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Abstract. This paper addresses an algorithm for solving a linear program with an additional rank-two reverse convex constraint. Unlike the existing methods which generate an approximately optimal solution, the algorithm provides a globally optimal solution of the nonconvex problem by a finite number of dual pivot operations. Computational results indicate that this algorithm can solve fairly large scale problems efficiently.

Key words: Global optimization, reverse convex program, rank-two quasiconcave function, parametric simplex algorithm.

1. Introduction

In this paper, we describe a practical method for solving a special class of reverse convex programming problems [3, 4, 18]:

$$\text{maximize}\{\, c^T x \mid x \in X \cap Y \,\}, \tag{1.1}$$

where $c \in R^n$, $X \subseteq R^n$ is a polytope, and $Y \subseteq R^n$ is defined by a rank-two quasiconcave function $f : R^n \to R^1$ as follows:

$$Y = \{\, x \in R^n \mid f(x) \leq 0 \,\}. \tag{1.2}$$

Since Y is the complement of a convex set $\{\, x \in R^n \mid f(x) > 0 \,\}$, the feasible region might be neither convex nor connected. Hence (1.1) can have multiple local maxima in $X \cap Y$. The detailed definition of rank-two property will be given in Section 2 (see also [14, 17]).

A typical example of problem (1.1) is a linear program with an additional linear multiplicative constraint [11, 17, 19]:

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$$\text{maximize}\{ c^T x \mid x \in X,\ (d_1{}^T x + d_{10})(d_2{}^T x + d_{20}) - d_{00} \leq 0 \}, \tag{1.3}$$

where $d_i \in R^n$, $i = 1, 2$, and $d_{i0} \in R^1$, $i = 0, 1, 2$. The product of two affine functions appears in many applications such as microeconomics [2], bond portfolio optimization [6] and geometrical optimization [8, 10] and so forth (see [13]). In [11, 19], we proposed a branch-and-bound method for obtaining an ϵ-optimal solution. We reduced (1.3) to a minimization of a univariate function, the value of which can be computed by solving an ordinary convex program. We extended this idea and solved more general class of problems than (1.3) in [12]. In [17], Thach et. al. converted (1.3) into a two-dimensional concave minimization problem and applied an outer approximation method.

In their recent article [14], Pferschy and Tuy proposed a fairy efficient algorithm for obtaining an ϵ-optimal solution of (1.1). The algorithm consists of two parts: In the first part, a local maximum x' is searched by using a procedure similar to the usual simplex algorithm. In the second part, the ϵ-optimality of x' is checked by minimizing the function f. Since f has rank-two property such as the product of two affine functions, one can minimize f on a polytope very efficiently by using the parametric methods proposed in [7, 9]. If x' is not a global maximum, it is discarded by adding the cutting plane $c^T x \geq c^T x' + \epsilon$, where ϵ is a positive tolerance. In this paper, we will propose a parametric dual simplex algorithm for solving (1.1). Our algorithm contrasts with the method by Pferschy and Tuy in two points: using no cutting planes and yielding a globally optimal solution.

The organization of the paper is as follows: In section 2, we parametrize (1.1) by introducing two auxiliary variables and define an equivalent master problem. We also show that an optimal solution x^* exists among the intersection points of the boundaries of X and Y. To find such an intersection point, we apply a parametric dual simplex procedure to a linear program associated with (1.1) in Section 3. Section 4 is devoted to the algorithm for searching an optimal solution x^*. We show that x^* can be obtained after applying finitely many dual simplex pivots to the master problem. Results of computational experiments of our algorithm are presented in Section 5.

2. Parametrization of the Problem

The nonconvex problem we consider in this paper is

$$\text{(P)}\left|\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax = b,\ x \geq 0, \\ & f(x) \leq 0, \end{array}\right. \tag{2.1}$$

where $A \in R^{m \times n}$, $b \in R^m$ and $c \in R^n$. We assume that the set:

$$X = \{ x \in R^n \mid Ax = b,\ x \geq 0 \} \tag{2.2}$$

is nonempty and bounded. The function $f : R^n \to R^1$ is continuous and quasiconcave on X. It also possesses *rank-two property* on X [14]. Namely, there exist two linearly independent vectors d_1, $d_2 \in R^n$ such that

$$y \in R^n,\ {d_i}^T y \geq 0,\ i = 1, 2 \Longrightarrow f(x + y) \geq f(x),\ \forall x \in X. \tag{2.3}$$

Let

$$Y = \{\, x \in R^n \mid f(x) \leq 0 \,\}, \tag{2.4}$$

then the feasible region of problem (P) is denoted by $X \cap Y$.

If we remove the last constraint $f(x) \leq 0$ from (P), we have an ordinary linear programming problem:

$$(\bar{\mathrm{P}}) : \mathrm{maximize}\{\, c^T x \mid x \in X \,\}, \tag{2.5}$$

which has an optimal solution $\bar{x}$ since X is nonempty and bounded. If $\bar{x} \in Y$, then $\bar{x}$ is a globally optimal solution of (P). To exclude this trivial case, we assume throughout the paper that

$$\max\{\, c^T x \mid x \in X \,\} > \max\{\, c^T x \mid x \in X \cap Y \,\}. \tag{2.6}$$

The following is a well-known result on reverse convex programming (see e.g. [18]):

Lemma 2.1. *If $X \cap Y \neq \emptyset$, among boundary points of Y exists a globally optimal solution x^* of (P).* □

In our problem, the linearity of the objective function strengthens this lemma. When X is of one-dimension, the problem can be easily solved. Then we assume in the sequel that $\dim X \geq 2$. We denote the set of relative interior points of X by $\mathrm{int}\, X$ and the set of boundary points of X by ∂X, i.e., $\partial X = X \setminus \mathrm{int}\, X$. For Y, ∂Y denotes the set of boundary points in the usual topological sense.

Lemma 2.2. *$\partial X \cap \partial Y \neq \emptyset$ if $X \cap Y \neq \emptyset$.*

Proof: Let us denote by Y^c the complement of Y. Then by the assumption (2.6) and the linearity of the objective function, $\partial X \cap Y^c$ contains an optimal solution $\bar{x}$ of $(\bar{\mathrm{P}})$ and hence is nonempty. On the other hand, $\partial X \cap Y$ is also nonempty if $X \cap Y \neq \emptyset$. In fact, $\partial X \subseteq Y^c$ would imply

$$X = \mathrm{co}\, \partial X \subseteq \mathrm{co}\, Y^c = Y^c,$$

where $\mathrm{co} \cdot$ denotes the convex hull. Thus ∂Y intersects ∂X since $\dim X \geq 2$ and hence ∂X is connected. □

Theorem 2.3. *If $X \cap Y \neq \emptyset$, then there exists a globally optimal solution x^* of (P) in $\partial X \cap \partial Y$.*

Proof: Let x^* be a globally optimal solution of (P) and suppose $x^* \in \operatorname{int} X \cap \partial Y$. Choose an arbitrary point, say x^1 from $\partial X \cap \partial Y$ and let δ be a sufficiently small positive number such that $x^2 = x^* + \delta(x^* - x^1)$ lies in X. We shall show that $x^2 \in Y$: Let us denote by W the closure of the convex set Y^c and assume the contrary. Then we have

$$x^* = \frac{\delta}{1+\delta}x^1 + \frac{1}{1+\delta}x^2,\ x^1 \in \partial Y \subseteq W \text{ and } x^2 \in Y^c = \operatorname{int} W,$$

where the last equality follows from the convexity of Y^c. Using the accessibility lemma (Stoer and Witzgall [16] (3.2.11)), we see that $x^* \in \operatorname{int} W = Y^c$. This is a contradiction. Then by the linearity of the objective function, the equality $c^T x^1 = c^T x^2 = c^T x^*$ holds, which implies that $x^1 \in \partial X \cap \partial Y$ is also globally optimal to (P). □

