitution and a rewrite rule, respectively, used for each single-step reduction. By the definition of $\xrightarrow{n}$,

$$\sigma_j \overline{Q_j} \overset{n-1}{\twoheadrightarrow} \top \text{ for } j = 1, \ldots, k.$$

By the induction hypothesis there exists a standard i-reduction derivation

$$B^{\ell}_j : \sigma_j \overline{Q_j} \rightarrowtail\!\!\!\twoheadrightarrow_{\mathcal{R}} \top \text{ for } j = 1, \ldots, k.$$

Namely, every trace in $\mathcal{TR}(B^{\ell}_j)$ for $j = 1, \ldots, k$ is standard. By combining A^{ℓ} and B^{ℓ}_j for $j = 1, \ldots, k$ we form an i-reduction derivation

$$\{\ell{:}S\} \rightarrowtail \{\ell{:}S_1\} \cup \sigma_1 \overline{Q_1} \rightarrowtail\!\!\!\twoheadrightarrow \{\ell{:}S_1\} \cup \top \rightarrowtail \{\ell{:}S_2\} \cup \sigma_2 \overline{Q_2} \rightarrowtail\!\!\!\twoheadrightarrow \cdots \rightarrowtail\!\!\!\twoheadrightarrow \top.$$

Combining the i-reduction derivation issuing from $\{\ell{:}S\}$ leading to $\top$ for every equation $\ell{:}S$ in S yields an i-reduction $A : S \rightarrowtail\!\!\!\twoheadrightarrow_{\mathcal{R}} \top$. Furthermore A is standard since every trace in $\mathcal{TR}(A)$ is standard. ∎

4.3 Conditional narrowing

Now we turn to conditional narrowing. Similarly to i-reduction we define conditional narrowing and conditional narrowing derivation. A narrowing derivation over terms is extended straightforwardly to deal with goals.

Definition 4.11 Let $\mathcal{R}$ be a CTRS. Narrowing relation $\rightsquigarrow$ over goals is defined as follows. Suppose S and T are goals.[9] $S \rightsquigarrow T$ if there exists an equation $e \in S$, a non-variable position $u \in \overline{\mathcal{O}}(e)$, a new variant $l \to r \Leftarrow Q$ of a rewrite rule in $\mathcal{R}$, and a substitution σ such that

- σ is a most general unifier of $e|_u$ and l,
- $T = (\sigma(S - \{e\})) \cup \{\sigma(e[r]_u)\} \cup \sigma\overline{Q}$.

[9] We use $\rightsquigarrow$ for narrowing relation both over goals and over terms (equations).

For orthogonal CTRSs, the elementary narrowing derivation over goals is defined by the 4-tuple (S, e, u, σ), and is denoted by $S \rightsquigarrow_{[e,u,\sigma]} T$. To make explicit the rewrite rule used in the derivation we also write $S \rightsquigarrow_{[e,u,l\to r\Leftarrow Q,\sigma]} T$. A (non-elementary) narrowing derivation over goals is defined in the same way as the one over terms.

A narrowing trace with respect to narrowing derivations over goals is defined similarly to the trace of an i-reduction derivation. A set of all the traces extracted from the narrowing derivation A (over goals) is denoted also by $\mathcal{TR}(A)$.

For an elementary narrowing derivation $S \rightsquigarrow_{[e,u,\sigma]} T$, there exists a corresponding elementary i-reduction derivation $\sigma S \rightarrowtail_{[e,u,\sigma]} T$. The correspondence between a narrowing derivation and i-reduction derivation is generalized. Namely, for a non-elementary narrowing derivation $S \rightsquigarrow^*_\sigma T$, there exists a corresponding non-elementary i-reduction derivation $\sigma S \rightarrowtail\!\!\twoheadrightarrow T$.

We define LOI narrowing derivations as follows.

Definition 4.12 Let $\mathcal{R}$ be an orthogonal s-CTRS. A narrowing derivation over goals for $\mathcal{R}$ is called LOI if the corresponding i-reduction derivation is standard.

