ISE

ISE-TR-93-102

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August 17, 1993

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Abstract.

We consider the multiclass M/G/1 queues with feedback. J classes of customers arrive at the stations from the outside of the system according to a Poisson process. The service time distribution for each class is arbitrary and the service discipline at each station is either the FCFS or the preemptive LCFS. After receiving a service, the customer at the station i either departs from the system with probability p_{i0}, or feeds back to the system and proceeds to the station k with probability p_{ik} $(i, k = 1, \cdots, J)$. We consider the cost functions naturally stem from the analysis of the system. They include the mean sojourn times for customers arriving at every station. First we consider the system at the arbitrary states to obtain the explicit formulae of the cost functions and then derive the steady state solutions by using the generalized Little's formula and PASTA. We also attempt to generalize the results to the other service disciplines.

Key words. Multiclass M/G/1 queues, feedbacks of customers, mean sojourn times, scheduling disciplines and Little's formula.

1 Introduction.

In this paper, we consider the multiclass M/G/1 queueing system with feedback. A single server serves J types of customers at J stations. Customers arrive at the station i $(i = 1, \ldots, J)$ from outside of the system according to a Poisson process. The service times of the customers at the every station are arbitrarily distributed. The customers in the station with the lowest number have the highest priority. The service discipline at each station is either the FCFS or the preemptive LCFS. After receiving a service, the customer at the station i or the *customer* i either departs from the system, or feeds back to the system and proceeds to the station k $(i, k = 1, \ldots, J)$. We derive the explicit formulae of the two cost functions of the specific *tagged customer* that represent the total amount of the *mean sojourn times* and the *cumulative works.*

In the previous paper [11], we consider the system at the arbitrary embedded Markov points and derive the mean sojourn times. In this paper, we consider the system at the arbitrary system states in order to derive the cost functions. Then we consider the system under the steady state. In Section 2, we define the system and the cost functions in detail. The set of the equations satisfied by the cost functions is derived. Section 3 is devoted to derive the two other cost functions naturally stem from the analysis of the system. We derive the statistics with regard to the system states in Sections 4 and 5. In Section 6, the set of the equations is solved to derive the explicit formulae of the two objective cost functions at the arbitrary system states. The steady state of the system is defined in Section 7. The uniqueness of the cost functions is established in Section 8. Finally, in Section 9, we evaluate the steady state values of the cost functions by the generalized Little's formula ($H = \lambda G$) and PASTA (Poisson arrival see time averages) property. The generalization of the service disciplines also goes abreast.

Disney [4] and Disney et al. [5] have been concerned with sojourn times in M/G/1 queues with instantaneous, Bernoulli feedback. Berg, et al. [1] considered the system in which each customer requires N services. Fed back customers return instantaneously, joining the end of the queue. The service discipline is FCFS. They derived the set of linear equations for the mean sojourn times per visit can be explicitly solved. The M/G/1 queue under the foreground-background processor-sharing discipline is investigated in [17]. Simon [20] considered the system with c types of customers and m levels of priority. Type j customers may require service $N(j)$ times. The k^{th} time a type j customer enters the queue it is assigned priority level $f(j,k)$. He obtained the set of linear equations for the mean waiting times. Doshi and Kaufman [6] studied the sojourn time of a tagged customer who has just completed his m^{th} pass in an M/G/1 queue with Bernoulli feedback. They also considered the model with multiple customer classes. The relationship between the M/M/1 feedback queue and M/G/1 queue with processor sharing is investigated in [2]. Recently, Epema [8] investigated the general single server (M/G/1) time-sharing model with multiple queues and customer classes, priorities and feedback. Customers are served in passes, receiving a complete quantum of service on every pass, or their remaining service demand, whichever is the lesser. If a customer completes his service demand during the pass, it leaves the system. He derived a set of linear equations in the mean waiting times of the customer passes for all classes and queues. The optimal scheduling of the multi-class queue with feedback has been studied in [15, 21]. The problem is known as the multiarmed bandit problem [22]. The priority queues are well investigated in [12], [10] and [26]. For further related topics on the field, see [13, 16].

2 The model description.

A single server serves J types of customers at J stations. Customers arrive at the station i from the outside of the system according to a Poisson process $\{A_i(t) : t \geq 0\}$ with the rate λ_i $(i = 1, \ldots, J)$. The customer at the station i is called the *class* i *customer*, or simply the *customer* i. The service time S_i of the class i customers is arbitrarily distributed.

The customers are serviced according to the predetermined *service disciplines*. We assume the following overall disciplines:

- The customers are preferentially serviced in the ascending order of their classes. That is, the customer j has priority over the customer i if j is less than i.

- The service discipline is *preemptive* over the class. When the customer arrives at the station i, all the customers belonging to the classes between $i+1$ and J are preempted from the service. The preempted customers *resume* services later on.

The service discipline for each class (station) is either the FCFS or the preemptive LCFS (PR-LCFS). The station i with the FCFS discipline serves customers according to the first come first served basis if no customers are in the stations from 1 to $i-1$. The station i with the preemptive LCFS discipline serves customers according to the preemptive resume last come first served basis if no customers are in the stations from 1 to $i-1$.

After receiving a service, the customer i either departs the system with probability p_{i0}, or feeds back to the system and proceeds to the station k with probability p_{ik} $(i,k=1,\dots,J)$. Let the matrix $P_m=(p_{ij}:1\le i,j\le m)$ $(1\le m\le J)$. The arrival processes, the service times and the feedback processes are assumed to be independent of each other.

Let v denote the *current work*, that is, a customer will receive v seconds of service potentially at the currently entered station on the visit. The customer departs the system or feeds back to the system after receiving v seconds of the service. Let $T_{ij}(v)$ be the total amount of service times that a customer is currently at the station i with the current work v receives until the customer departs from the system or leaves for one of the stations between $j+1$ and J for the first time $(1\le i,j\le J)$. Then,

$$T_{ij}(v)=\begin{cases} v+T_{lj}(S_l) & \text{with probability } p_{il}, \quad l=1,\dots,j, \\ v & \text{with probability } p_{il}, \quad l=j+1,\dots,J, \text{ or } 0. \end{cases} \tag{2.1}$$

Note that, even if $i>j$, the customer is assumed to receive at least the service v. The expected value of $T_{ij}(v)$ is given by

$$E[T_{ij}(v)]=v+\sum_{l=1}^{j}p_{il}E[T_{lj}(S_l)], \qquad 1\le i,j\le J. \tag{2.2}$$

Specifically, if we let $T_{ij}=T_{ij}(S_i)$, then

$$E[T_{ij}]=E[S_i]+\sum_{l=1}^{j}p_{il}E[T_{lj}], \qquad 1\le i,j\le J. \tag{2.3}$$

So we can obtain its solution in the vector form if $(I-P_j)^{-1}$ exists. We define the intensity ρ_j in the following manner:

$$\begin{aligned}\rho_0 &= 0,\\ \rho_j &= \sum_{i=1}^{j}\lambda_i E[T_{ij}], \qquad j=1,\dots,J.\end{aligned}$$

Then we put the following assumption.

Assumption 2.1.

1. $P_J^n\to 0$ as $n\to\infty$.

2. $\rho_J < 1$.

The first assumption is the sufficient condition for the existence of $(I - P_j)^{-1}$ for $j = 1, \ldots, J$. The number of customers in the station i is denoted by n_i and its vector is denoted by $n = (n_1, \ldots, n_J)$. The customers in each station are arranged in the order of their arrivals. Let v_{im_i} be the current work of the m_i^{th} customer i ($i = 1, \ldots, J$ and $m_i = 1, \ldots, n_i$) and let $V = \{v_{im_i} : i = 1, \ldots, J \text{ and } m_i = 1, \ldots, n_i\}$ be the set of these current works. We call the pair (V, n) the *system state* or simply the *state*. We use the term *work* also to denote the current work.

Now we give our attention to a specific customer called a *tagged customer*. Specifically, if the customer is in the station i, we call him the tagged customer i. We would like to derive the two types of the cost functions defined below. The first type of the cost functions represents the mean sojourn time of the tagged customer. We define

$$W_{ij}(V,n) = \begin{cases} \text{the } \textit{mean sojourn time} \text{ of the tagged customer, who currently} \\ \text{arrives at the station } i \text{ when the initial system state just} \\ \text{prior to the arrival is } (V,n)\text{, spent at the station } j \text{ until} \\ \text{the customer departs from the system, } 1 \le i,j \le J. \end{cases} \tag{2.4}$$

For convenience, we define that $W_{0j}(V,n) = 0$. Note that we exclude the tagged customer from the initial system state. Further, we define

$$W_i(V,n) = \begin{cases} \text{the } \textit{initial (mean) sojourn time} \text{ of the tagged customer, who} \\ \text{currently arrives at the station } i \text{ when the initial system state} \\ \text{just prior to the arrival is } (V,n)\text{, spent until } \textit{his first} \\ \textit{service completion} \text{ at the station, } 1 \le i \le J. \end{cases} \tag{2.5}$$

Then the mean sojourn time $W_{ij}(\cdot)$ of the tagged customer i ($i = 1, \ldots, J$) for the station j ($j = 1, \ldots, J$) is decomposed into the two parts, the initial mean sojourn time and the mean sojourn time after spending the initial sojourn time. We mathematically express the fact as follows:

$$W_{ij}(V,n) = \begin{cases} E[W_{Kj}(V^*, n^*)|(i),(V,n)], & i \neq j, \\ W_j(V,n) + E[W_{Kj}(V^*, n^*)|(j),(V,n)], & i = j, \end{cases} \tag{2.6}$$

where the condition $\{(i),(V,n)\}$ denotes that the tagged customer currently arrives at the station i when the system state just prior to the arrival is (V,n), and where K is the station number for which the tagged customer leaves after staying the station i and (V^*, n^*) is the system state on his first service completion at the station i. Note that the two statistics $W_j(V,n)$ and (V^*, n^*) may be different for every service discipline.