Remark. Under the same condition as Theorem 2.3, we see that a globally optimal solution x^* of (P) lies in the intersection of an edge of X and the boundary ∂Y of Y. If $x^* \in \operatorname{int} F \cap \partial Y$ for some two or higher dimensional face F of X, then we can show that some $x^1 \in \partial F \cap \partial Y$ is also a globally optimal solution in exactly the same way as in the proof of Theorem 2.3. Repeating this argument if necessary, we will obtain the desired solution. □

The following lemma furnishes an insight into the rank-two property:

Lemma 2.4. *Suppose that $f : R^n \to R^1$ is continuous and quasiconcave, and has rank-two property on X with respect to two vectors d_1 and d_2. Then there exists a function $g : R^2 \to R^1$ which is continuous and quasiconcave on $\mathrm{Z} = \{ (d_1{}^T x, d_2{}^T x) \mid x \in X \}$, and satisfies*

$$f(x) = g(d_1{}^T x, d_2{}^T x),\ \forall x \in X, \tag{2.7}$$

$$\eta \in R^2,\ \eta \geq 0 \Longrightarrow g(\zeta + \eta) \geq g(\zeta),\ \forall \zeta \in \mathrm{Z}. \tag{2.8}$$

Proof: If f is not expressed as (2.7), for some x^1, $x^2 \in X$ we have

$$d_i{}^T x^1 = d_i{}^T x^2,\ i = 1, 2;\ \ f(x^1) < f(x^2).$$

However, it follows from (2.3) that

$$f(x^1) < f(x^2) \Longrightarrow \exists i,\ d_i{}^T x^1 < d_i{}^T x^2,$$

which is a contradiction. It is easy to see that (2.8) holds.

Let us show that g is quasiconcave on Z: Choose arbitrary ζ^1, $\zeta^2 \in \mathrm{Z}$. Then we have $\zeta^1 = (d_1{}^T x^3, d_2{}^T x^3)$, $\zeta^2 = (d_1{}^T x^4, d_2{}^T x^4)$ for some x^3, $x^4 \in X$, and

$$\begin{aligned} g((1-\lambda)\zeta^1 + \lambda\zeta^2) &= f((1-\lambda)x^3 + \lambda x^4) \\ &\geq \min\{ f(x^3),\ f(x^4) \} = \min\{ g(\zeta^1),\ g(\zeta^2) \} \end{aligned}$$

for any $\lambda \in [0, 1]$. The continuity of g is obvious. □

Therefore (P) can be reformulated as follows by the function $g : R^2 \rightarrow R^1$:

$$(\mathrm{P}') \left| \begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & x \in X, \\ & g({d_1}^T x, {d_2}^T x) \leq 0. \end{array} \right. \qquad (2.9)$$

Introducing two auxiliary variables ζ_1, ζ_2, we can transform (P$'$) into an equivalent master problem:

$$(\mathrm{MP}) \left| \begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & x \in X, \\ & {d_1}^T x = \zeta_1, \quad {d_2}^T x = \zeta_2, \\ & g(\zeta) \leq 0, \end{array} \right. \qquad (2.10)$$

where $\zeta = (\zeta_1, \zeta_2)$. The following theorem can be readily obtained:

Theorem 2.5. *If (x^*, ζ^*) is an optimal solution of (MP), then x^* solves (P).* □

Let us denote

$$\mathrm{H} = \{\, \zeta \in R^2 \mid g(\zeta) \leq 0 \,\}. \qquad (2.11)$$

Lemma 2.6. *$x \in \partial Y$ if and only if $\zeta = ({d_1}^T x, {d_2}^T x) \in \partial \mathrm{H}$.*

Proof: We first show the 'only if' part: Note that $\zeta = ({d_1}^T x, {d_2}^T x)$ lies in H. Given an arbitrary positive δ, let $\delta' = \delta / \|(d_1, d_2)\|$. Since $x \in \partial Y$, there is a point $x' \in B_{\delta'}(x) \cap Y^c$, where $B_{\delta'}(x)$ is the δ'-neighborhood of x. Let $\zeta' = ({d_1}^T x', {d_2}^T x')$. Then $\zeta' \in \mathrm{H}^c$ and

$$\begin{aligned} \|\zeta' - \zeta\| &= (({d_1}^T (x' - x))^2 + ({d_2}^T (x' - x))^2)^{1/2} \\ &\leq \|(d_1, d_2)\| \|x' - x\| < \|(d_1, d_2)\| \delta' = \delta. \end{aligned}$$

Thus $B_\delta(\zeta) \cap \mathrm{H}^c \neq \emptyset$ and hence $\zeta \in \partial \mathrm{H}$.

To show the 'if' part let us consider the $2 \times n$ matrix D of rows ${d_1}^T$ and ${d_2}^T$. Since these vectors are linearly independent, we assume without loss of generality that the 2×2 submatrix $\tilde{D}$ of the first two columns is nonsingular. Given an arbitrary positive number δ, let $\delta' = \delta / \|\tilde{D}^{-1}\|$ and consider the δ'-neighborhood of ζ. Since $\zeta \in \partial \mathrm{H}$, we can find a point ζ' of $B_{\delta'}(\zeta) \cap \mathrm{H}^c$. Let

$$y = \begin{pmatrix} \tilde{D}^{-1}(\zeta' - \zeta) \\ 0 \end{pmatrix} \in R^n,$$

then $\|y\| \leq \|\tilde{D}^{-1}\| \|\zeta' - \zeta\| < \delta$ and also $D(x + y) = \zeta'$. Thus δ-neighborhood of x contains a point $x + y$ of Y^c. □

For a given $\zeta = (\zeta_1, \zeta_2)$ let us consider the following problem:

$$
(\mathrm{P}(\zeta)) \left| \begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & x \in X, \\ & d_1{}^T x = \zeta_1, \quad d_2{}^T x = \zeta_2 \end{array} \right. \tag{2.12}
$$

We refer to a ζ as an *active point* if $\zeta \in \partial\mathrm{H}$ and problem $(\mathrm{P}(\zeta))$ is feasible. Let us denote an optimal solution of $(\mathrm{P}(\zeta))$ by $x^*(\zeta)$, then the argument thus far is summarized into the following theorem:

Theorem 2.7. *The point $x^* = x^*(\zeta^*)$ which maximizes $c^T x^*(\zeta)$ over all active points ζ is a globally optimal solution of (P).* □

Thus problem (P) can be solved by solving $(\mathrm{P}(\zeta))$ as varying ζ over all active points. This could be done easily if the boundary $\partial\mathrm{H}$ is parametrized by one parameter, e.g., the implicit function $\zeta_2 = \phi(\zeta_1)$ is known for $\zeta = (\zeta_1, \zeta_2) \in \partial\mathrm{H}$. But such a favorable situation is not expected in general.

In the rest of this paper, we assume the following for the sake of simplicity:

Assumption 2.1. *No vertices of X are boundary points of Y, i.e.,*

$$
V(X) \cap \partial Y = \emptyset, \tag{2.13}
$$

where $V(\cdot)$ represents the set of vertices. □

Then we immediately see from Lemma 2.6 that ζ is not active if $x^*(\zeta)$ is a vertex of X.

3. How to Find an Active Point

In this section we will propose a method for finding an active point ζ, which will serve as a starting point of our algorithm.