5 Completeness results of conditional narrowing

Conditional narrowing is complete for orthogonal 1s-CTRSs with respect to normalizable solutions. This is a consequence of a more general result of Kaplan [11]. Namely, conditional narrowing is complete for confluent 1-CTRSs with respect to normalizable solutions, where equality in conditions is interpreted as joinability. We further expect the completeness of LOI conditional narrowing for orthogonal 1s-CTRSs. This is indeed the case. The proof of the completeness is given in Sect.5.1.

As for 2-CTRSs, Giovannetti and Moiso showed that completeness of 2-CTRSs is not sufficient for the completeness of conditional narrowing. The following is a counterexample taken from [7]:

$$\mathcal{R} = \begin{cases} a \to b \\ a \to c \\ b \to c \Leftarrow x = b, x = c. \end{cases}$$

$\mathcal{R}$ is a complete 2-CTRS. A goal $\{b =^? c\}$ is solvable since $\emptyset\{b =^? c\} \twoheadrightarrow_{\mathcal{R}} \top$, where $=$ is interpreted as joinability. However, we can not solve a goal $b =^? c$ by conditional narrowing. Hence, we need a stronger property than completeness of 2-CTRSs. It is shown in [12] that conditional narrowing is complete for level-semi-complete 2-CTRSs. However, this result presently has little practical implication from the programming language point of view, since we do not know a computationally effective sufficient condition for level-semi-completeness. Thus we are guided to look for a class of CTRSs whose properties are ensured by syntactic means. We will show that orthogonal 2s-CTRSs are complete with respect to normalizable solutions. We further expect the completeness of LOI conditional narrowing with respect to normalizable solutions for orthogonal 2s-CTRSs. This is proved in Sect.5.2.

5.1 Completeness result for 1s-CTRSs

We begin by giving the formal definition of the notion of completeness of conditional narrowing.

Definition 5.1 Let θ_1 and θ_2 be substitutions. $\theta_1 =_{\mathcal{R}} \theta_2$ if $\theta_1 x \twoheadleftarrow\!\!\!\twoheadrightarrow_{\mathcal{R}} \theta_2 x$ for all $x \in \mathcal{V}$, where $\twoheadleftarrow\!\!\!\twoheadrightarrow_{\mathcal{R}}$ is a reflexive, transitive and symmetric closure of $\rightarrow_{\mathcal{R}}$, and $\theta_1 \preceq_{\mathcal{R}} \theta_2$ if $\sigma\theta_1 =_{\mathcal{R}} \theta_2$ for some substitution σ.

Definition 5.2 Let $\mathcal{R}$ be a CTRS and S a goal. Conditional narrowing is *complete* if for every substitution σ such that $\mathcal{R} \vdash \sigma S$ holds, there exists a conditional narrowing derivation $S \rightsquigarrow^*_\tau \top$ such that $\tau \preceq_{\mathcal{R}} \sigma[\mathcal{V}(S)]$.

The following lifting lemma for 1-CTRSs is essential to prove the completeness.[10]

Lemma 5.1 (Lifting lemma for 1-CTRSs [2]) Let $\mathcal{R}$ be a 1-CTRS. Suppose we have goals S and T, a normalized substitution θ of S and a set V of variables such that $\mathcal{V}(S) \cup \mathcal{D}(\theta) \subseteq V$ and $T = \theta S$. For an i-reduction derivation $T \rightarrowtail\!\!\!\!\rightarrow T'$ there exist a goal S', substitutions θ' and σ such that

- $S \rightsquigarrow^*_\sigma S'$,
- $\theta' S' = T'$,
- $\theta' \sigma = \theta[V]$,
- θ' is a normalized substitution.

The narrowing derivation $S \rightsquigarrow^*_\sigma S'$ and the i-reduction derivation $T \rightarrowtail\!\!\!\!\rightarrow T'$ employ the same rewrite rules at the same positions in the corresponding goals.

Lemma 5.3 below establishes the correspondence between a standard i-reduction derivation and an LOI narrowing derivation. To prove it we need the following lemma.

Lemma 5.2 Let θ and σ be substitutions and s be a term such that $\sigma \preceq \theta[\mathcal{V}(s)]$. Then $\mathrm{Red}(\sigma s) \subseteq \mathrm{Red}(\theta s)$.