The second type of the cost functions represents the *cumulative current work* of the tagged customer. Let $f_j(t)$ be the *current work* or the *current remaining service time* at time t of the tagged customer at the station j. For example, if the tagged customer enters the station j at t_0, then $f_j(t_0) = S_j$. The value of $f_j(t)$ gradually decreases as the server serves the customer. If the tagged customer completes his current service at t_1, $f_j(t_1) = 0$. The customer again enters the station j at $t_2 \ge t_1$, then $f_j(t_2) = S_j$. The value of $f_j(t)$ gradually decreases until the server completes the service of the the tagged customer at t_3. Then, $f_j(t_3) = 0$ and so forth. Then we define

$$\begin{aligned} G_{ij}(V,n) &= \begin{cases} \text{the } \textit{cumulative current work} \text{ of the tagged customer, who} \\ \text{currently arrives at the station } i \text{ when the initial system state} \\ \text{just prior to the arrival is } (V,n)\text{, accumulated at the station } j \\ \text{until the customer departs from the system, } 1 \le i,j \le J, \end{cases} \\ &= E[\int_0^\infty f_j(t)dt|(i),(V,n)]. \end{aligned} \tag{2.7}$$

For convenience, we define that $G_{0j}(V,n)=0$. Note that we exclude the tagged customer's work S_i from the component V of the initial system state. Further, we define

$$G_i(V,n) = \begin{cases} \text{the } \textit{initial cumulative current work} \text{ of the tagged customer, who} \\ \text{currently arrives at the station } i \text{ when the initial system state} \\ \text{just prior to the arrival is } (V,n) \text{, accumulated until } \textit{his} \\ \textit{first service completion} \text{ at the station, } 1 \le i \le J, \end{cases} \quad (2.8)$$
$$= E[\int_0^{W_i} f_i(t)dt|(i),(V,n)].$$

where 0 is the arrival epoch of the tagged customer at the station i, and where W_i is his first service completion epoch (the initial sojourn time) at the station.

Then the cost function (the cumulative current work) $G_{ij}(\cdot)$ of the tagged customer i ($i=1,\ldots,J$) for the station j ($j=1,\ldots,J$) is decomposed into the two parts, the cost accumulated during the initial sojourn time and the cost accumulated after spending the initial sojourn time. We mathematically express the fact as follows:

$$G_{ij}(V,n) = \begin{cases} E[G_{Kj}(V^*,n^*)|(i),(V,n)], & i \neq j, \\ G_j(V,n) + E[G_{Kj}(V^*,n^*)|(j),(V,n)], & i = j, \end{cases} \quad (2.9)$$

where K is the station number for which the tagged customer i leaves after staying the station i and (V^*,n^*) is the system state on his service completion at the station i. Note that the two statistics $G_j(V,n)$ and (V^*,n^*) may be different for every service discipline.

After stating some assumptions, we will explicitly solve these equations (2.6) and (2.9) in Section 6. We also attempt to generalize these results to the service disciplines other than the above mentioned.

These statistics are shown to be closely related to the busy periods. So we close the section with the definitions of the statistics. Let B^j be the first time until the system is cleared of the customers from classes 1 through $j, j=1,\ldots,J$. Let

$$B^j(v) = \begin{cases} \text{the first time until the system is cleared of the customers from classes} \\ \text{1 through } j \text{ with the `exceptional' service time } v,\ j=1,\ldots,J. \end{cases} \quad (2.10)$$

For notational convenience, let $B^0(v)=v$. In the usual queueing parlance, B^j is the *busy period* composed of customers 1 through j and $B^j(v)$ is the *Exceptional First Service Busy Period* (EFSBP) composed of customers 1 through j [24]. We will call B^j and $B^j(v)$ simply the *class j busy period* and the *class j busy period with the exceptional service v*, respectively. Their expected values are given by

$$E[B^j] = \frac{\sum_{i=1}^{j} \lambda_i E[T_{ij}]}{(\sum_{i=1}^{j} \lambda_i)(1-\rho_j)}, \quad (2.11)$$
$$E[B^j(v)] = \frac{v}{1-\rho_j}. \quad (2.12)$$

Further, we define

$$B^j(V,n) = \begin{cases} \text{the first time until the system is cleared of the customers from classes 1} \\ \text{through } j \text{ starting from the initial system state } (V,n),\ j=1,\ldots,J. \end{cases} \quad (2.13)$$

Note that the works v_{im_i} $(i=j+1,\ldots,J$ and $m_i=1,\ldots,n_i)$ are not performed in this period. We will call $B^j(V,n)$ the *class j busy period with the initial state* (V,n). We now consider a set of customers currently in the system and denote it by X. For example, if the m_i^{th} customer i currently in the system belongs to the set, we express it as $(i, m_i) \in X$. The set of customers who are initially in the system and are not in X when the system state is (V, n) is denoted by $\overline{X}$. We then define

$$B^0(V,n;X) = \begin{cases} \text{the first time until the current works of customers} \\ \text{belonging to } X \text{ have been performed,} \end{cases} \tag{2.14}$$

$$B^j(V,n;X) = \begin{cases} \text{the first time starting from the initial system state } (V,n) \\ \text{until the current works of customers belonging to } X \text{ have} \\ \text{been performed and the system is cleared of the customers} \\ \text{from classes 1 through } j \text{ except the customers in } \overline{X},\ j = 1,\ldots,J. \end{cases} \tag{2.15}$$

The main difference between $B^j(V,n)$ and $B^j(V,n;X)$ is that the works v_{im_i} $(i = j+1,\ldots,J$ and $m_i = 1,\ldots,n_i)$ can be performed in the latter period by appropriately choosing the set X. We will call it the *class j busy period with the initial state* $\{(V,n);X\}$. Note that these EFSBPs are invariant for all *work conserving* service disciplines [14]. Then it can be shown that

$$B^0(V,n;X) = \sum_{(i,m_i)\in X} v_{im_i}, \tag{2.16}$$

$$B^j(V,n;X) = \sum_{(i,m_i)\in X} B^j(T_{ij}(v_{im_i})). \tag{2.17}$$

If we set $X_j = \{(i,m_i) : i = 1,\ldots,j \text{ and } m_i = 1,\ldots,n_i\}$, then $B^j(V,n) = B^j(V,n;X_j)$ for $j = 1,\ldots,J$.

3 Initial cost functions.

In this section, we derive the initial cost functions $W_i(\cdot)$ and $G_i(\cdot)$ of the tagged customer i $(i = 1,\ldots,J)$. First we consider the statistics for the two specific service disciplines, FCFS and PR-LCFS, and then attempt to generalize the results.

The initial cost functions $W_i(\cdot)$ and $G_i(\cdot)$ are different for each service discipline. We first derive them for the FCFS discipline. The set X_i^F $(i = 1,\ldots,J)$ of customers is composed of the customers belonging to the classes 1 through i who are initially in the system. We distinguish the FCFS discipline by the superscript F. The initial sojourn time $W_i^F(V,n)$ of the tagged customer i for the FCFS discipline with the initial state (V,n) is composed of the class $i-1$ busy period with the initial state $\{(V,n);X_i^F\}$, and the class $i-1$ busy period with the exceptional service S_i of the tagged customer. Hence,

$$W_i^F(V,n) = E[B^{i-1}(V,n;X_i^F) + B^{i-1}(S_i)], \quad i = 1,\ldots,J. \tag{3.18}$$

As we have shown in the last section, the EFSBP is the sum of the EFSBPs starting with every customer in X_i^F. Its expected value is easily obtained by the usual method [24]. Then

$$W_i^F(V,n) = \sum_{j=1}^{i} \sum_{m_j=1}^{n_j} \frac{E[T_{ji-1}(v_{jm_j})]}{1-\rho_{i-1}} + \frac{E[S_i]}{1-\rho_{i-1}}, \quad i = 1,\ldots,J. \tag{3.19}$$

On the other hand, the cumulative current work initially equal to S_i gradually decreases to 0. Then we carefully calculate the values. For the FCFS discipline, the current work is equal to S_i until the service of the tagged customer begins. Then it decreases at the rate of 1 second a second until the customer is preempted by an arriving customer in one of the classes between 1 and $i-1$. The current work of the tagged customer keeps its last level. After completing the class $i-1$ busy period initiated by the arriving customer, it then decreases at the rate of 1 second a second until the customer is preempted by another arriving customer in one of the classes between 1 and $i-1$, and so on. Let t_l ($l = 1, 2, \ldots$) denotes the attained service time of the tagged customer on his l^{th} preemption. We obtain the cost function $G_i^F(V, n)$ by conditioning on the service time S_i of the tagged customer, the number $A^i = \sum_{k=1}^{i-1} A_k(S_i)$ of the customers who preempt the tagged customer and the time $t_l, l = 1, \ldots, A^i$. Then we obtain

$$G_i^F(V, n|S_i, A^i = m, \{t_l\}) = S_i \left\{E[B^{i-1}(V, n; X_i^F)]\right\} + \sum_{l=1}^{m}(S_i - t_l)E[B_l^{i-1}] + \frac{S_i^2}{2} \tag{3.20}$$

where B_l^{i-1} is the l^{th} class $i-1$ busy period. By the nature of the Poisson process, the time $\{t_l\}$ have the same distribution as the order statistics corresponding to m independent random variables uniformly distributed on the interval S_i [18]. Hence we obtain

$$E[t_l|S_i, A^i = m] = \frac{l}{m+1}S_i, \quad l = 1, \ldots, m. \tag{3.21}$$

Then we have

$$G_i^F(V, n) = E[S_i]\sum_{j=1}^{i}\sum_{m_j=1}^{n_j} \frac{E[T_{ji-1}(v_{jm_j})]}{1-\rho_{i-1}} + \frac{1}{2}\Lambda_{i-1}E[S_i^2]E[B^{i-1}] + \frac{E[S_i^2]}{2} \tag{3.22}$$

where $\Lambda_{i-1} = \sum_{k=1}^{i-1} \lambda_k$.