For an interval I of real numbers we denote

$$
X(I) = \{\, x \mid x \in X,\ d_1{}^T x \in I \,\}.
$$

When $I = [v, v]$, i.e., a degenerate interval, we simply write $X(v)$. The following two parametric linear programs play an important role:

$$
(\mathrm{Q}_-(v)) \left| \begin{array}{ll} \text{minimize} & d_2{}^T x \\ \text{subject to} & x \in X(v), \end{array} \right. \tag{3.1}
$$

$$
(\mathrm{Q}_+(v)) \left| \begin{array}{ll} \text{maximize} & d_2{}^T x \\ \text{subject to} & x \in X(v). \end{array} \right. \tag{3.2}
$$

We impose here the dual nondegeneracy assumption:

Assumption 3.1. *Both of $(Q_-(v))$ and $(Q_+(v))$ have a unique optimal solution unless $X(v)$ is empty.* □

We denote by $\check{x}(v)$ and $\hat{x}(v)$ optimal solutions of $(\mathrm{Q}_-(v))$ and $(\mathrm{Q}_+(v))$, respectively, if they exist, and let

$$h_-(v) = {d_2}^T \check{x}(v), \quad h_+(v) = {d_2}^T \hat{x}(v). \tag{3.3}$$

Lemma 3.1. *An optimal solution $\check{x}(v)$ (resp. $\hat{x}(v)$) of $(Q_-(v))$(resp. $(Q_+(v))$) minimizes (resp. maximizes) $f(x)$ on $X(v)$.*

Proof: Since $h_-(v) \le {d_2}^T x$ for any $x \in X(v)$, from Lemma 2.4 we have

$$f(\check{x}(v)) = g({d_1}^T \check{x}(v), {d_2}^T \check{x}(v)) = g(v, h_-(v)) \le g({d_1}^T x, {d_2}^T x) = f(x).$$

We can show the assertion about $\hat{x}(v)$ similarly. □

Corollary 3.2. *If $g(v, h_-(v)) \le 0 < g(v, h_+(v))$, then there exists at least one active point $\zeta = (\zeta_1, \zeta_2)$ with $\zeta_1 = v$. If $g(v, h_-(v)) \le 0 = g(v, h_+(v))$, then either $\zeta = (v, h_+(v))$ is an active point or there are no active points with $\zeta_1 = v$. Otherwise, there are no active points with $\zeta_1 = v$.* □

For a given value $\tilde{v}$ of the parameter there can be an active point $\zeta = (\zeta_1, \zeta_2)$ with $\zeta_1 = \tilde{v}$ when

$$g(\tilde{v}, h_-(\tilde{v})) \le 0 \le g(\tilde{v}, h_+(\tilde{v})) \tag{3.4}$$

holds. If such an active point actually exists, we will find it in the course of solving $(\mathrm{Q}_+(\tilde{v}))$ starting from $\check{x}(\tilde{v})$ by the usual simplex algorithm. The procedure we will present receives a value $\tilde{v}$ such that $X(\tilde{v}) \neq \emptyset$ and yields an active point $\tilde{\zeta}$ with $\tilde{\zeta}_1 > \tilde{v}$. The first component $\tilde{\zeta}_1$ is the least value among all ζ's with $\zeta_1 > \tilde{v}$ which potentially provide a globally optimal solution of (P).

3.1. Role of the minimization problem $(\mathrm{Q}_-(v))$

Suppose that we are given a value $\tilde{v}$ satisfying

$$\textbf{case 1:} \quad X((\tilde{v}, \tilde{v} + \delta]) \cap Y = \emptyset \text{ for some } \delta > 0. \tag{3.5}$$

We solve problem $(\mathrm{Q}_-(v))$ for every v in the interval $[\tilde{v}, \infty)$ by using a parametric right-hand-side simplex algorithm. Then we will obtain a sequence of intervals $[v_0, v_1]$, $[v_1, v_2]$, ..., $[v_{p-1}, v_p]$, and the sequence of associated optimal bases B_0, B_1, ..., $B_{p-1} \in R^{(m+1)\times(m+1)}$, where $v_0 = \tilde{v}$ and $v_p = \max\{ {d_1}^T x \mid x \in X \}$. The following lemma shows that

$$\textbf{case 1.1:} \quad g(v_i, h_-(v_i)) > 0, \ \ i = 1, \ldots, p \tag{3.6}$$

implies that there are no feasible solutions of (P) satisfying ${d_1}^T x > \tilde{v}$.

Lemma 3.3. *$X((\tilde{v},\infty))\cap Y=\emptyset$ if and only if (3.6) holds.*

Proof: The 'only if' part is trivial. To show the 'if' part, let us assume that there is a feasible solution $x'\in X\cap Y$ such that ${d_1}^T x' > \tilde{v}$. Let $\zeta'=(\zeta_1',\zeta_2')=({d_1}^T x',{d_2}^T x')$, then $\zeta_1'>\tilde{v}$ and $\zeta_2'\geq h_-(\zeta_1')$, and consequently

$$g(\zeta_1',h_-(\zeta_1'))\leq g(\zeta_1',\zeta_2')=f(x')\leq 0.$$

Let $[v_k,v_{k+1}]$ be an interval containing ζ_1', then $\zeta_1'=\lambda v_k+(1-\lambda)v_{k+1}$ for some $\lambda\in[0,1]$. By definition h_- is linear on $[v_k,v_{k+1}]$, and by the quasiconcavity of g we have

$$g(\zeta_1',h_-(\zeta_1'))\geq \min\{\, g(v_k,h_-(v_k)),\ g(v_{k+1},h_-(v_{k+1}))\,\}.$$

When $k\geq 1$, the right-hand-side is positive, which leads to a contradiction. When $k=0$, replacing v_0 by $v_0+\delta'$ for an δ' such that $0<\delta'<\min\{\,\delta,\ {d_1}^T x'-\tilde{v}\,\}$, we again have a contradiction. □

Applying this lemma with $\tilde{v}=\min\{\,{d_1}^T x\mid x\in X\,\}$, we obtain the following corollary:

Corollary 3.4. *Let $\check{x}$ minimize ${d_2}^T x$ among the minimal solutions of ${d_1}^T x$ on X. Then $\check{x}\notin Y$ and (3.6) occurs if and only if (P) is infeasible.*

Proof: Obvious from Corollary 3.2 and Lemma 3.3. □

Unless (3.6) occurs, we will find an interval $[v_k,v_{k+1}]$ satisfying

$$\textbf{case 1.2:}\qquad g(v_i,h_-(v_i))>0,\ \ i=1,2,\ldots,k;\ \ g(v_{k+1},h_-(v_{k+1}))\leq 0. \tag{3.7}$$

Let $\tilde{\zeta}$ be an intersection of the line segment $(v_k,h_-(v_k))$-$(v_{k+1},h_-(v_{k+1}))$ and ∂H. If the intersection is not unique, we take the one with the smallest first component.

Lemma 3.5. *The point $\tilde{\zeta}$ obtained in case 1.2 has the least first component among active points ζ with $\zeta_1>\tilde{v}$.*

Proof: Assume that there is an active point, say $\zeta'=(\zeta_1',\zeta_2')$ such that $\tilde{v}<\zeta_1'<\tilde{\zeta}_1$. Then $g(\zeta_1',h_-(\zeta_1'))\leq g(\zeta')\leq 0$. By the quasiconcavity of g, we see that ζ_1' falls in the same interval as $\tilde{\zeta}_1$. This contradicts the choice of $\tilde{\zeta}$. □

3.2. Role of the maximization problem $(\mathrm{Q}_+(v))$

Left is the case where

$$\textbf{case 2:}\qquad X((\tilde{v},\tilde{v}+\delta])\cap Y\neq\emptyset \text{ for any } \delta>0. \tag{3.8}$$

As will be shown in the next section, the procedure is not applied to this case when an active point ζ with $\zeta_1=\tilde{v}$ is found. Then we assume that there are no active points ζ with $\zeta_1=\tilde{v}$. By corollary 3.2 we have

$$g(\tilde{v}, h_+(\tilde{v})) \leq 0. \tag{3.9}$$

We increase the parameter v from $\tilde{v}$ and solve ($\mathrm{Q}_+(v)$). Then a sequence of intervals $[v_0, v_1]$, $[v_1, v_2]$, ..., $[v_{p'-1}, v_{p'}]$ is generated as before.