Proof: Straightforward. ∎

Lemma 5.3 Let $\mathcal{R}$ be an orthogonal 1s-CTRS. Suppose we have an i-reduction derivation $A : T \rightarrowtail\!\!\!\!\rightarrow T'$ and its lifted narrowing derivation $B : S \rightsquigarrow^*_\sigma S'$, where $T = \theta S$ and θ is normalized. If A is standard then B is LOI.

Proof: By the lifting lemma (Lemma 5.1), for an i-reduction A there exists a lifted narrowing derivation B. By the definition of i-reduction, there exists an i-reduction derivation $\sigma B : \sigma S \rightarrowtail\!\!\!\!\rightarrow S'$ and furthermore $\sigma \preceq \theta[V]$, where $V = \mathcal{V}(S) \cup \mathcal{D}(\theta)$. In each step of the derivations A, B and σB, the same equations are selected, the same positions are contracted (narrowed, in the case of B), and the same rewrite rules are used. For every trace $A' \in \mathcal{TR}(A)$, there exists a corresponding trace $A'' \in \mathcal{TR}(\sigma B)$. We have $\mathcal{X}(A') = \mathcal{X}(A'')$, and $\mathrm{Red}(A') \supseteq \mathrm{Red}(A'')$ by Lemma 5.2. From the above facts we can infer that if A' is outside-in then A'' is outside-in, and furthermore that for every contraction at u in A', if u in A' is leftmost then u in A'' is leftmost by Lemma 5.2. ∎

[10] Bockmayr first gave this lemma, and a rigorous proof was later given by Middeldorp and Hamoen[12].

Theorem 5.1 LOI conditional narrowing is complete with respect to normalizable solutions for orthogonal 1s-CTRSs.

Proof: Let $\mathcal{R}$ be an orthogonal 1s-CTRS. Suppose we have a normalizable solution σ of a goal S, i.e., $\sigma S \twoheadrightarrow_{\mathcal{R}} \top$. By the confluence of $\rightarrow_{\mathcal{R}}$, $\hat{\sigma} S \twoheadrightarrow_{\mathcal{R}} \top$, where $\hat{\sigma}$ is a normalized substitution obtained by reducing all the terms in $\mathcal{C}od(\sigma)$. By Lemma 4.3 there exists a standard i-reduction derivation $\hat{\sigma} S \rightarrowtail\!\!\!\twoheadrightarrow_{\mathcal{R}} \top$. By Lemma 5.3 there exists an LOI narrowing derivation $S \rightsquigarrow_{\tau}^{*} \top$ such that $\tau \preceq \hat{\sigma}[\mathcal{V}(S)]$. Hence $\tau \preceq_{\mathcal{R}} \sigma[\mathcal{V}(S)]$. ∎

This result extends the completeness result of the leftmost-outside-in narrowing for constructor-based orthogonal TRSs, which was obtained by Darlington and Guo[3] who followed the Huet and Lévy's definition of the standard derivation.

5.2 Completeness results for 2s-CTRSs

The key to prove the completeness of conditional narrowing for orthogonal 2s-CTRSs is a lifting lemma for 2s-CTRSs as in 1s-CTRSs. By a lifting lemma, we want to establish the correspondence between narrowing derivations and i-reduction derivations. A problem arises here that during the i-reduction extra variables may be bound to a term that becomes a redex at some later i-reduction step. When this occurs, it is impossible to make correspondence between narrowing and i-reduction derivations. We have to exclude this kind i-reduction derivations.

We define a restricted i-reduction $\overset{n}{\rightarrowtail\!\!\!\ominus\!\!\!\rightarrow}$ over solvable goals as follows.

Definition 5.3 A substitution σ is called a $\overset{n}{\rightarrow}$*-normalized* if for any $x \in \mathcal{D}(\sigma)$, σx is a $\overset{n}{\rightarrow}$-normal form.

Definition 5.4 Let $\mathcal{R}$ be a CTRS, and S and T be solvable goals.