Second, we derive the initial sojourn time for the PR-LCFS discipline. Let $X_1^{PL} = \phi$. The set X_i^{PL} ($i = 2, \ldots, J$) of customers composed of the customers belonging to the classes 1 through $i-1$ who are initially in the system. We distinguish the PR-LCFS discipline by the superscript PL. Note the difference of the definitions from the case of the FCFS discipline. The initial sojourn time $W_i^{PL}(V, n)$ of the tagged customer i for the PR-LCFS discipline with the initial system state (V, n) is composed of the class i busy period with the initial state (V, n) and the initial set of customers X_i^{PL}, and the class i busy period with the exceptional first service S_i. Hence,

$$\begin{aligned} W_i^{PL}(V, n) &= E[B^i(V, n; X_i^{PL}) + B^i(S_i)] & (3.23)\\ &= \sum_{j=1}^{i-1}\sum_{m_j=1}^{n_j} \frac{E[T_{ji}(v_{jm_j})]}{1-\rho_i} + \frac{E[S_i]}{1-\rho_i}, \quad i = 1, \ldots, J. & (3.24) \end{aligned}$$

Similarly as the FCFS discipline, the cost function $G_i^{PL}(V, n)$ is obtained as the following expression.

$$G_i^{PL}(V, n) = E[S_i]\sum_{j=1}^{i-1}\sum_{m_j=1}^{n_j} \frac{E[T_{ji}(v_{jm_j})]}{1-\rho_i} + \frac{1}{2}\Lambda_i E[S_i^2]E[B^i] + \frac{E[S_i^2]}{2} \tag{3.25}$$

where $\Lambda_i = \sum_{k=1}^{i} \lambda_k$.

Next we attempt to generalize the results.

Generalizations of the initial cost functions.

As we have seen above, the initial sojourn times are the sum of the EFSBPs starting with the tagged customer i and every initial customer in X_i. The initial cumulative current works are also decomposed into the values associated with the EFSBPs.

Then we may generally assume the followings:

$$\begin{aligned} W_i(V, n) &= \sum_{j=1}^{J}\{\phi_{1j}^i \sum_{m_j=1}^{n_j} v_{jm_j} + \phi_{2j}^i n_j\} + w^i, \quad i = 1, \ldots, J, & (3.26)\\ G_i(V, n) &= \sum_{j=1}^{J}\{\eta_{1j}^i \sum_{m_j=1}^{n_j} v_{jm_j} + \eta_{2j}^i n_j\} + g^i, \quad i = 1, \ldots, J. & (3.27) \end{aligned}$$

For example, in the case of the FCFS discipline, we can obtain from (2.2), (3.2) and (3.5),

$$
\begin{aligned}
\phi_{1j}^{iF} &= \begin{cases} 1/(1-\rho_{i-1}), & j=1,\dots,i, \\ 0, & j=i+1,\dots,J, \end{cases} \\
\phi_{2j}^{iF} &= \begin{cases} \sum_{l=1}^{i-1} p_{jl} E[T_{l,i-1}]/(1-\rho_{i-1}), & j=1,\dots,i, \\ 0, & j=i+1,\dots,J, \end{cases} \\
w^{iF} &= E[S_i]/(1-\rho_{i-1}), \\
\eta_{1j}^{iF} &= \begin{cases} E[S_i]/(1-\rho_{i-1}), & j=1,\dots,i, \\ 0, & j=i+1,\dots,J, \end{cases} \\
\eta_{2j}^{iF} &= \begin{cases} E[S_i]\sum_{l=1}^{i-1} p_{jl} E[T_{l,i-1}]/(1-\rho_{i-1}), & j=1,\dots,i, \\ 0, & j=i+1,\dots,J, \end{cases} \\
g^{iF} &= E[S_i^2](\Lambda_{i-1}E[B^{i-1}]+1)/2 = E[S_i^2]/\{2(1-\rho_{i-1})\},
\end{aligned}
$$

where $i=1,\dots,J$ and we distinguish the FCFS discipline by the superscript F. The expected values $E[T_{l,i-1}]$ and $E[B^{i-1}]$ are given by (2.3) and (2.11). In the case of the PR-LCFS discipline, we can obtain from (2.2), (3.6) and (3.8),

$$
\begin{aligned}
\phi_{1j}^{iPL} &= \begin{cases} 1/(1-\rho_{i}), & j=1,\dots,i-1, \\ 0, & j=i,\dots,J, \end{cases} \\
\phi_{2j}^{iPL} &= \begin{cases} \sum_{l=1}^{i} p_{jl} E[T_{li}]/(1-\rho_{i}), & j=1,\dots,i-1, \\ 0, & j=i,\dots,J, \end{cases} \\
w^{iPL} &= E[S_i]/(1-\rho_{i}), \\
\eta_{1j}^{iPL} &= \begin{cases} E[S_i]/(1-\rho_{i}), & j=1,\dots,i-1, \\ 0, & j=i,\dots,J, \end{cases} \\
\eta_{2j}^{iPL} &= \begin{cases} E[S_i]\sum_{l=1}^{i} p_{jl} E[T_{li}]/(1-\rho_{i}), & j=1,\dots,i-1, \\ 0, & j=i,\dots,J, \end{cases} \\
g^{iPL} &= E[S_i^2](\Lambda_{i}E[B^{i}]+1)/2 = E[S_i^2]/\{2(1-\rho_{i})\},
\end{aligned}
$$

where $i=1,\dots,J$ and we distinguish the PR-LCFS discipline by the superscript PL. The expected values $E[T_{l,i}]$ and $E[B^i]$ are given by (2.3) and (2.11).

For notational convenience, we define the following vectors:

$$
\begin{aligned}
\mathcal{W}^i &= (\phi_{11}^i,\dots,\phi_{1J}^i,\phi_{21}^i,\dots,\phi_{2J}^i)' \in R^{2J\times 1}, \quad i=1,\dots,J, && (3.28) \\
\mathcal{G}^i &= (\eta_{11}^i,\dots,\eta_{1J}^i,\eta_{21}^i,\dots,\eta_{2J}^i)' \in R^{2J\times 1}, \quad i=1,\dots,J, && (3.29) \\
\overline{V} &= (\overline{V}_1,\dots,\overline{V}_J) \in R^{1\times J}, && (3.30)
\end{aligned}
$$

where $'$ denotes the transposition and where $\overline{V}_j = \sum_{m_j=1}^{n_j} v_{jm_j}$ is the *total amount of works* at the station j ($j=1,\dots,J$). Then, the assumption (3.9) is arranged as follows:

$$
\begin{aligned}
W_i(V,n) &= (\overline{V},n)\mathcal{W}^i + w^i, \quad i=1,\dots,J, && (3.31) \\
G_i(V,n) &= (\overline{V},n)\mathcal{G}^i + g^i, \quad i=1,\dots,J. && (3.32)
\end{aligned}
$$

The expressions will be cited later to derive the cost functions $W_{ij}(\cdot)$ and $G_{ij}(\cdot)$.

The important things to consider about the assumption are that the *aggregated system state* $(\overline{V},n)$ should be sufficient for estimating the cost functions $W_i(V,n)$ and $G_i(V,n)$, and that the cost functions should be the linear function of the aggregated system state. Of course, the coefficients $\mathcal{W}^i, \mathcal{G}^i, w^i$ and g^i ($i=1,\dots,J$) can be different for every service discipline.

4 System states just after the busy periods.

In this and the next sections, we derive the expected value of the system states on the completion epoch of the initial sojourn times W_i. As we mentioned before, these statistics are closely related to the busy periods. So we consider the system states after the EFSBPs in this section.