Since we have assumed that the maximal solution of ($\mathrm{Q}_+(v)$) is unique for each v, the set $X(v) \cap \{ x \mid {d_2}^T x = h_+(v) \}$ consists of a single point $\hat{x}(v)$ for each v. Then for $v = (1-\lambda)v_k + \lambda v_{k+1}$ in the interval $[v_k, v_{k+1}]$ it holds that

$$x^*(v, h_+(v)) = (1-\lambda)x^*(v_k, h_+(v_k)) + \lambda x^*(v_{k+1}, h_+(v_{k+1})).$$

Consequently we have

$$c^T x^*(v, h_+(v)) \leq \max\{ c^T x^*(v_k, h_+(v_k)),\ c^T x^*(v_{k+1}, h_+(v_{k+1})) \}.$$

Therefore if both $x^*(v_k, h_+(v_k))$ and $x^*(v_{k+1}, h_+(v_{k+1}))$ belong to Y, i.e., $g(v_k, h_+(v_k)) \leq 0$ and $g(v_{k+1}, h_+(v_{k+1})) \leq 0$, then we can discard ζ's with $\zeta_1 \in (v_k, v_{k+1})$ without overlooking any points which potentially provide a globally optimal solution of (P). It might happen that

$$g(v_k, h_+(v_k)) \leq 0,\ g(v_{k+1}, h_+(v_{k+1})) \leq 0 \text{ and}$$
$$g(v, h_+(v)) > 0 \text{ for some } v \in [v_k, v_{k+1}],$$

and hence the line segment $(v_k, h_+(v_k))$-$(v_{k+1}, h_+(v_{k+1}))$ intersects $\partial\mathrm{H}$ at an active point. But such an active point need not be considered.

Thus in the above process two cases are possible:

$$\textbf{case 2.1:} \quad g(v_i, h_+(v_i)) \leq 0,\ \ i = 1, \ldots, p', \tag{3.10}$$

$$\textbf{case 2.2:} \quad g(v_i, h_+(v_i)) \leq 0,\ \ i = 1, \ldots, k;\ \ g(v_{k+1}, h_+(v_{k+1})) > 0. \tag{3.11}$$

If (3.11) occurs, then choose a point $\tilde{\zeta}$ with the least first component among intersection points of $(v_k, h_+(v_k))$-$(v_{k+1}, h_+(v_{k+1}))$ and $\partial\mathrm{H}$.

Lemma 3.6. *If (3.10) holds, no active points ζ such that $\zeta_1 > \tilde{v}$ provide a globally optimal solution of (P).*

Proof: Suppose there is an active point $\zeta' = (\zeta_1', \zeta_2')$ such that $\zeta_1' > \tilde{v}$. Let ζ_1' be in the interval $[v_i, v_{i+1}]$ for some $i \leq p'-1$. Then as discussed above $x^*(\zeta')$ does not provide a better objective function value than either $x^*(v_i, h_+(v_i))$ or $x^*(v_{i+1}, h_+(v_{i+1}))$. Since neither $(v_i, h_+(v_i))$ nor $(v_{i+1}, h_+(v_{i+1}))$ is active under Assumption 2.1, ζ' can be ignored by Theorem 2.7. □

Lemma 3.7. *The point $\tilde{\zeta}$ obtained in case 2.2 has the least first component among active points ζ with $\zeta_1 > \tilde{v}$ which potentially provide a globally optimal solution of (P).*

Proof: Suppose there is an active point $\zeta' = (\zeta_1', \zeta_2')$ such that $\zeta_1' \in [v_i, v_{i+1}] \cap (v_0, \tilde{\zeta}_1)$. When $0 \leq i < k$, we see by the same argument as in the proof of the previous lemma that ζ' does not provide a globally optimal solution. When $i = k$, both ζ_1' and $\tilde{\zeta}_1$ lie in the same interval $[v_k, v_{k+1}]$. This contradicts the choice of $\tilde{\zeta}$. □

3.3. Procedure for finding an active point

We are now ready to present the procedure for finding an active point $\tilde{\zeta}$ with $\tilde{\zeta}_1 > \tilde{v}$ for a given $\tilde{v}$ such that $X(\tilde{v}) \neq \emptyset$:

Procedure ACT($\tilde{v}$).

Case 1. If (3.5) holds, then do the following:

1° Solve ($\mathrm{Q}_-(v)$) parametrically as increasing the value v from $\tilde{v}$ and generate a sequence of intervals $[v_0, v_1]$, $[v_1, v_2]$, ..., $[v_{p-1}, v_p]$, where $v_0 = \tilde{v}$ and $v_p = \max\{ d_1{}^T x \mid x \in X \}$.

2° Find an interval $[v_k, v_{k+1}]$ satisfying

$$g(v_i, h_-(v_i)) > 0, \quad i = 1, 2, \ldots, k; \quad g(v_{k+1}, h_-(v_{k+1})) \leq 0. \tag{3.12}$$

If such an interval is not found, then stop.

3° Compute a point $\tilde{\zeta}$ with the least first component among intersection points of $(v_k, h_-(v_k))$-$(v_{k+1}, h_-(v_{k+1}))$ and ∂H.

Case 2. If (3.8) holds, then do the following:

1° Solve ($\mathrm{Q}_+(v)$) parametrically as increasing the value v from $\tilde{v}$ and generate a sequence of intervals $[v_0, v_1]$, $[v_1, v_2]$, ..., $[v_{p'-1}, v_{p'}]$, where $v_0 = \tilde{v}$ and $v_p = \max\{ d_1{}^T x \mid x \in X \}$.

2° Find an interval $[v_k, v_{k+1}]$ satisfying

$$g(v_i, h_+(v_i)) \leq 0, \quad i = 1, \ldots, k; \quad g(v_{k+1}, h_+(v_{k+1})) > 0. \tag{3.13}$$

If such an interval is not found, then stop.

3° Compute a point $\tilde{\zeta}$ with the least first component among intersection points of $(v_k, h_+(v_k))$-$(v_{k+1}, h_+(v_{k+1}))$ and ∂H. □

Under the dual nondegeneracy assumption, the number of pivot operations required by the parametric right-hand-side simplex algorithm is finite (see e.g. [1]). Hence the above procedure provides an active point $\tilde{\zeta}$ in finite time if it exists. The associated problem (P($\tilde{\zeta}$)) has a unique optimal solution $x^*(\tilde{\zeta})$ on some edge of X.

Remark. In both the cases of procedure ACT, the intervals $[v_{i-1}, v_i]$, $i = 1, 2, \ldots$, are successively generated. If an interval $[v_k, v_{k+1}]$ satisfies (3.12) (or (3.13)), then we can immediately terminate this process even though the value of v does not reach $\max\{ d_1{}^T x \mid x \in X \}$. □

4. How to Find an Optimal Solution

Suppose we have an active point ζ^0 such that $x^*(\zeta^0)$ lies on some edge of the polyhedron X. Such an active point can be obtained by using the procedure developed in the previous section.

Let us recall the constraint of $(\mathrm{P}(\zeta^0))$:

$$\tilde{A}x = \tilde{b} - e^1\zeta_1^0 - e^2\zeta_2^0,\ x \geq 0, \tag{4.1}$$

where $e^i \in R^{m+2}$ represents the $m+i$th unit column vector for $i = 1, 2$, $\tilde{b}^T = (b^T, 0, 0)$ and

$$\tilde{A} = \begin{pmatrix} A \\ {d_1}^T \\ {d_2}^T \end{pmatrix}.$$

Let $\tilde{B}_0 \in R^{(m+2)\times(m+2)}$ be an optimal basis of $(\mathrm{P}(\zeta^0))$. The optimal dictionary associated with $\tilde{B}_0$ is defined below:

$$\left| \begin{array}{l} x_B = \bar{b} - \bar{e}^1\zeta_1^0 - \bar{e}^2\zeta_2^0 - \bar{N}_0 x_N \\ z = c_B^T(\bar{b} - \bar{e}^1\zeta_1^0 - \bar{e}^2\zeta_2^0) + \bar{c}_N^T x_N, \end{array} \right. \tag{4.2}$$

where

$$\begin{array}{l} [\tilde{B}_0, \tilde{N}_0] = \tilde{A},\quad \bar{b} = \tilde{B}_0^{-1}\tilde{b},\quad \bar{e}^i = \tilde{B}_0^{-1}e^i,\quad i = 1, 2, \\ \bar{N}_0 = \tilde{B}_0^{-1}\tilde{N}_0,\quad \bar{c}_N^T = (c_N^T - c_B^T\bar{N}_0), \end{array} \tag{4.3}$$

and the indices B and N represent the basic and the nonbasic parts, respectively. Since $x^*(\zeta^0)$ lies on an edge of X, the dictionary (4.2) is degenerate, i.e., some components of $\bar{b} - \bar{e}^1\zeta_1^0 - \bar{e}^2\zeta_2^0$ are equal to zero.