1. $\overset{0}{\rightarrowtail\!\!\!\ominus\!\!\!\rightarrow} = \emptyset$,
2. $S \overset{n+1}{\rightarrowtail\!\!\!\ominus\!\!\!\rightarrow} T$ if there exist an equation $e \in S$, a position $u \in \mathcal{O}(e)$, a new variant $l \rightarrow r \Leftarrow Q$ of a rewrite rule in $\mathcal{R}$, and a substitution σ such that
   - $e|_u \equiv \sigma l$,
   - $T = (S - \{e\}) \cup \{e[\sigma r]_u\} \cup \sigma\overline{Q}$,
   - $\sigma\overline{Q} \twoheadrightarrow_{\mathcal{R}_n} \top$,
   - $\sigma\lceil_{\mathrm{Ext}(R)}$ is $\rightarrow_{\mathcal{R}_n}$-normalized,
   - $n+1$ does not exceed the level of e.

$\rightarrowtail\!\!\!\ominus\!\!\!\rightarrow$ is defined as $\cup_{n \geqslant 0} \overset{n}{\rightarrowtail\!\!\!\ominus\!\!\!\rightarrow}$.

The reflexive and transitive closures of $\rightarrowtail\!\!\!\ominus\!\!\!\rightarrow$ and $\overset{n}{\rightarrowtail\!\!\!\ominus\!\!\!\rightarrow}$ are denoted by $\rightarrowtail\!\!\!\ominus\!\!\!\twoheadrightarrow$ and $\overset{n}{\rightarrowtail\!\!\!\ominus\!\!\!\twoheadrightarrow}$, respectively. The notion of the restricted i-reduction is due to Middeldorp and Hamoen[12].

A restricted i-reduction derivation can be lifted to a narrowing derivation, as shown below.

Definition 5.5 A solution σ of a goal S is called *sufficiently normalized* (*normalizable*) if $\sigma\upharpoonright_{\mathcal{V}(e)}$ is $\mathcal{R}_n$-normalized ($\mathcal{R}_n$-normalizable) where n is the level of σe, for every equation $e \in S$.

Lemma 5.4 (Lifting lemma for 2-CTRSs[12]) Let $\mathcal{R}$ be a level-confluent 2-CTRS. Suppose we have goals S and T, a sufficiently normalized solution θ of S and a set V of variables such that $\mathcal{V}(S) \cup \mathcal{D}(\theta) \subseteq V$ and $T = \theta S$. For a restricted i-reduction derivation $T \twoheadrightarrow\!\!\!\circ\!\!\!\twoheadrightarrow T'$ there exist a goal S', substitutions θ' and σ such that

- $S \rightsquigarrow_\sigma^* S'$,
- $\theta' S' = T'$,
- $\theta'\sigma = \theta[V]$,
- θ' is a sufficiently normalized solution of S'.

The narrowing derivation $S \rightsquigarrow_\sigma^* S'$ and the i-reduction derivation $T \twoheadrightarrow\!\!\!\circ\!\!\!\twoheadrightarrow T'$ employ the same rewrite rules at the same positions in the corresponding goals.

The reduction derivation that corresponds to this restricted i-reduction is defined as follows.

Definition 5.6 Let $\mathcal{R}$ be an arbitrary CTRS, and $\mathcal{S}_n$ be associated TRSs that are inductively defined as follows:

$$\mathcal{S}_0 = \mathcal{R}_0,$$
$$\mathcal{S}_{n+1} = \{(\sigma l, \sigma r) \mid R : l \to r \Leftarrow Q \in \mathcal{R}, \sigma\overline{Q} \twoheadrightarrow_{\mathcal{S}_n} \top \text{ and } \sigma\upharpoonright_{\mathrm{Ext}(R)} \text{ is } \to_{\mathcal{S}_n}\text{-normalized}, \sigma \in \Theta\}.$$

As in $\to_{\mathcal{R}}$, we define the reduction relation $\to_{\mathcal{S}}$ associated with TRSs $\mathcal{S}_n$ as $\bigcup_{n \geqslant 0} \to_{\mathcal{S}_n}$.