Let $V_l^j(v)$ and $N_l^j(v)$ $(0 \leq j < l \leq J)$ be respectively the total amount of works and the number of the customers at the station l on the completion epoch of $B^j(v)$. It is assumed for these variables that the initial (tagged) customer with his current work (or exceptional service) v is ignored from temporary considerations after receiving the service v as if it was rejected from the system . Let $V_l^j(V,n;X)$ and $N_l^j(V,n;X)$ $(0 \leq j < l \leq J)$ be respectively the total amount of works and the number of customers at the station l on the completion epoch of $B^j(V,n;X)$ where in this case all feedbacks of customers are taken into consideration (including the customers initially in the system). Recall that $B^j(V,n;X)$ is a class j busy period with the initial state $\{(V,n);X\}$. Similarly as $B^j(V,n;X)$, these statistics are the sums of the statistics generated at the sub-busy periods initiated by the every customer in X. Then

$$
\begin{aligned}
V_l^j(V,n;X) &= \sum_{m_l \in \{m:(l,m)\notin X\}} v_{lm_l} + \sum_{(k,m_k)\in X} V_l^j(v_{km_k}, e_k; \{(k,m_k)\}), && (4.33)\\
N_l^j(V,n;X) &= \sum_{m_l \in \{m:(l,m)\notin X\}} 1 + \sum_{(k,m_k)\in X} N_l^j(v_{km_k}, e_k; \{(k,m_k)\}), && (4.34)
\end{aligned}
$$

where $0 \leq j < l \leq J$. The empty sum, which often occurs at $j = 0$, is defined to be 0 from now on. These statistics consist of the customers initially in the system, the customers arrived from the outside of the system and the customers arrived by the feedbacks. The every random variable in the right-hand side of each equation are mutually independent.

The derivation procedure of these solutions are similar to those in the previous paper [11]. Since the EFSBPs are invariant for all work conserving service disciplines, the statistics defined above are also invariant for these service disciplines. Similarly as the derivation of the moments of the ordinary busy period, we consider a service discipline which serves the customers nonpreemptively according to the last come first served basis.

For notational convenience, we define

$$
\begin{aligned}
V_{kl}^j(v_{km_k}) &\equiv V_l^j(v_{km_k}, e_k; \{(k,m_k)\}),\\
N_{kl}^j(v_{km_k}) &\equiv N_l^j(v_{km_k}, e_k; \{(k,m_k)\}),
\end{aligned}
$$

and define

$$
\mathbf{1}_{lkj}(v) = \begin{cases} 1 & \text{if the customer } k \text{ with the current work } v \text{ enters the station } l \\ & \text{on the completion epoch of the service } T_{kj}(v), \\ 0 & \text{otherwise.} \end{cases}
$$

Then we have

$$
\begin{aligned}
V_{kl}^j(v_{km_k}) &= \sum_{m_l=1}^{A_l(T_{kj}(v_{km_k}))} S_{lm_l} + \sum_{i=1}^{j} \sum_{m_i=1}^{A_i(T_{kj}(v_{km_k}))} V_{il}^j(S_{im_i}) + S_l \mathbf{1}_{lkj}(v_{km_k}),\\
N_{kl}^j(v_{km_k}) &= A_l(T_{kj}(v_{km_k})) + \sum_{i=1}^{j} \sum_{m_i=1}^{A_i(T_{kj}(v_{km_k}))} N_{il}^j(S_{im_i}) + \mathbf{1}_{lkj}(v_{km_k}),
\end{aligned}
$$

where S_{im_i} is the service time of m_i^{th} customer i $(0 \leq j < l \leq J$ and $1 \leq i,k \leq J)$. The expected values are defined as follows:

$$
\begin{aligned}
\overline{V}_{kl}^j &= E[V_{kl}^j(S_k)],\\
\overline{N}_{kl}^j &= E[N_{kl}^j(S_k)].
\end{aligned}
$$

(Notice the distinction between the statistics $\overline{V}^j_{kl}$ and the aggregated state $\overline{V}_j$.) By conditioning on the completion epoch of the current work v and after some calculations, we obtain

$$E[V^j_{kl}(v)] = v\left\{\lambda_l E[S_l] + \sum_{i=1}^{j} \lambda_i \overline{V}^j_{il}\right\} + p_{kl}E[S_l] + \sum_{i=1}^{j} p_{ki}\overline{V}^j_{il}, \tag{4.35}$$

$$E[N^j_{kl}(v)] = v\left\{\lambda_l + \sum_{i=1}^{j} \lambda_i \overline{N}^j_{il}\right\} + p_{kl} + \sum_{i=1}^{j} p_{ki}\overline{N}^j_{il}, \tag{4.36}$$

where $0 \le j < l \le J$ and $1 \le k \le J$. Specifically, $\overline{V}^j_{kl}$ and $\overline{N}^j_{kl}$ satisfy the following equations:

$$\overline{V}^j_{kl} = E[S_k]\left\{\lambda_l E[S_l] + \sum_{i=1}^{j} \lambda_i \overline{V}^j_{il}\right\} + p_{kl}E[S_l] + \sum_{i=1}^{j} p_{ki}\overline{V}^j_{il}, \quad k = 1,\ldots,j, \tag{4.37}$$

$$\overline{N}^j_{kl} = E[S_k]\left\{\lambda_l + \sum_{i=1}^{j} \lambda_i \overline{N}^j_{il}\right\} + p_{kl} + \sum_{i=1}^{j} p_{ki}\overline{N}^j_{il}, \quad k = 1,\ldots,j \tag{4.38}$$

where $1 \le j < l \le J$. These equations are easily solved by the usual techniques in the vector forms under **Assumption 2.1.** The expected values of the statistics defined in this section can be expressed by these results. We can show

$$V^j_l(v) = \sum_{m_l=1}^{A_l(v)} S_{lm_l} + \sum_{i=1}^{j} \sum_{m_i=1}^{A_i(v)} V^j_{il}(S_{im_i}),$$

$$N^j_l(v) = A_l(v) + \sum_{i=1}^{j} \sum_{m_i=1}^{A_i(v)} N^j_{il}(S_{im_i}).$$

We now define the following constants:

$$\overline{\xi}^j_l = \lambda_l E[S_l] + \sum_{i=1}^{j} \lambda_i \overline{V}^j_{il} = \xi^j_l E[S_l], \tag{4.39}$$

$$\overline{\chi}^j_{kl} = p_{kl}E[S_l] + \sum_{i=1}^{j} p_{ki}\overline{V}^j_{il} = \chi^j_{kl}E[S_l], \tag{4.40}$$

$$\xi^j_l = \lambda_l + \sum_{i=1}^{j} \lambda_i \overline{N}^j_{il}, \tag{4.41}$$

$$\chi^j_{kl} = p_{kl} + \sum_{i=1}^{j} p_{ki}\overline{N}^j_{il}, \tag{4.42}$$

where $0 \le j < l \le J$ and $1 \le k \le J$. $\overline{V}^j_{il}$ and $\overline{N}^j_{il}$ are the solutions of the equations (4.5) and (4.6). Hence their expected values are

$$E[V^j_l(v)] = v\overline{\xi}^j_l, \tag{4.43}$$

$$E[N^j_l(v)] = v\xi^j_l. \tag{4.44}$$

We can finally obtain

$$\begin{aligned} E[V^j_l(V,n;X)] &= \sum_{m_l \in C_l(X)} v_{lm_l} + \sum_{(k,m_k)\in X} E[V^j_{kl}(v_{km_k})] \\ &= \sum_{m_l \in C_l(X)} v_{lm_l} + \sum_{(k,m_k)\in X} \left\{v_{km_k}\overline{\xi}^j_l + \overline{\chi}^j_{kl}\right\}, \end{aligned} \tag{4.45}$$

$$E[N_l^j(V,n;X)] = \sum_{m_l \in C_l(X)} 1 + \sum_{(k,m_k)\in X} E[N_{kl}^j(v_{km_k})]$$
$$= \sum_{m_l \in C_l(X)} 1 + \sum_{(k,m_k)\in X} \left\{ v_{km_k}\xi_l^j + \chi_{kl}^j \right\}, \quad (4.46)$$

where $C_l(X) = \{m : (l,m) \notin X\}$ and $0 \le j < l \le J$. These results are used to obtain the expected system state after the initial sojourn times.

5 System states just after the initial sojourn times.

In this sections, we derive the expected value of the system states on the completion epoch of the initial sojourn times W_i. Since these statistics are different for every discipline, we first consider the FCFS discipline and then consider the PR-LCFS discipline. The aggregated system state is a pair $(\overline{V}, n)$ where $\overline{V} = (\overline{V}_1, \ldots, \overline{V}_J)$ is the vector of the total amount of the current works in each station and $n = (n_1, \ldots, n_J)$ is the vector of the number of the customers in each station.

As we have defined before, for the FCFS discipline, the set X_i^F $(i = 1, \ldots, J)$ of customers is composed of the customers belonging to the classes 1 through i who are initially in the system. We recall the relation (3.1) between the initial sojourn time and the busy period:

$$W_i^F(V,n) = E[B^{i-1}(V,n;X_i^F) + B^{i-1}(S_i)], \quad i = 1,\ldots,J. \quad (5.47)$$

Let (i) denote the tagged customer i with the initial work S_i. The (aggregated) system states $(\overline{V}^*, n^*)$ for the FCFS discipline on the completion epoch of the initial sojourn time are obtained from (4.1) and (4.2),

$$E^F[\overline{V}_l^*|(i),(V,n)] = E[V_l^{i-1}(V,n;X_i^F) + V_l^{i-1}(S_i)]$$
$$= \begin{cases} 0, & l = 1,\ldots,i-1, \\ \sum_{k=1}^{i}\sum_{m_k=1}^{n_k} E[V_{ki}^{i-1}(v_{km_k})] + E[V_i^{i-1}(S_i)], & l = i, \\ \overline{V}_l + \sum_{k=1}^{i}\sum_{m_k=1}^{n_k} E[V_{kl}^{i-1}(v_{km_k})] + E[V_l^{i-1}(S_i)], & l = i+1,\ldots,J, \end{cases} \quad (5.48)$$

$$E^F[n_l^*|(i),(V,n)] = E[N_l^{i-1}(V,n;X_i^F) + N_l^{i-1}(S_i)]$$
$$= \begin{cases} 0, & l = 1,\ldots,i-1, \\ \sum_{k=1}^{i}\sum_{m_k=1}^{n_k} E[N_{ki}^{i-1}(v_{km_k})] + E[N_i^{i-1}(S_i)], & l = i, \\ n_l + \sum_{k=1}^{i}\sum_{m_k=1}^{n_k} E[N_{kl}^{i-1}(v_{km_k})] + E[N_l^{i-1}(S_i)], & l = i+1,\ldots,J, \end{cases} \quad (5.49)$$

where $1 \le i \le J$. The concrete expressions for these equations are given below.