Remark. A dictionary associated with an optimal basis B_k of $(\mathrm{Q}_-(\zeta^0))$ (or $(\mathrm{Q}_+(\zeta^0))$) generates (4.2) very efficiently. In the dictionary of $(\mathrm{Q}_-(\zeta^0))$ the last row is as follows:

$$h_-(\eta_1^0) = {d_{2B}}^T B_k^{-1}(\hat{b} - e^1\zeta_1^0) + ({d_{2N}}^T - {d_{2B}}^T B_k^{-1}N_k), \tag{4.4}$$

where $e^1 \in R^{m+1}$ is the $m+1$st unit column vector, $({d_{2B}}^T, {d_{2N}}^T)$ is the partition of ${d_2}^T$ and $\hat{b}^T = (b^T, 0)$. Let us introduce an artificial variable x_{n+1} (≥ 0) and replace (4.4) by

$$x_{n+1} = h_-(\eta_1^0) - {d_{2B}}^T B_k^{-1}(\hat{b} - e^1\zeta_1^0) - ({d_{2N}}^T - {d_{2B}}^T B_k^{-1}N_k). \tag{4.5}$$

Also add the objective function row to this system of linear equations. If we apply a single dual pivot at the $m+2$nd row (4.5), then we have a feasible dictionary of $(\mathrm{P}(\zeta^0))$. This dictionary is also optimal under the dual nondegeneracy assumption imposed on both $(\mathrm{Q}_-(v))$ and $(\mathrm{Q}_+(v))$, since the set $X(\zeta_1^0) \cap \{\, x \mid {d_2}^T x = \zeta_2^0 \,\}$ which coincides with the feasible region of $(\mathrm{P}(\zeta^0))$ is a singleton. □

It is well known (see e.g. [1]) that the basis $\tilde{B}_0$ is optimal to (P(ζ)) for all ζ satisfying

$$\bar{b} - \bar{e}^1\zeta_1 - \bar{e}^2\zeta_2 \geq 0. \tag{4.6}$$

Hence the maximal value of z while $\tilde{B}_0$ remains optimal can be computed by solving a two-dimensional linear programming problem:

$$(\bar{\mathrm{P}}_0) \left| \begin{array}{ll} \text{maximize} & -c_B^T\bar{e}^1\zeta_1 - c_B^T\bar{e}^2\zeta_2 \\ \text{subject to} & \bar{e}^1\zeta_1 + \bar{e}^2\zeta_2 \leq \bar{b}. \end{array} \right. \tag{4.7}$$

To find an optimal solution of (P), however, we need the additional constraint $g(\zeta) \leq 0$:

$$(\mathrm{P}_0) \left| \begin{array}{ll} \text{maximize} & -c_B^T\bar{e}^1\zeta_1 - c_B^T\bar{e}^2\zeta_2 \\ \text{subject to} & \bar{e}^1\zeta_1 + \bar{e}^2\zeta_2 \leq \bar{b}, \\ & g(\zeta) \leq 0. \end{array} \right. \tag{4.8}$$

Let

$$\mathrm{Z}_0 = \{\, \zeta \in R^2 \mid \bar{e}^1\zeta_1 + \bar{e}^2\zeta_2 \leq \bar{b} \,\}. \tag{4.9}$$

Note that Z_0 is a bounded polyhedron, since for any $\zeta \in \mathrm{Z}_0$ we have

$$\min\{\, {d_i}^T x \mid x \in X\} \leq \zeta_i \leq \max\{\, {d_i}^T x \mid x \in X \,\}, \quad i = 1, 2. \tag{4.10}$$

Thus (P_0) is of the same form as (P), and also has a feasible solution ζ^0. If ζ is a vertex of Z_0, then $x^*(\zeta)$ is a vertex of X. Hence Assumption 2.1 implies that no vertex of Z_0 lies in ∂H.

Let $\bar{\zeta}$ be an optimal solution of ($\bar{\mathrm{P}}_0$). Then two cases can occur:

(i) $g(\bar{\zeta}) \leq 0$: The point $\bar{\zeta}$ solves (P_0) but does not lie in ∂H. Then it can be ignored by Theorem 2.7.

(ii) $g(\bar{\zeta}) > 0$: Applying Theorem 2.3 to (P_0) yields the existence of an optimal solution of (P_0) in $\partial\mathrm{Z}_0 \cap \partial$H.

In both cases, it is sufficient to search the intersection points of $\partial\mathrm{Z}_0$ and ∂H.

4.1. Solution method for the two-dimensional problem

Since Z_0 is a polytope in the plane defined by $m+2$ linear inequalities, we can enumerate all vertices of Z_0 in $O(m \log m)$ time by using a technique of computational geometry (see e.g. [15]).

Owing to the degeneracy of the dictionary (4.2), the active point ζ^0 lies on the boundary of Z_0. Let $w^1, \ldots, w^q, w^{q+1}(= w^1)$ be the vertices ordered counterclockwise from ζ^0. In general there can be more than two intersection points of $\partial\mathrm{Z}_0$ and ∂H, but we choose the first one as ζ^1. Namely, we find k such that

$$g(w^i) \leq 0, \quad i = 1, \ldots, k; \quad g(w^{k+1}) > 0 \tag{4.11}$$

and let ζ^1 be an intersection point of the edge w^k-w^{k+1} and $\partial\mathrm{H}$ (see Figure 4.2).

Lemma 4.1. *When (4.11) occurs, the edge w^k-w^{k+1} has a unique intersection point with ∂H.*

Proof: Assume that w^k-w^{k+1} intersects $\partial\mathrm{H}$ at more than one point. Let ζ^1 and ζ' be two distinct points of them and suppose ζ' is a convex combination of ζ^1 and w^{k+1}. Let Θ denote the closure of the open convex set H^c, then

$$w^{k+1} \in \mathrm{H}^c = \operatorname{int}\Theta \text{ and } \zeta^1 \in \partial\mathrm{H} \subseteq \Theta.$$

By the accessibility lemma [16] we obtain $\zeta' \in \operatorname{int}\Theta = \mathrm{H}^c$. This is contrary to the choice of ζ'. □

As shown in Section 2, the slope of the tangent to $\partial\mathrm{H}$ is always nonpositive, and hence we have $\zeta_1^0 \leq \zeta_1^1$ and $\zeta_2^0 \geq \zeta_2^1$. By Assumption 2.1, we see that

$$\zeta^0 \neq \zeta^1. \tag{4.12}$$

If we replace ζ^0 by ζ^1 in the dictionary (4.2), then for the i_kth row corresponding to the edge w^k-w^{k+1} we have

$$\bar{b}_{i_k} - \bar{e}^1_{i_k}\zeta^1_1 - \bar{e}^2_{i_k}\zeta^1_2 = 0. \tag{4.13}$$

Choosing some nonbasic column as an incoming basic vector and carrying out a single dual pivot, we obtain an alternative basis matrix $\tilde{B}^1$, which is optimal to $(\mathrm{P}(\zeta^1))$. If we cannot find such a column, i.e., every element of the i_kth row of $\bar{N}_0$ is nonnegative:

$$e^{i_k}\bar{N}_0 \geq 0, \tag{4.14}$$

then for some $\delta > 0$ there is no active point ζ such that $\zeta_1 \in (\zeta_1^1, \zeta_1^1 + \delta]$. To find a new active point we apply procedure $\mathrm{ACT}(\zeta_1^1)$ in Section 3.

In this way, starting from an active point ζ^0 and the associated polytope Z_0, a sequence ζ^0, Z_0, ζ^1, Z_1, ζ^2, $\ldots$, ζ^{r-1}, Z_{r-1}, ζ^r is generated until (4.14) holds. Some polytope may appear more than once in the sequence but all ζ^j's are distinct. Furthermore, the union $\bigcup_{i=0}^{r-1} \mathrm{Z}_i$ covers the boundary $\partial\mathrm{H}$ between ζ^0 and ζ^r. Hence ζ^j with $c^T x^*(\zeta^j) = \max\{\, c^T x^*(\zeta^i) \mid i = 0, 1, \ldots, r \,\}$ serves as the best incumbent among ζ's between ζ^0 and ζ^r. Note that we also have a sequence of associated bases $\tilde{B}_0, \tilde{B}_1, \ldots, \tilde{B}_{r-1}$ such that $\tilde{B}_i$ is optimal to both $(\mathrm{P}(\zeta^i))$ and $(\mathrm{P}(\zeta^{i+1}))$.