We now look for a class of 2-CTRSs $\mathcal{R}$ whose induced reduction relation $\to_{\mathcal{R}_n}$ is equal to $\to_{\mathcal{S}_n}$ for each level n.

Definition 5.7 A CTRS $\mathcal{R}$ is called *level-normal* if $\to_{\mathcal{R}_n} = \to_{\mathcal{S}_n}$ for all $n \geqslant 0$.

In level-normal 2-CTRSs, for solvable goals the existence of a restricted i-reduction derivation ending at $\top$ is guaranteed. Namely we have the following lemma.

Lemma 5.5 Let $\mathcal{R}$ be a level-normal 2-CTRS and S a goal. We have $S \twoheadrightarrow_{\mathcal{R}} \top$ if and only if there exists a restricted i-reduction $S \twoheadrightarrow\!\!\!\circ\!\!\!\twoheadrightarrow_{\mathcal{R}} \top$.

Proof:

($\Rightarrow$) By double induction on the level n and the length of the reduction derivation. The case of $n = 0$ is trivial. Suppose $S \overset{n+1}{\to}_{[e,u,l\to r\Leftarrow Q,\sigma]} S' \overset{n+1}{\twoheadrightarrow} \top$, where $S' = S - \{e\} \cup \{e[\sigma r]_u\}$. Let m be the level of e. We distinguish two cases.

Case $n + 1 \leqslant m$

There exists an i-reduction derivation:

$$S \overset{n+1}{\rightarrowtail\!\!\circ\!\!\twoheadrightarrow} S' \cup \sigma\overline{Q} \overset{n}{\twoheadrightarrow\!\!\circ\!\!\twoheadrightarrow} S' \cup \top \overset{n+1}{\twoheadrightarrow\!\!\circ\!\!\twoheadrightarrow} \top.$$

The first step is by the definition of $\overset{n+1}{\rightarrowtail\!\!\circ\!\!\twoheadrightarrow}$ and by the level-normality, the second steps are by the definition of $\overset{n+1}{\to}$ and by the induction hypothesis on the level, and the third steps are

by the induction hypothesis on the length of the reduction derivation.
Case $n+1>m$
There exists an i-reduction derivation:

$$S \overset{m}{\rightarrowtail} S-\{e\} \cup \mathsf{T} \overset{n+1}{\rightarrowtail} \mathsf{T}.$$

The first steps are by the induction hypothesis on the level, and the second steps are by $S-\{e\} \subseteq S'$ and by the induction hypothesis on the length of the derivation.
($\Leftarrow$) Trivial. ∎

Middeldorp and Hamoen proved the above result for a level-complete 2-CTRS in [12].

Theorem 5.2 Conditional narrowing is complete with respect to sufficiently normalizable solutions for level-confluent and level-normal 2-CTRSs.

Proof: The proof is similar to the proof of Theorem 5.1. Use Lemmas 5.5 and 5.4. ∎

This result partly answers the open problem of Giovanetti and Moiso's; Conditional narrowing is complete with respect to normalized solutions that "do not depend" on non-normalizable solutions for level-confluent CTRSs[7].[11]

The level-normality is not a strange property that is only enjoyed by a very restricted class of CTRSs. We show that orthogonal 2-CTRSs are level-normal.

Proposition 5.1 An orthogonal 2s-CTRS is level-normal.

Proof: The proof is given in Appendix. ∎

As a corollary of Theorem 5.2, we obtain the completeness result for orthogonal 2s-CTRSs.

Corollary 5.1 Conditional narrowing is complete with respect to sufficiently normalizable solutions for orthogonal 2s-CTRSs.

The next question is whether the completeness is retained in the case of LOI conditional narrowing. The additional conditions of the last two clauses of Definition 5.4 of $\rightarrowtail$ do not interfere with the choice of positions as dictated by the LOI narrowing. Therefore, in the lifting lemma (Lemma 5.4) we have that if the i-reduction derivation is standard then the narrowing derivation is LOI, as in Lemma 5.3. Thus we obtain the completeness result of LOI conditional narrowing for 2s-CTRSs.

Theorem 5.3 LOI conditional narrowing is complete with respect to sufficiently normalizable solutions for orthogonal 2s-CTRSs.