Next, for the PR-LCFS discipline, let $X_1^{PL} = \phi$ and the set X_i^{PL} $(i = 2,\ldots,J)$ of customers is composed of the customers belonging to the classes 1 through $i-1$ who are initially in the system. We recall the relation (3.6) between the initial sojourn time and the busy period:

$$W_i^{PL}(V,n) = E[B^i(V,n;X_i^{PL}) + B^i(S_i)], \quad i = 1,\ldots,J. \quad (5.50)$$

The (aggregated) system states $(\overline{V}^*, n^*)$ for the PR-LCFS discipline on the completion epoch of the initial sojourn time are obtained from (4.1), (4.2), (4.3), (4.4), (4.11) and (4.12).

$$E^{PL}[\overline{V}_l^*|(i),(V,n)] = E[V_l^i(V,n;X_i^{PL}) + V_l^i(S_i)]$$
$$= \begin{cases} 0, & l = 1,\ldots,i-1, \\ \overline{V}_i, & l = i, \\ \overline{V}_l + \sum_{k=1}^{i-1}\sum_{m_k=1}^{n_k} E[V_{kl}^i(v_{km_k})] + E[V_l^i(S_i)], & l = i+1,\ldots,J, \end{cases} \quad (5.51)$$

$$E^{PL}[n_l^*|(i),(V,n)] = E[N_l^i(V,n;X_i^{PL}) + N_l^i(S_i)]$$
$$= \begin{cases} 0, & l=1,\ldots,i-1, \\ n_i, & l=i, \\ n_l + \sum_{k=1}^{i-1}\sum_{m_k=1}^{n_k} E[N_{kl}^i(v_{km_k})] + E[N_l^i(S_i)], & l=i+1,\ldots,J, \end{cases} \quad (5.52)$$

where $1 \le i \le J$.

Next we attempt to generalize the results.

Generalizations of the results.

As we have seen above, the aggregated system state after the initial sojourn times are the sum of the state starting with the tagged customer i and the every initial customer in X_i. Then we may generally assume the followings:

$$E[\overline{V}_l^*|(i),(V,n)] = \sum_{k=1}^{J}\{\overline{\alpha}_{kl}^i\overline{V}_k + \overline{\beta}_{kl}^i n_k\} + \overline{\gamma}_l^i, \quad (5.53)$$
$$E[n_l^*|(i),(V,n)] = \sum_{k=1}^{J}\{\alpha_{kl}^i\overline{V}_k + \beta_{kl}^i n_k\} + \gamma_l^i, \quad (5.54)$$

where $1 \le i,l \le J$. The explicit values of these constants for the FCFS discipline and the PR-LCFS discipline can be obtained by the results in the last section. In the case of the FCFS discipline, we can obtain from (4.11), (4.12), (4.13) and (4.14),

$$E^F[\overline{V}_l^*|(i),(V,n)] = E[V_l^{i-1}(V,n;X_i^F) + V_l^{i-1}(S_i)]$$
$$= \begin{cases} 0, & l=1,\ldots,i-1, \\ \sum_{k=1}^{i}\left\{\overline{\xi}_i^{i-1}\overline{V}_k + \overline{\chi}_{ki}^{i-1}n_k\right\} + \overline{\xi}_i^{i-1}E[S_i], & l=i, \\ \overline{V}_l + \sum_{k=1}^{i}\left\{\overline{\xi}_l^{i-1}\overline{V}_k + \overline{\chi}_{kl}^{i-1}n_k\right\} + \overline{\xi}_l^{i-1}E[S_i], & l=i+1,\ldots,J, \end{cases} \quad (5.55)$$

$$E^F[n_l^*|(i),(V,n)] = E[N_l^{i-1}(V,n;X_i^F) + N_l^{i-1}(S_i)]$$
$$= \begin{cases} 0, & l=1,\ldots,i-1, \\ \sum_{k=1}^{i}\left\{\xi_i^{i-1}\overline{V}_k + \chi_{ki}^{i-1}n_k\right\} + \xi_i^{i-1}E[S_i], & l=i, \\ n_l + \sum_{k=1}^{i}\left\{\xi_l^{i-1}\overline{V}_k + \chi_{kl}^{i-1}n_k\right\} + \xi_l^{i-1}E[S_i], & l=i+1,\ldots,J, \end{cases} \quad (5.56)$$

where $1 \le i \le J$. We distinguish the FCFS discipline by the superscript F. In the case of the PR-LCFS discipline, we can obtain from (4.11), (4.12), (4.13) and (4.14),

$$E^{PL}[\overline{V}_l^*|(i),(V,n)] = E[V_l^i(V,n;X_i^{PL}) + V_l^i(S_i)]$$
$$= \begin{cases} 0, & l=1,\ldots,i-1, \\ \overline{V}_i, & l=i, \\ \overline{V}_l + \sum_{k=1}^{i-1}\left\{\overline{\xi}_l^i\overline{V}_k + \overline{\chi}_{kl}^i n_k\right\} + \overline{\xi}_l^i E[S_i], & l=i+1,\ldots,J, \end{cases} \quad (5.57)$$

$$E^{PL}[n_l^*|(i),(V,n)] = E[N_l^i(V,n;X_i^{PL}) + N_l^i(S_i)]$$
$$= \begin{cases} 0, & l=1,\ldots,i-1, \\ n_i, & l=i, \\ n_l + \sum_{k=1}^{i-1}\left\{\xi_l^i\overline{V}_k + \chi_{kl}^i n_k\right\} + \xi_l^i E[S_i], & l=i+1,\ldots,J, \end{cases} \quad (5.58)$$

where $1 \le i \le J$. We distinguish the PR-LCFS discipline by the superscript PL. For the notational convenience, we define the following matrices and vectors.

$$\begin{aligned} \overline{A}^i &= (\overline{\alpha}_{kl}^i : k,l=1,\ldots,J), & \overline{B}^i &= (\overline{\beta}_{kl}^i : k,l=1,\ldots,J), \\ A^i &= (\alpha_{kl}^i : k,l=1,\ldots,J), & B^i &= (\beta_{kl}^i : k,l=1,\ldots,J), \\ \overline{\Gamma}^i &= (\overline{\gamma}_1^i,\ldots,\overline{\gamma}_J^i), & \Gamma^i &= (\gamma_1^i,\ldots,\gamma_J^i), \end{aligned} \quad (5.59)$$

where $1 \leq i \leq J$. We further aggregate the matrices and the vectors as follows.

$$\mathcal{U}^i = \begin{pmatrix} \overline{A}^i & A^i \\ \overline{B}^i & B^i \end{pmatrix}, \tag{5.60}$$

$$\mathbf{u}^i = (\overline{\Gamma}^i, \Gamma^i). \tag{5.61}$$

Then, the assumptions (5.7) and (5.8) are arranged as follows.

$$E[(\overline{V}^*, n^*)|(i), (V, n)] = (\overline{V}, n)\mathcal{U}^i + \mathbf{u}^i, \tag{5.62}$$

where $1 \leq i \leq J$. The expression will be cited later to derive the cost functions.

Similarly as the initial cost functions, the important things to consider about the assumption are that the aggregated system state $(\overline{V}, n)$ should be sufficient for estimating the aggregated system state $(\overline{V}^*, n^*)$ after the initial sojourn time W_i, and that the expected value of the aggregated system state after the initial sojourn time should be the linear function of the aggregated system state. Of course, the coefficients $\mathcal{U}^i$ and $\mathbf{u}^i$ ($i = 1, \ldots, J$) can be different for every service discipline.

6 Expressions of the cost functions.

In this section, we derive the explicit formulae of the cost functions $W_{ij}(\cdot)$ and $G_{ij}(\cdot)$ under the assumptions stated in the last sections. As we have defined, $W_{ij}(V, n)$ is the mean sojourn time of the tagged customer, who is currently at the station i when the current system state is (V, n), spent at the station j until he departs from the system, and $G_{ij}(V, n)$ is the cumulative current works of the same tagged customer accumulated at the station j until he departs from the system, $1 \leq i, j \leq J$.

We put again the assumptions that are derived from the analysis in the previous sections.

Assumption 6.1.

$$W_i(V, n) = (\overline{V}, n)\mathcal{W}^i + w^i, \tag{6.63}$$

$$G_i(V, n) = (\overline{V}, n)\mathcal{G}^i + g^i, \tag{6.64}$$

$$E[(\overline{V}^*, n^*)|(i), (V, n)] = (\overline{V}, n)\mathcal{U}^i + \mathbf{u}^i \quad \in R^{1\times 2J}, \tag{6.65}$$

where $1 \leq i \leq J$.

Of course, the assumption is satisfied by the FCFS and PR-LCFS disciplines.