4.2. Algorithm for finding an optimal solution of (P)

The algorithm for obtaining a globally optimal solution x^* of (P) is summarized below:

Algorithm OPT.

Phase 1. Find an initial active point ζ^0 as follows:

(i) Let $\tilde{v} = \min\{ d_1{}^T x \mid x \in X\}$. Solve $(\mathrm{Q}_+(\tilde{v}))$ starting from $\tilde{x}(\tilde{v})$ and search for an active point $\zeta^0 = (\zeta_1^0, \zeta_2^0)$ with $\zeta_1^0 = \tilde{v}$. If ζ^0 is found, then go to Phase 2.

(ii) Call procedure ACT$(\tilde{v})$ for obtaining an active point $\zeta^0 = (\zeta_1^0, \zeta_2^0)$ with $\zeta_1^0 > \tilde{v}$. If ζ^0 is not found, then terminate.

Phase 2. Obtain an optimal basis $\tilde{B}_0$ of $(\mathrm{P}(\zeta^0))$. Let $x^* = x^*(\zeta^0)$ and $j = 0$. Update the incumbent x^* as follows:

1° Construct a polytope $\mathrm{Z}_j \subset R^2$ associated with $\tilde{B}_j$ according to (4.2), (4.3) and (4.9). Generate the sequence of vertices $w^1, \ldots, w^q, w^{q+1}(= w^1)$ of Z_j counterclockwise from ζ^j. Find an edge w^k-w^{k+1} of Z_j such that

$$g(w^i) \leq 0, \; i = 1, \ldots, k; \;\; g(w^{k+1}) > 0.$$

Let ζ^{j+1} be an intersection point of w^k-w^{k+1} and $\partial\mathrm{H}$. If $c^T x^*(\zeta^{j+1}) > c^T x^*$, then let $x^* = x^*(\zeta^{j+1})$.

2° Let i_k be the index of the inequality defining the edge w^k-w^{k+1}. If

$$e^{i_k} \tilde{B}_j^{-1} \tilde{N}_j \geq 0,$$

where $\tilde{N}_j$ represents the nonbasic columns of $\tilde{A}$, then do the following:

(i) Call procedure ACT(ζ_1^{j+1}) for obtaining an active point $\zeta^{j+2} = (\zeta_1^{j+2}, \zeta_2^{j+2})$ with $\zeta_1^{j+2} > \zeta_1^{j+1}$. If ζ^{j+2} is not found, then terminate.

(ii) Obtain an optimal basis $\tilde{B}_{j+2}$ of $(\mathrm{P}(\zeta^{j+2}))$. If $c^T x^*(\zeta^{j+2}) > c^T x^*$, then let $x^* = x^*(\zeta^{j+2})$. Let $j = j + 2$ and go to 1°.

3° Obtain an alternative optimal basis $\tilde{B}_{j+1}$ of $(\mathrm{P}(\zeta^{j+1}))$ by performing a dual simplex pivot at the i_kth row of the optimal dictionary associated with $\tilde{B}_j$. Let $j = j + 1$ and go to 1°. □

Theorem 4.2. *Under Assumption 2.1 algorithm OPT terminates after finitely many iterations and provides a globally optimal solution of (P) if it exists.*

Proof: When ζ^0 is found in Phase 1, it is the most left active point among those that potentially provide a globally optimal solution by either Lemma 3.5 or 3.7. When ζ^0 is not found, case 1.1 of Section 3 occurs and (P) is infeasible by Corollary 3.4.

Suppose OPT generates a sequence $\mathrm{Z}_0, \mathrm{Z}_1, \ldots, \mathrm{Z}_{r-1}$ of two-dimensional polytopes and active points $\zeta^0, \zeta^1, \ldots, \zeta^r$ before ACT is called again. Note that the union of Z_j's covers the boundary $\partial\mathrm{H}$ between ζ^0 and ζ^r. Therefore when ζ^r is found, we keep a solution as x^* which is best among solutions $x^*(\zeta)$ for ζ with $\zeta_1 \leq \zeta_1^r$. When ACT is called again, case 1 of Section 3 occurs and the most left active point ζ^{r+1} on the right of ζ^r is provided if it exists. No active points are overlooked. Note that

- there is only a finite number of Z_j's,
- each active point to be generated is an intersection of an edge w^k-w^{k+1} of some Z_j and ∂H such that $g(w^k) \leq 0$ and $g(w^{k+1}) > 0$, and
- such an edge has a unique intersection with ∂H by Lemma 4.1.

Thus OPT terminates after a finite number of iterations and provides a globally optimal solution of (P). □

Remarks. In both the procedures ACT and OPT, we need not know the function g explicitly. The value $g(w^i)$ computed in step 1° of OPT is equal to that of f at an optimal solution $x^*(w^i)$ of $(P(w^i))$. Since the optimal basis B^j of $(P(w^i))$ is common for every w^i, the computational burden for obtaining $x^*(w^i)$'s is only a little.

In step 1° of OPT, we might miss some ζ' on the edge w^i-w^{i+1} such that

$$g(w^i) \leq 0, \ g(w^{i+1}) \leq 0 \text{ and } g(\zeta') = 0.$$

However, we can see by the same reason as case 2 of Section 3 that such a ζ' cannot be optimal to (P_0). Hence $x^*(\zeta')$ is not an optimal solution of (P). □

4.3. Numerical example

Before concluding this section, let us illustrate algorithm OPT by using a problem with three variables:

$$\left|\begin{array}{ll} \text{maximize} & x_3 \\ \text{subject to} & x_1 + 2x_2 + x_3 \leq 6, \\ & 8x_1 + 4x_2 + 5x_3 \leq 30, \\ & -26x_1 - 8x_2 + 18x_3 \leq 9, \\ & x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0, \\ & (3x_1 - x_2 + 3)(-x_1 + 3x_2 + 4) - 18 \leq 0. \end{array}\right. \tag{4.15}$$

We see from Figure 4.1 that

$$3x_1 - x_2 + 3 \geq 0; \quad -x_1 + 3x_2 + 4 \geq 0 \tag{4.16}$$

for all $x \in X$. Hence the product of two affine function:

$$f(x) = (3x_1 - x_2 + 3)(-x_1 + 3x_2 + 4) - 18 \tag{4.17}$$

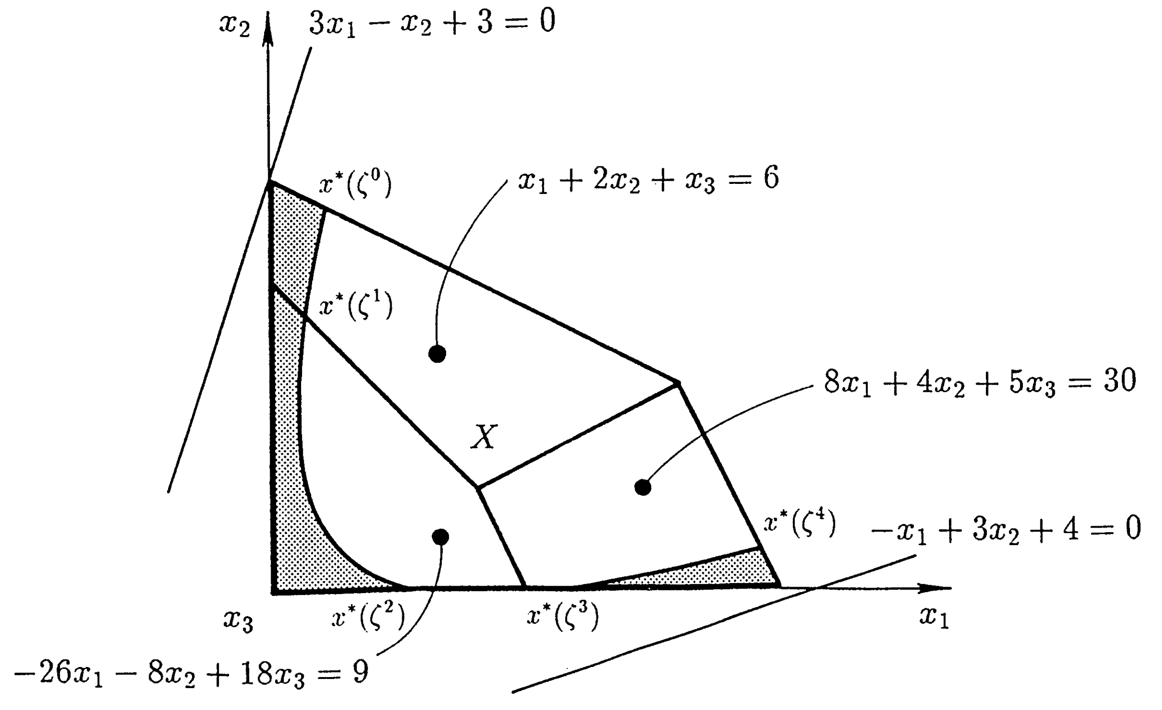

Figure 4.1. Three-dimensional example (4.15) of (P).

is a quasiconcave function on X [7]. It is easy to check that f has rank-two property on X for linearly independent vectors $d_1 = (3, -1)^T$ and $d_2 = (-1, 3)^T$, and that we have

$$g(\zeta) = (\zeta_1 + 3)(\zeta_2 + 4) - 18. \tag{4.18}$$

In Phase 1, we first solve a linear program: minimize$\{ 3x_1 - x_2 \mid x \in X \}$. Then we have $x^0 = (0, 3)$ as its optimal solution. Since $f(x^0) = -18 < 0$, we need to solve the following problem in order to obtain an initial active point ζ^0:

$$\left|\begin{array}{ll} \text{maximize} & -x_1 + 3x_2 \\ \text{subject to} & x \in X, \\ & 3x_1 - x_2 = v. \end{array}\right. \tag{4.19}$$

procedure ACT(x^0) solves (4.19) parametrically by increasing the value of v from -3, and yields an active point ζ^0 after a single pivot:

$$\zeta^0 = (-1.490, 7.922); \quad x^*(\zeta^0) = (0.431, 2.784, 0.000).$$

In Phase 2, we solve the following problem parametrically by changing ζ:

$$\left|\begin{array}{ll} \text{maximize} & x_3 \\ \text{subject to} & x \in X, \\ & 3x_1 - x_2 = \zeta_1, \ -x_1 + 3x_2 = \zeta_2. \end{array}\right. \tag{4.20}$$

The optimal dictionary of (4.20) at $\zeta = (-1.490, 7.922)$ is as follows:

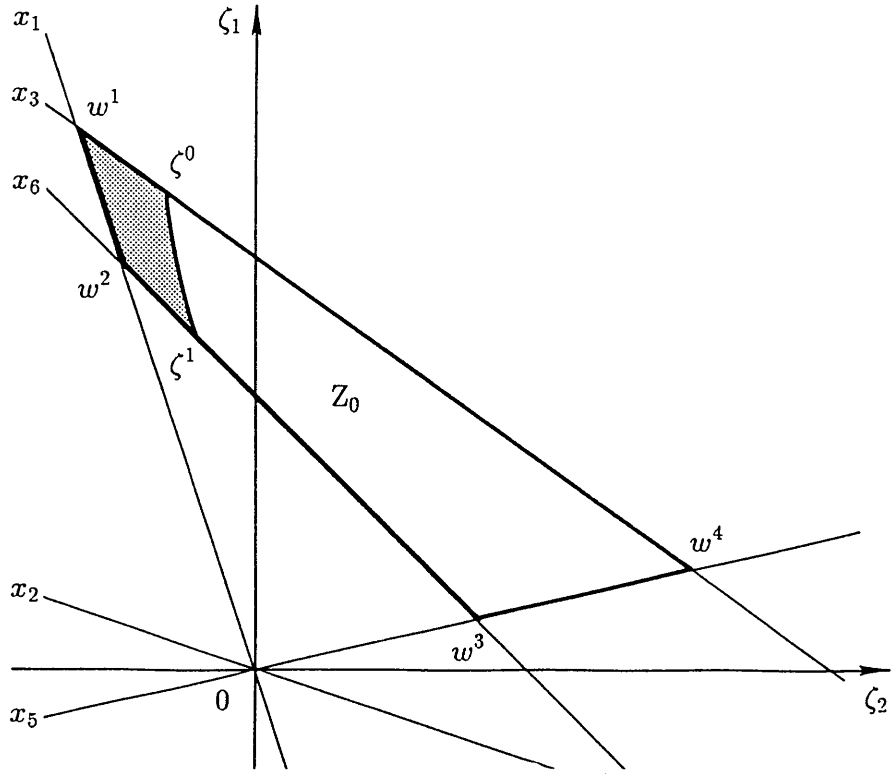

Figure 4.2. The polytope Z_0 associated with the dictionary (4.21).

$$\left| \begin{aligned} x_2 &= 0.000 + 0.125\zeta_1 + 0.375\zeta_2, \\ x_5 &= 0.000 - 0.375\zeta_1 + 1.875\zeta_2 + 5.000x_4, \\ x_6 &= -99.000 + 22.000\zeta_1 + 22.000\zeta_2 + 18.000x_4, \\ x_3 &= 6.000 - 0.625\zeta_1 - 0.875\zeta_2 - x_4, \\ x_1 &= 0.000 + 0.375\zeta_1 + 0.125\zeta_2, \\ z &= 6.000 - 0.625\zeta_1 - 0.875\zeta_2 - x_4. \end{aligned} \right. \tag{4.21}$$

Hence we define

$$Z_0 = \left\{ \zeta \in R^2 \,\middle|\, \begin{array}{ll} 0.125\zeta_1 + 0.375\zeta_2 \geq 0, & -0.375\zeta_1 + 1.875\zeta_2 \geq 0, \\ 22\zeta_1 + 22\zeta_2 \geq 99, & 0.625\zeta_1 + 0.875\zeta_2 \leq 6, \\ 0.375\zeta_1 + 0.125\zeta_2 \geq 0 & \end{array} \right\}. \tag{4.22}$$

We obtain an alternative active point ζ^1 by computing the intersection of $g(\zeta) = 0$ and the edge w^2-w^3 of Z_0 (see Figure 4.2). Applying a dual pivot to (4.21) at the third row corresponding to w^2-w^3, we have:

$$\zeta^1 = (-1.131, 5.631); \quad x^*(\zeta^1) = (0.280, 1, 970, 1.775).$$

In the same way, we have

$$\zeta^2 = (3.000, -1.000); \quad x^*(\zeta^2) = (1.000, 0.000, 1.944).$$

However, there is no active point ζ such that $3.000 < \zeta_1 \leq 3.000 + \delta$ for some positive δ. We need again to solve (4.19) by using $\mathrm{ACT}(\zeta^2)$. Then we have active points:

Table 5.1. Computational results for (5.1).

m	100	100	100	150	150
n	80	100	120	120	150
Algorithm OPT.					
Average number of pivots.					
Total:	80.2	82.8	96.6	118.8	121.6
	(24.091)	(25.262)	(23.161)	(19.379)	(24.352)
Type 1:	11.5	14.5	16.9	21.3	19.2
	(7.075)	(6.546)	(7.217)	(6.001)	(10.274)
Type 2:	7.7	10.1	17.0	15.4	15.6
	(4.627)	(4.134)	(13.550)	(7.186)	(13.669)
Type 3:	61.0	58.2	62.7	82.1	86.8
	(24.763)	(23.241)	(29.581)	(15.443)	(27.845)
Average CPU time in seconds.					
Total:	8.845	9.435	11.038	23.308	28.733
	(3.449)	(3.650)	(3.773)	(4.677)	(9.335)
P-T method.					
Average number of pivots.					
Total:	218.8	280.2	334.8	456.2	519.3
	(182.757)	(227.947)	(260.272)	(339.748)	(574.074)
Average CPU time in seconds.					
Total:	12.557	18.197	23.217	53.288	69.870
	(10.688)	(15.039)	(18.765)	(40.456)	(80.815)

$$\zeta^3 = (6.000, -2.000); \quad x^*(\zeta^3) = (2.000, 0.000, 2.800),$$
$$\zeta^4 = (10.474, -2.664); \quad x^*(\zeta^4) = (3.595, 0.310, 0.000).$$

The maximum of x_3 is attained at $x^*(\zeta^3)$.