6 Concluding remarks and future research

We have presented LOI conditional narrowing and shown its completeness with respect to normalizable solutions for orthogonal 1s-CTRSs and 2s-CTRSs. The result bears practical

[11] Quotation was put by the authors of the paper [7].

significance since an orthogonal s-CTRS with LOI conditional narrowing can be viewed as a computation model for functional-logic programming languages.

A next step in our research is to implement LOI conditional narrowing. Since it is not possible to find a set of external narex positions during conditional narrowing derivation, we have to restrict ourselves to a smaller class of orthogonal s-CTRSs; those that enable us to find an external narex position during a narrowing derivation without look-ahead. A strongly sequential CTRS of Huet and Lévy is a good candidate for that purpose[12], and indeed a constructor-based strongly sequential term rewriting system has been used as a model for functional-logic programming languages.

The proofs of the completeness of LOI conditional narrowing do not suggest the implementation method of LOI conditional narrowing. A method to implement LOI conditional narrowing will be to decompose it into more basic operations and to design a rule-based inference system like Hölldobler's rule set TRANS[8]. Further investigations are needed in this direction of research.

Lastly, we briefly discuss the completeness results of conditional narrowing for orthogonal 3s-CTRSs.[13] To the best of our knowledge the only published result of the completeness for 3-CTRSs is the completeness for level-complete 3-CTRSs obtained by Middeldorp and Hamoen[12]. This result is not applicable to orthogonal 3-CTRSs in an interesting way. For 3-CTRSs, orthogonality is not sufficient to realize level-confluence. Take orthogonal 3s-CTRSs

$$\mathcal{R} = \begin{cases} a \to x \Leftarrow f(x) \equiv k \\ f(b) \to k \\ f(c) \to k, \end{cases}$$

for example. We see that $a \to_{\mathcal{R}_2} b$ and $a \to_{\mathcal{R}_2} c$ hold while $b \to_{\mathcal{R}_2} c$ does not hold. Hence $\mathcal{R}$ is not level-confluent. Therefore all the arguments in the previous sections that hinge on the level-confluence derived from orthogonality of CTRSs break down in the cases of 3-CTRSs. In other words, orthogonality plays no active role in securing level-completeness of 3-CTRSs. To find effective means to realize (or check) level-completeness for 3-CTRSs is a future research theme.

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[12]Huet and Lévy defined the strong sequentiality for TRSs but the notion can be extended to CTRSs.

[13]Note that orthogonal CTRSs are well-defined since the orthogonality relies on only left-hand sides of rewrite rules.

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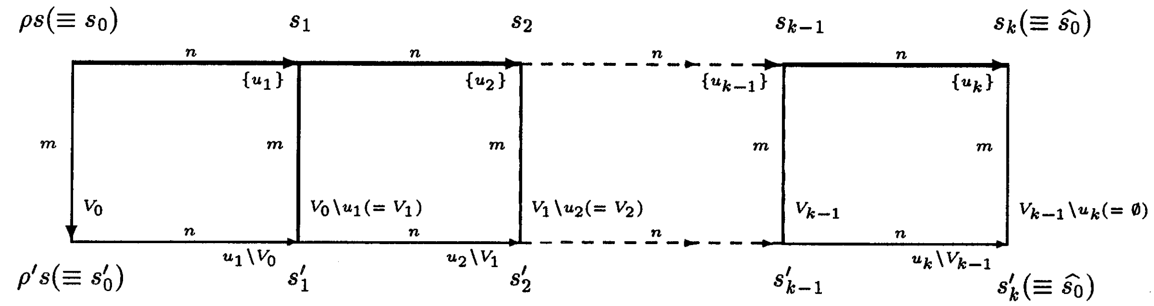
Appendix: Proof of Proposition 5.1

In this appendix all the derivations refer to reduction derivations. To prove Proposition 5.1 we need the following lemma.