By fixing the index j ($j = 1, \ldots, J$), let us consider the cost functions for the station j. Now we define the following matrices. Let $\mathcal{J} = 2J^2$.

$$\overline{W} = (0, \ldots, 0, \mathcal{W}^{j\prime}, 0, \ldots, 0)' \quad \in R^{\mathcal{J}\times 1},$$

$$\overline{G} = (0, \ldots, 0, \mathcal{G}^{j\prime}, 0, \ldots, 0)' \quad \in R^{\mathcal{J}\times 1},$$

$$Q = \begin{pmatrix} p_{11}I_0 & p_{12}I_0 & \cdots & p_{1J}I_0 \\ p_{21}I_0 & p_{22}I_0 & \cdots & p_{2J}I_0 \\ \vdots & \vdots & \ddots & \vdots \\ p_{J1}I_0 & p_{J2}I_0 & \cdots & p_{JJ}I_0 \end{pmatrix} \in R^{\mathcal{J}\times\mathcal{J}},$$

$$I_0 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \in R^{2J\times 2J},$$

$$U = \begin{pmatrix} \mathcal{U}^1 & 0 & \cdots & 0 \\ 0 & \mathcal{U}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{U}^J \end{pmatrix} \in R^{\mathcal{J}\times\mathcal{J}},$$

where $'$ denotes the transposition. Then we have

$$UQ = \begin{pmatrix} \mathcal{U}^1 p_{11} & \mathcal{U}^1 p_{12} & \cdots & \mathcal{U}^1 p_{1J} \\ \mathcal{U}^2 p_{21} & \mathcal{U}^2 p_{22} & \cdots & \mathcal{U}^2 p_{2J} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{U}^J p_{J1} & \mathcal{U}^J p_{J2} & \cdots & \mathcal{U}^J p_{JJ} \end{pmatrix} \in R^{\mathcal{J}\times\mathcal{J}}.$$

We now suppose $(I - UQ)^{-1}$ exists where I is an identity matrix in $R^{\mathcal{J}\times\mathcal{J}}$. Then we can define

$$W = \begin{pmatrix} \mathcal{W}_{1j} \\ \vdots \\ \mathcal{W}_{Jj} \end{pmatrix} = (I - UQ)^{-1}\overline{W} \in R^{\mathcal{J}\times 1}, \tag{6.66}$$

$$G = \begin{pmatrix} \mathcal{G}_{1j} \\ \vdots \\ \mathcal{G}_{Jj} \end{pmatrix} = (I - UQ)^{-1}\overline{G} \in R^{\mathcal{J}\times 1}. \tag{6.67}$$

Further we define

$$\overline{w} = \begin{pmatrix} \mathbf{u}^1 \sum_{k=1}^J p_{1k}\mathcal{W}_{kj} \\ \vdots \\ \mathbf{u}^{j-1} \sum_{k=1}^J p_{j-1k}\mathcal{W}_{kj} \\ w^j + \mathbf{u}^j \sum_{k=1}^J p_{jk}\mathcal{W}_{kj} \\ \mathbf{u}^{j+1} \sum_{k=1}^J p_{j+1k}\mathcal{W}_{kj} \\ \vdots \\ \mathbf{u}^J \sum_{k=1}^J p_{Jk}\mathcal{W}_{kj} \end{pmatrix} \in R^{J\times 1},$$

$$\overline{g} = \begin{pmatrix} \mathbf{u}^1 \sum_{k=1}^J p_{1k}\mathcal{G}_{kj} \\ \vdots \\ \mathbf{u}^{j-1} \sum_{k=1}^J p_{j-1k}\mathcal{G}_{kj} \\ g^j + \mathbf{u}^j \sum_{k=1}^J p_{jk}\mathcal{G}_{kj} \\ \mathbf{u}^{j+1} \sum_{k=1}^J p_{j+1k}\mathcal{G}_{kj} \\ \vdots \\ \mathbf{u}^J \sum_{k=1}^J p_{Jk}\mathcal{G}_{kj} \end{pmatrix} \in R^{J\times 1}.$$

From **Assumption 2.1**, $(I - P_J)^{-1}$ exists. Then we can define

$$w = \begin{pmatrix} w_{1j} \\ \vdots \\ w_{Jj} \end{pmatrix} = (I - P_J)^{-1}\overline{w} \in R^{J\times 1}, \tag{6.68}$$

$$g = \begin{pmatrix} g_{1j} \\ \vdots \\ g_{Jj} \end{pmatrix} = (I - P_J)^{-1}\overline{g} \in R^{J\times 1}. \tag{6.69}$$

As stated in the previous sections, the coefficients $\mathcal{W}^i, w^i$, $\mathcal{G}^i$ and g^i can be different for every service discipline. So these vectors and matrices are also different for every service discipline.

The following theorem is now derived.

Theorem 1. By fixing the index j $(j = 1, \ldots, J)$, let us consider the cost functions for the station j. We put **Assumption 2.1** and **Assumption 6.1**. Further if we assume that $(I - UQ)^{-1}$ exists, then

$$W_{ij}(V,n) = (\overline{V},n)\mathcal{W}_{ij} + w_{ij}, \quad i = 1, \ldots, J, \tag{6.70}$$
$$G_{ij}(V,n) = (\overline{V},n)\mathcal{G}_{ij} + g_{ij}, \quad i = 1, \ldots, J, \tag{6.71}$$

are the solutions of the the equations (2.6) and (2.9), respectively.

Proof. We show that (6.8) and (6.9) satisfy the equations (2.6) and (2.9), respectively. For $i \neq j$,

$$\begin{aligned}
& E[W_{Kj}(V^*,n^*)|(i),(V,n)] \\
=\ & \sum_{k=1}^{J} p_{ik} E[(\overline{V}^*,n^*)\mathcal{W}_{kj} + w_{kj}|(i),(V,n)] \\
=\ & \sum_{k=1}^{J} p_{ik} \left[E[(\overline{V}^*,n^*)|(i),(V,n)]\mathcal{W}_{kj} + w_{kj}\right] \\
=\ & \sum_{k=1}^{J} p_{ik} \left[\left\{(\overline{V},n)\mathcal{U}^i + \mathbf{u}^i\right\}\mathcal{W}_{kj} + w_{kj}\right] \\
=\ & (\overline{V},n)\mathcal{U}^i \sum_{k=1}^{J} p_{ik}\mathcal{W}_{kj} + \left\{\mathbf{u}^i \sum_{k=1}^{J} p_{ik}\mathcal{W}_{kj} + \sum_{k=1}^{J} p_{ik} w_{kj}\right\} \\
=\ & W_{ij}(V,n).
\end{aligned} \tag{6.72}$$

The last equation follows from the definition of the constants $\mathcal{W}_{ij}$ and w_{ij}, that is,

$$\mathcal{W}_{ij} = \mathcal{U}^i \sum_{k=1}^{J} p_{ik}\mathcal{W}_{kj}, \tag{6.73}$$
$$w_{ij} = \mathbf{u}^i \sum_{k=1}^{J} p_{ik}\mathcal{W}_{kj} + \sum_{k=1}^{J} p_{ik} w_{kj}. \tag{6.74}$$

Hence $W_{ij}(V,n)$ satisfies the equation (2.6). In the same manner, we can show that $G_{ij}(V,n)$ satisfies the equation (2.9) for $i \neq j$.

For $i = j$,

$$\begin{aligned}
& W_j(V,n) + E[W_{Kj}(V^*,n^*)|(j),(V,n)] \\
=\ & (\overline{V},n)\mathcal{W}^j + w^j + \sum_{k=1}^{J} p_{jk} E[(\overline{V}^*,n^*)\mathcal{W}_{kj} + w_{kj}|(j),(V,n)] \\
=\ & (\overline{V},n)\mathcal{W}^j + w^j + \sum_{k=1}^{J} p_{jk} \left\{E[(\overline{V}^*,n^*)|(j),(V,n)]\mathcal{W}_{kj} + w_{kj}\right\} \\
=\ & (\overline{V},n)\mathcal{W}^j + w^j + \sum_{k=1}^{J} p_{jk} \left[\left\{(\overline{V},n)\mathcal{U}^j + \mathbf{u}^j\right\}\mathcal{W}_{kj} + w_{kj}\right] \\
=\ & (\overline{V},n)\left\{\mathcal{W}^j + \mathcal{U}^j \sum_{k=1}^{J} p_{jk}\mathcal{W}_{kj}\right\} + \left\{w^j + \mathbf{u}^j \sum_{k=1}^{J} p_{jk}\mathcal{W}_{kj} + \sum_{k=1}^{J} p_{jk} w_{kj}\right\} \\
=\ & W_{jj}(V,n).
\end{aligned} \tag{6.75}$$

The last equation follows from the definition of the constants $\mathcal{W}_{jj}$ and w_{jj}, that is,

$$\mathcal{W}_{jj} = \mathcal{W}^j + \mathcal{U}^j \sum_{k=1}^{J} p_{jk} \mathcal{W}_{kj}, \tag{6.76}$$

$$w_{jj} = w^j + \mathbf{u}^j \sum_{k=1}^{J} p_{jk} \mathcal{W}_{kj} + \sum_{k=1}^{J} p_{jk} w_{kj}. \tag{6.77}$$

Hence $W_{jj}(V,n)$ satisfies the equation (2.6). In the same manner, we can show that $G_{jj}(V,n)$ satisfies the equation (2.9). □

7 The system in the steady state.

We have considered the system in the arbitrary states. In the following sections, we consider the system with the steady state.