5. Computational Experiments

We will report the results of computational experiments on algorithm OPT. We solved the following subclass of (P):

$$\left|\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax \leq b,\ x \geq 0, \\ & d_1^T x \geq d_{10},\ d_2^T x \geq d_{20}, \\ & (d_1^T x - d_{10})(d_2^T x - d_{20}) - d_{00} \leq 0. \end{array}\right. \tag{5.1}$$

where c, $d_i \in R^n (i = 1, 2)$, $d_{i0} \in R^1 (i = 0, 1, 2)$, $b \in R^m$ and $A \in R^{m \times n}$. Elements of c, d_i's and A were randomly generated between -1.000 and 1.000, and those of b, d_{i0}'s

Table 5.2. Computational results for (5.1).

m	150	200	200	200	220
n	180	180	200	220	250
Algorithm OPT.					
Average number of pivots.					
Total:	165.2	152.3	172.2	151.9	176.4
	(40.877)	(36.064)	(33.893)	(39.333)	(55.579)
Type 1:	38.1	28.3	24.3	23.1	29.0
	(15.719)	(10.508)	(10.479)	(13.141)	(9.623)
Type 2:	25.1	34.2	19.7	15.8	29.8
	(17.535)	(14.020)	(15.646)	(8.483)	(14.379)
Type 3:	102.0	89.8	128.2	113.0	117.6
	(35.296)	(48.099)	(34.790)	(49.649)	(57.395)
Average CPU time in seconds.					
Total:	36.902	52.750	65.847	61.875	93.880
	(9.909)	(20.613)	(16.973)	(26.774)	(38.794)
P-T method.					
Average number of pivots.					
Total:	774.5	402.1	668.2	586.8	752.9
	(1015.864)	(552.154)	(827.674)	(608.934)	(770.500)
Average CPU time in seconds.					
Total:	112.035	83.108	149.338	131.598	241.232
	(150.187)	(116.895)	(188.593)	(140.885)	(252.917)

were between 0.000 and 1.000. The size of problems ranged from $(m, n) = (100, 80)$ to $(220, 250)$. For each size we selected ten examples which were feasible and had no trivial solutions. We coded OPT and the algorithm proposed by Pferschy and Tuy [14] (abbr. P-T method) in C language, and tested them on a SUN SPARCstation ELC computer (20.5 mips). The tolerance ϵ required by the latter algorithm for obtaining an ϵ-optimal solution was fixed at 10^{-5}.

Table 5.1 shows the computational results when the size of (m, n) is $(100, 80)$ to $(150, 150)$. For each size of (m, n), the average number of (primal or dual) simplex pivots and the average CPU time in seconds (and their respective standard deviations in the brackets) needed for solving ten examples are listed. Here, Type 1 pivots mean primal ones which were carried out in Phase 1 for solving a linear program: minimize$\{ {d_1}^T x \mid x \in X \}$. Types 2 and 3 stand for dual simplex pivots applied in procedure ACT and **step** 3° of Phase 2, respectively. Table 5.2 shows the results when (m, n) is between

(150, 180) and (220, 250).

We see from these results that algorithm OPT is fairly efficient compared to P-T method for randomly generated problems (5.1). In particular, the total number of pivots required by OPT is only about 25 % of that by P-T method. Moreover, the variance of the former is far less than the latter. Since P-T method discards local maxima by cutting off the feasible region, unfortunate cuts sometimes delay the convergence considerably. Contrary to this, algorithm OPT uses no cuts and hence the convergence is relatively stable. It should also be emphasized that OPT yields not an ϵ-optmal solution but a globally optimal one.

References

[1] Chvătal, V., *Linear Programming*, (Freeman and Company, New York, 1983).

[2] Henderson, J. M. and R. E. Quandt, *Microeconomic Theory*, (Mcgraw-Hill, New York, 1971).

[3] Hillestad, R. J. and S. E. Jacobsen, "Reverse convex programming," *Applied Mathematics and Optimization* **6** (1980), 63 – 78.

[4] Hillestad, R. J. and S. E. Jacobsen, "Linear programs with an additional reverse convex constraint," *Applied Mathematics and Optimization* **6** (1980), 257 – 269.

[5] Horst, R. and H. Tuy, *Global Optimization: Deterministic Approaches,* (Springer, Berlin, 1990).

[6] Konno, H. and M. Inori, "Bond portfolio optimization by bilinear fractional programming," *Journal of the Operations Research Society of Japan* **32** (1988), 143 – 158.

[7] Konno, H. and T. Kuno, "Linear multiplicative programming," *Mathematical Programming* **56** (1992), 51 – 6

[8] Konno, H., T. Kuno, S. Suzuki, P. T. Thach and Y. Yajima, "Global optimization techniques for problem in the plane" Report IHSS91-36, Institute of Human and Social Sciences, Tokyo Institute of Technology (Tokyo, 1991).

[9] Konno, H., Y. Yajima and T. Matsui, "Parametric simplex algorithms for solving a special class of nonconvex minimization problems," *Journal of Global Optimization* **1** (1991), 65 – 82.

[10] Kuno, T., "Globally determining a minimum-area rectangle enclosing the projection of a higher-dimensional set," *Operations Research Letters* **13** (1993), 295 – 303.

[11] Kuno, T., H. Konno and Y. Yamamoto, "A parametric successive underestimation method for convex programming problems with an additional convex multiplicative constraint," *Journal of the Operations Research Society of Japan* **35** (1992), 290 – 299.

[12] Kuno, T., Y. Yajima, Y. Yamamoto and H. Konno, "Convex programs with an additional constraint on the product of several convex functions," Report IHSS91-38, Institute of Human and Social Sciences, Tokyo Institute of Technology (Tokyo, 1991) to appear in *European Journal of Operational Research.*

[13] Pardalos, P. M., "Polynomial time algorithms for some classes of constrained nonconvex

quadratic problems," *Optimization* **21** (1990), 843 - 853.

[14] Pferschy, U. and H. Tuy, "Linear Programs with an additional rank two reverse convex constraint," Draft, Institut für Mathematik, Technische Universtät Graz (Graz, 1992).

[15] Preparata, F. P. and M. I. Shamos, *Computational Geometry*, (Springer, Berlin, 1985).

[16] Stoer, J. and C. Witzgall, *Convexity and Optimization in Finite Dimensions I*, (Springer, Berlin,1970).

[17] Thach, P. T., R. E. Burkard and W. Oettli, "Mathematical programs with a two-dimensional reverse convex constraint," *Journal of Global Optimization* **1** (1991), 145 - 154.

[18] Tuy, H., "Convex Programs with an additional reverse convex constraint," *Journal of Optimization Theory and Applications* **52** (1987), 463 - 486.

[19] Yamamoto, Y., "Finding an ϵ-approximate solution of convex programs with a multiplicative constraint," Discussion Paper No.456, Institute of Socio-Economic Planning, University of Tsukuba (Tsukuba, Ibaraki, 1991).

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TITLE

A Parametric Simplex Algorithm for Solving a Class of Rank-Two Reverse Convex Programs

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ABSTRACT

This paper addresses an algorithm for solving a linear program with an additional rank-two reverse convex constraint. Unlike the existing methods which generate an approximately optimal solution, the algorithm provides a globally optimal solution of the nonconvex problem by a finite number of dual pivot operations. Computational results indicate that this algorithm can solve fairly large scale problems efficiently.

SUPPLEMENTARY NOTES