- where $V_0 = \{v \mid (\rho s)|_v \equiv \Delta, v \in \mathcal{O}(\rho s)\}$

Figure 1: Derivations A and A'

Lemma .1 Let $\mathcal{R}$ be an orthogonal 2s-CTRS. Suppose that Δ is a $\xrightarrow{m}$-redex and Δ' is its contractum, and that ρ and ρ' are substitutions such that

- there exists $x \in \mathcal{D}(\rho)$ such that $\rho x \equiv \mathcal{C}[\Delta]$,
- $\rho' = \rho - \{x \mapsto \mathcal{C}[\Delta]\} \cup \{x \mapsto \mathcal{C}[\Delta']\}$.

If there exists a derivation

$$A : \rho s \xrightarrow{n}\!\!\!\!\rightarrow \widehat{\rho s}, \text{ where } \widehat{\rho s} \text{ is the normal form of } \rho s$$

then there exists a derivation

$$A' : \rho' s \xrightarrow{n}\!\!\!\!\rightarrow \widehat{\rho s} \text{ such that } |A| = |A'|.$$

Moreover, if Δ or its descendant is reduced in A, we can construct a derivation A' whose length is less than k.

Proof: By the diagram chase of Fig.1, we have

$$A' : \rho' s \xrightarrow{n}\!\!\!\!\rightarrow \widehat{\rho' s} \text{ such that } |A| = |A'|.$$

The construction of the diagram is possible by Lemma 4.1. The length of the derivation A' is made shorter when there exists u_i that is in V_0 or is a descendant of elements of V_0. In this case $u_i \backslash V_{i-1} = \emptyset$ and we can eliminate the reduction step $\rightarrow_{u_i \backslash V_{i-1}}$ in A'. ∎

Proposition 1: An orthogonal 2s-CTRS is level-normal.

Proof: We prove by induction on the level n of the reduction. Since $\rightarrow_{\mathcal{S}_0} = \rightarrow_{\mathcal{R}_0}$, we turn to the induction step. Suppose $\rightarrow_{\mathcal{S}_n} = \rightarrow_{\mathcal{R}_n}$. By definition and the induction hypothesis, $\rightarrow_{\mathcal{S}_{n+1}} \subseteq \rightarrow_{\mathcal{R}_{n+1}}$. Therefore, we have only to prove that for any $s \rightarrow_{\mathcal{R}_{n+1}} t$, we have $s \rightarrow_{\mathcal{S}_{n+1}} t$. Suppose $s \rightarrow_{\mathcal{R}_{n+1}} t$, i.e., there exist a rewrite rule $R : l \rightarrow r \Leftarrow Q$, a substitution τ and a context $\mathcal{C}[\ ]$ such that $s \equiv \mathcal{C}[\tau l], t \equiv \mathcal{C}[\tau r]$ and $\tau\overline{Q} \xrightarrow{n}\!\!\!\!\rightarrow \top$. From τ we will construct a substitution $\widehat{\tau}$ such that $\widehat{\tau}\restriction_{\mathrm{Ext}(R)}$ is $\xrightarrow{n}$-normalized, $s \equiv \mathcal{C}[\widehat{\tau} l], t \equiv \mathcal{C}[\widehat{\tau} r]$ and $\widehat{\tau}\overline{Q} \xrightarrow{n}\!\!\!\!\rightarrow \top$.

Define $\hat{\tau}$ as follows:

$$\hat{\tau}x = \begin{cases} \tau x & \text{if } x \in \mathcal{V}(l) \\ \sigma x & \text{if } x \in \mathrm{Ext}(R). \end{cases}$$

Then, the task of constructing $\hat{\tau}$ is reduced to the task of constructing a substitution σ such that its domain is $\mathrm{Ext}(R)$ and it satisfies the following properties:

- for any $e \in \overline{Q}, \sigma(\tau\restriction_{\mathcal{V}(l)} e) \stackrel{n}{\twoheadrightarrow}$ true $=^?_\ell$ true, where ℓ is the label of e, and
- σ is $\stackrel{n}{\rightarrow}$-normalized.

We have a stronger result; σ is $(\rightarrow_{\mathcal{R}})$-normalized. We show the construction of σ in the case that $\overline{Q} = \{e\}$. In the case of $\overline{Q} = \emptyset$, there is nothing to prove further. It is straightforward to extend the construction to the case that $\overline{Q}$ consists of more than one equations.