Let us consider the system operated under some fixed service discipline. $\{(V(t), n(t)) : t \geq 0\}$ is the stochastic process associated with the system state (V, n). The state space is denoted by $\mathcal{E}$. Each component $\{v_{jm_j}(t) : t \geq 0\}$ ($j = 1, \ldots, J$ and $m_j = 1, 2, \ldots$) of the process $\{V(t) : t \geq 0\}$ is a process which is left continuous with right-hand limits. It is assumed that these components v_{jm_j} are appropriately arranged in order to know the service order of the customers. For example, these may be arranged for every class of customers in the order of the arrivals in the case of the FCFS and PR-LCFS disciplines. Each component $\{n_j(t) : t \geq 0\}$ ($j = 1, \ldots, J$) of the process $\{n(t) : t \geq 0\}$ is a jump process which is left continuous with right-hand limits at the upward jump and which is right continuous with left-hand limits at the downward jump. Further we assume that $\{(V(t), n(t)) : t \geq 0\}$ is a Markov process. From the nature of the Poisson processes, the overall arrival process $\{A(t) = \sum_{i=1}^{J} A_i(t) : t \geq 0\}$ is also a Poisson process. The total arrival rate of the customers is equal to $\Lambda = \sum_{j=1}^{J} \lambda_j$. Let T_m be the arrival epoch of the m^{th} arriving customer and let W_j^m ($1 \leq j \leq J$) be the total amount of the sojourn time of the customer spend at the station j. G_j^m ($1 \leq j \leq J$) is defined as the cumulative current works at the station j of the m^{th} arriving customer. We assume that W_j^m and G_j^m can be expressed as

$$W_j^m = \mathbf{W}_j(V(T_m), n(T_m)), \tag{7.78}$$

$$G_j^m = \mathbf{G}_j(V(T_m), n(T_m)), \tag{7.79}$$

where $m = 1, 2, \ldots$ and $j = 1, \ldots, J$. From the Markov property, the FCFS and PR-LCFS disciplines satisfy these assumptions. Further we define the stochastic process $\{(\overline{V}(t), n(t)) : t \geq 0\}$ associated with the aggregated system state $(\overline{V}, n)$, and let $\Longrightarrow$ denote convergence in distribution. Each component $\{\overline{V}_j(t) : t \geq 0\}$ ($j = 1, \ldots, J$) of $\{\overline{V}(t) : t \geq 0\}$ is a process which is left continuous with right-hand limits, and $\{n(t) : t \geq 0\}$ is a jump process defined above.

Now we are thinking about the steady state, then we are willing to assume that

$$(V(T_m), n(T_m)) \Longrightarrow (V(T_\infty), n(T_\infty)) \quad \text{as } m \to \infty, \tag{7.80}$$

$$(\overline{V}(T_m), n(T_m)) \Longrightarrow (\overline{V}(T_\infty), n(T_\infty)) \quad \text{as } m \to \infty, \tag{7.81}$$

as well as

$$\lim_{N \to \infty} \frac{1}{N} \sum_{m=1}^{N} W_j^m = E[\mathbf{W}_j(V(T_\infty), n(T_\infty))] < \infty, \tag{7.82}$$

$$\lim_{N \to \infty} \frac{1}{N} \sum_{m=1}^{N} G_j^m = E[\mathbf{G}_j(V(T_\infty), n(T_\infty))] < \infty, \tag{7.83}$$

and

$$\lim_{t\to\infty} \frac{1}{A(t)} \int_0^t n_j(s)dA(s) = E[n_j(T_\infty)] < \infty, \quad (7.84)$$

$$\lim_{t\to\infty} \frac{1}{A(t)} \int_0^t \overline{V}_j(s)dA(s) = E[\overline{V}_j(T_\infty)] < \infty, \quad (7.85)$$

where $1 \leq j \leq J$. The time average values $(\tilde{V}, \tilde{n})$ of the aggregated system state are defined as follows:

$$\tilde{n}_j = \lim_{t\to\infty} \frac{1}{t} \int_0^t n_j(s)ds, \quad (7.86)$$

$$\tilde{V}_j = \lim_{t\to\infty} \frac{1}{t} \int_0^t \overline{V}_j(s)ds, \quad (7.87)$$

where $1 \leq j \leq J$.

In the following discussions, the assumptions in this section are assumed to be satisfied.

8 Uniqueness of the cost functions.

We have obtained the set of the solutions of the cost functions W_{ij} and G_{ij} in Theorem 1. Now we prove the uniqueness of the solutions under the appropriate assumptions.

Let $\mathcal{T}$ be the set of the stopping times [7] for the process $\{(V(t), n(t)) : t \geq 0\}$. The set $\mathcal{T}$ is restricted to the subset $\mathcal{T}^0$ such that for some matrix $\mathcal{B}^0 \geq 0$ and some vector $\mathbf{b}^0 \geq 0$,

$$\mathcal{T}^0 = \left\{T \in \mathcal{T} : E[(\overline{V}(T), n(T))|(V, n)] \leq (\overline{V}, n)\mathcal{B}^0 + \mathbf{b}^0, \quad \forall(V, n) \in \mathcal{E}\right\} \quad (8.88)$$

where the initial condition $\{(V(0), n(0)) = (V, n)\}$ is simply denoted by (V, n). T_∞ can be included in $\mathcal{T}^0$. Since from the steady state assumption (7.7) and (7.8), there exists a vector $\mathbf{b}_0$ such that

$$E[(\overline{V}(T_\infty), n(T_\infty))|(V, n)] = \mathbf{b}_0, \quad (8.89)$$

for all $(V, n) \in \mathcal{E}$. Also $0 \in \mathcal{T}^0$. Further, for all $t \geq 0$, let $\tau^i_m(t)$ $(i = 1, \ldots, J; m = 1, 2, \ldots)$ be the time such that if the tagged customer were to arrive at the station i at time t, then the customer would complete his m^{th} *current service* (service per visit to one of the stations) at that time. For convenience, let $\tau^i_0(t) = t$ $(i = 1, \ldots, J)$. Then we define

$$\mathcal{T}^1 = \{\tau^i_m(T) : T \in \mathcal{T}^0; i = 1, \ldots, J; m = 0, 1, 2, \ldots\}. \quad (8.90)$$

Before proving the uniqueness of the cost functions, we need the following lemma.

Lemma. We put **Assumption 2.1** and **Assumption 6.1**. Further we assume that $(UQ)^m \to 0$ as $m \to \infty$. Then there exist a matrix $\mathcal{B}^1 \in R^{2J\times 2J}$ and a vector $\mathbf{b}^1 \in R^{1\times 2J}$ such that

$$\sum_{m=0}^{\infty} E[(\overline{V}(\tau^i_m(t)), n(\tau^i_m(t)))|(V(t), n(t))] \leq (\overline{V}(t), n(t))\mathcal{B}^1 + \mathbf{b}^1, \quad (8.91)$$

for all $t \geq 0$ and $i = 1, \ldots, J$.

Proof. From **Assumption 6.1**, it can be easily shown that

$$E[(\overline{V}(\tau_m), n(\tau_m))|(i), (V, n)] = (\overline{V}, n)Y^i(QU)^{m-1}\overline{E} + \mathbf{e}_i \sum_{k=0}^{m-1} P_J^{m-k-1}X(QU)^k\overline{E} \in R^{1\times 2J} \quad (8.92)$$

where $\tau_m = \tau_m^i(t)$ $(m \geq 1)$ and $(V,n) = (V(t), n(t))$, and where

$$\begin{aligned}
Y^i &= (\mathbf{O},\ldots,\mathbf{O},\mathcal{U}^i,\mathbf{O},\ldots,\mathbf{O}) \in R^{2J\times\mathcal{J}},\\
X &= \begin{pmatrix} \mathbf{u}^1 & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{u}^2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{u}^J \end{pmatrix} \in R^{J\times\mathcal{J}},\\
\mathbf{e}_i &= (0,\ldots,0,1,0,\ldots,0) \in R^{1\times J},\\
\overline{E} &= \begin{pmatrix} I_0 \\ \vdots \\ I_0 \end{pmatrix} \in R^{\mathcal{J}\times 2J},
\end{aligned}$$

and the other constants are defined in Section 6. Hence we have

$$\begin{aligned}
&\sum_{m=1}^{\infty} E[(\overline{V}(\tau_m), n(\tau_m))|(i),(V,n)]\\
&= (\overline{V},n)Y^i \sum_{m=1}^{\infty}(QU)^{m-1}\overline{E} + \mathbf{e}_i \sum_{m=1}^{\infty}\sum_{k=0}^{m-1} P_J^{m-k-1}X(QU)^k\overline{E}\\
&= \{(\overline{V},n)Y^i + \mathbf{e}_i(I-P_J)^{-1}X\}\{I + Q(I-UQ)^{-1}U\}\overline{E}
\end{aligned}$$

The last equality holds from **Assumption 2.1** and the assumption that $(UQ)^m \to 0$ as $m \to \infty$.
□

Now we introduce the normed linear spaces of functions [19]. Let f be a function defined on $\mathcal{E}$ into R, and let

$$||f||_{(V,n)} = \sup\left\{E\left[|f(V(T),n(T))|\,|(V,n)\right] : T \in \mathcal{T}^1\right\} \tag{8.93}$$

for every initial system state $(V(0), n(0)) = (V,n)$. Then the set of the functions is defined as follows:

$$\Pi_{(V,n)} = \{f | f : \mathcal{E} \to R \text{ and } ||f||_{(V,n)} < \infty\} \tag{8.94}$$

for every $(V,n) \in \mathcal{E}$. As usual, we shall consider that the function $f \in \Pi_{(V,n)}$ is equivalent to $0 \in \Pi_{(V,n)}$ if $||f||_{(V,n)} = 0$. Then the set $\Pi_{(V,n)}$ becomes a normed linear space for every $(V,n) \in \mathcal{E}$. Further we define the space Π such that

$$\Pi = \bigcap_{(V,n)\in\mathcal{E}} \Pi_{(V,n)}. \tag{8.95}$$

We then obtain the following uniqueness of the solutions.