Let an equation $t(\triangleq \tau\restriction_{\mathcal{V}(l)} e)$, a substitution $\mu(= \tau\restriction_{\mathrm{Ext}(R)})$ and a derivation

$$A : \mu t(\triangleq t_0) \stackrel{n}{\rightarrow}_{\{v_1\}} t_1 \stackrel{n}{\rightarrow}_{\{v_2\}} \cdots \stackrel{n}{\rightarrow}_{\{v_k\}} t_k$$

where $t_k \triangleq$ true $=^?_\ell$ true, be given. We prove that if there exists a derivation $\mu t \stackrel{n}{\rightarrow} t_k$, then there exists a derivation $\sigma t \stackrel{n}{\rightarrow} t_k$ such that σ is a normalized substitution. We prove it by transfinite induction on the lexicographic ordering on the pair of the length of the derivation and the number of $\rightarrow_{\mathcal{U}(\mathcal{R})}$-redexes in the substitution of the starting term. The result holds trivially for the base case.

Suppose $A : \mu t \stackrel{n}{\rightarrow}_{[u, l\rightarrow r \Leftarrow Q]} t' \stackrel{n}{\twoheadrightarrow} t_k$ and the result holds for the derivation less than (w.r.t. the lexicographic ordering) A. We distinguish the following cases.

(1) There exists a position $v \preceq u$ such that $(x \triangleq) t|_v \in \mathcal{D}(\mu)$.
Let $\mu' = \mu - \{x \mapsto \mu x\} \cup \{x \mapsto (\mu x)'\}$ where $\mu x \stackrel{n}{\rightarrow} (\mu x)'$. We have $\mu' t \stackrel{n}{\twoheadrightarrow} t_k$ whose length is less than $|A|$ by Lemma .1. By the induction hypothesis there exists a derivation $\sigma t \stackrel{n}{\twoheadrightarrow} t_k$ such that σ is normalized.

(2) $u \in \overline{\mathcal{O}}(t)$.

(2-1) For every v such that $u \prec v$, if $t|_v \in \mathcal{D}(\mu)$, then $v \notin \mathrm{Pattern}(t, u)$.
The derivation is rewritten as $\mu t \stackrel{n}{\rightarrow} \mu t[r']_u \stackrel{n}{\twoheadrightarrow} t_k$, where r' is a variant of r. By the induction hypothesis there exists a derivation $\sigma t[r']_u \stackrel{n}{\twoheadrightarrow} t_k$ such that σ is normalized. Hence $\sigma t \stackrel{n}{\twoheadrightarrow} t_k$.

(2-2) There exists v such that $u \prec v$, $(x \triangleq) t|_v \in \mathcal{D}(\mu)$ and $v \in \mathrm{Pattern}(t, u)$.
Let U_x be a set of positions of x in t, and Red_v be a set of $\rightarrow_{\mathcal{U}(\mathcal{R})}$-redex positions below $(\succ)$ v. For every $v \in U_x$ and for every $w \in \mathrm{Red}_v$ we check whether the descendants of w are contracted or not. We distinguish the following two cases.

1. If a descendant of w gets contracted by becoming a $\stackrel{n}{\rightarrow}$-redex position, we contract w beforehand. This gives rise to the situation considered in case (1). Hence the result holds.

2. If no descendant of w is contracted, we introduce a fresh variable z and define $\mu' = \mu - \{x \mapsto \mu x\} \cup \{x \mapsto \mu x[z]_{w/v}\}$.[14] This gives a new term $\mu' t$ where the number of the redexes in μ' is one less than that of the redexes in μ. The normalization of $\mu' t$ is not affected since w does not contribute to the reduction. By the induction hypothesis there exists a derivation $\sigma t \stackrel{n}{\twoheadrightarrow} t_k$ such that σ is normalized.

Note that no descendant of w will become an external position during the derivation because of the orthogonality of the CTRS, and hence the above two cases are exhaustive.∎

[14] A position w' such that $vw' = w$ is denoted by w/v.