Theorem 2. Let $\{W_{ij} : i = 1,\ldots,J\}$ and $\{G_{ij} : i = 1,\ldots,J\}$ respectively be the sets of the cost functions, which are defined by (2.4) and (2.7), satisfy the equations (2.6) and (2.9) ($j = 1,\ldots,J$). We put **Assumption 2.1** and **Assumption 6.1**. Further we assume that $(UQ)^m \to 0$ as $m \to \infty$. Then each components W_{ij} and G_{ij} of the sets are the unique elements of Π.

Proof. Fix the initial state $(V,n) \in \mathcal{E}$ and the index j $(j = 1,\ldots,J)$. We first show that W_{ij} and G_{ij} are the elements of $\Pi_{(V,n)}$ for $i = 1,\ldots,J$. For $T \in \mathcal{T}^1$, we have

$$W_{ij}(V(T),n(T)) \leq \sum_{j=1}^{J} W_{ij}(V(T),n(T))$$

$$
\begin{aligned}
&= E[\sum_{m=1}^{M} W_{X_m}(V(\tau^i_{m-1}(T)), n(\tau^i_{m-1}(T)))|(V(T), n(T))] \\
&= E[\sum_{m=1}^{M} \{(\overline{V}(\tau^i_{m-1}(T)), n(\tau^i_{m-1}(T)))\mathcal{W}^{X_m} + w^{X_m}\}|(V(T), n(T))] \\
&\leq E[\sum_{m=0}^{\infty} (\overline{V}(\tau^i_m(T)), n(\tau^i_m(T)))|(V(T), n(T))]\mathcal{W}^0 + E[M]w^0
\end{aligned}
$$

where M is the number of visits to the stations and X_m denotes the station staying at the m^{th} visit, and where $\mathcal{W}^0 = (\max_k \phi^k_{11}, \ldots, \max_k \phi^k_{1J}, \max_k \phi^k_{21}, \ldots, \max_k \phi^k_{2J})'$, $w^0 = \max_k w^k$ and $E[M] = E[M|(i), (V(T), n(T))] = \mathbf{e}_i(I - P_J)^{-1}\mathbf{1}$. Hence we have

$$
\begin{aligned}
E[W_{ij}(V(T), n(T))|(V, n)] &\leq \{E[(\overline{V}(T), n(T))|(V, n)]\mathcal{B}^1 + \mathbf{b}^1\}\mathcal{W}^0 + \mathbf{e}_i(I - P_J)^{-1}\mathbf{1}w^0 \\
&\leq \{\{\{(\overline{V}, n)\mathcal{B}^0 + \mathbf{b}^0\}\mathcal{B}^1 + \mathbf{b}^1\}\mathcal{B}^1 + \mathbf{b}^1\}\mathcal{W}^0 + \mathbf{e}_i(I - P_J)^{-1}\mathbf{1}w^0.
\end{aligned}
$$

These inequalities comes from the Lemma and the definition that $T \in \mathcal{T}^1$. The similar inequality holds for $E[G_{ij}(V(T), n(T))|(V, n)]$, $(\forall T \in \mathcal{T}^1$ and $i = 1, \ldots, J)$.

Let $\{W^1_{ij} \in \Pi : i = 1, \ldots, J\}$ and $\{W^2_{ij} \in \Pi : i = 1, \ldots, J\}$ be any two sets of the solutions of the equation (2.6). Then

$$
\begin{aligned}
& ||W^1_{ij} - W^2_{ij}||_{(V,n)} \\
=\ & \sup_{T \in \mathcal{T}^1} E\left[\left|W^1_{ij}(V(T), n(T)) - W^2_{ij}(V(T), n(T))\right| |(V, n)\right] \\
\leq\ & \sup_{T \in \mathcal{T}^1} \sum_{k=1}^{J} p_{ik} E\left[E\left[\left|W^1_{kj}(V(T^*), n(T^*)) - W^2_{kj}(V(T^*), n(T^*))\right| |(V(T), n(T))\right] |(V, n)\right] \\
=\ & \sup_{T \in \mathcal{T}^1} \sum_{k=1}^{J} p_{ik} E\left[\left|W^1_{kj}(V(T^*), n(T^*)) - W^2_{kj}(V(T^*), n(T^*))\right| |(V, n)\right] \\
\leq\ & \sup_{T \in \mathcal{T}^1} \sum_{k=1}^{J} p_{ik} \sup_{T^* \in \mathcal{T}^1} E\left[\left|W^1_{kj}(V(T^*), n(T^*)) - W^2_{kj}(V(T^*), n(T^*))\right| |(V, n)\right] \\
=\ & \sup_{T \in \mathcal{T}^1} \sum_{k=1}^{J} p_{ik} ||W^1_{kj} - W^2_{kj}||_{(V,n)} \\
=\ & \sum_{k=1}^{J} p_{ik} ||W^1_{kj} - W^2_{kj}||_{(V,n)}.
\end{aligned}
$$

The last inequality comes from the fact that T^* is a stopping time in $\mathcal{T}^1$, since T^* is the service completion epoch of the tagged customer. Let $\langle W_j \rangle = (||W_{1j}||_{(V,n)}, \ldots, ||W_{Jj}||_{(V,n)})'$. Then

$$
\begin{aligned}
0 &\leq \langle W^1_j - W^2_j \rangle \\
&\leq P_J \langle W^1_j - W^2_j \rangle \\
&\cdots \\
&\leq P^n_J \langle W^1_j - W^2_j \rangle \to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

The last expression comes from **Assumption 2.1**. Hence we have

$$
W^1_{ij} = W^2_{ij}, \quad i = 1, \ldots, J, \tag{8.96}
$$

as the elements of $\Pi_{(V,n)}$. The similar argument shows the uniqueness of the function G_{ij} $(i = 1, \ldots, J)$.

Since the state $(V, n) \in \mathcal{E}$ and the index j $(j = 1, \ldots, J)$ are arbitrary, the proof is completed. □

Corollary. Under the assumptions of Theorem 2, the sets of the solutions $\{W_{ij}(\cdot) : i = 1, \ldots, J\}$ and $\{G_{ij}(\cdot) : i = 1, \ldots, J\}$ in Theorem 1 are unique for all $j = 1, \ldots, J$.

Proof. Let $\{W_{ij}^1(\cdot) : i = 1, \ldots, J\}$ and $\{W_{ij}^2(\cdot) : i = 1, \ldots, J\}$ be any two sets of the solutions of the equation (2.6). By the assumptions of Theorem 2, any cost function $W_{ij}(\cdot)$ defined by (2.4) is in Π. Hence, $W_{ij}^1(\cdot) \in \Pi$ and $W_{ij}^2(\cdot) \in \Pi$. Then we have

$$\sup \left\{ E\left[\left| W_{ij}^1(V(T), n(T)) - W_{ij}^2(V(T), n(T)) \right| |(V,n) \right] : T \in \mathcal{T}^1 \right\} = 0 \qquad (8.97)$$

for all $(V, n) \in \mathcal{E}$ and $i = 1, \ldots, J$. In particular, $T = 0 \in \mathcal{T}^1$. Then

$$|W_{ij}^1(V,n) - W_{ij}^2(V,n)| = 0 \qquad (8.98)$$

for all $i, j = 1, \ldots, J$ and for all $(V, n) \in \mathcal{E}$. In the same manner, we can show the uniqueness of the function $G_{ij}(\cdot)$. This completes the proof. □

Remark. The reason why we consider the set $\mathcal{T}^1$ is to ensure that the every objective function f (e.g. $f = W_{ij}$, or G_{ij}) satisfies that $\|f\|_{(V,n)} < \infty$ for all initial state (V, n).

9 Steady state values of the cost functions.

Now we evaluate the steady state values of the cost functions W_{ij} and G_{ij}. The generalized Little's formula ($H = \lambda G$) and the Poisson Arrivals See Time Averages (PASTA) property are used to obtain the values. The power of these results is fully utilized. Finally, some numerical examples are provided.

Because the cost functions are unique over Π under the assumptions of Theorem 2 and $T_\infty \in \mathcal{T}^1$, we have

$$E[\mathbf{W}_j(V(T_\infty), n(T_\infty))] = \sum_{i=1}^{J} \frac{\lambda_i}{\Lambda} \left\{ E[(\overline{V}(T_\infty), n(T_\infty))] \mathcal{W}_{ij} + w_{ij} \right\}, \qquad (9.99)$$

$$E[\mathbf{G}_j(V(T_\infty), n(T_\infty))] = \sum_{i=1}^{J} \frac{\lambda_i}{\Lambda} \left\{ E[(\overline{V}(T_\infty), n(T_\infty))] \mathcal{G}_{ij} + g_{ij} \right\}, \qquad (9.100)$$

where $\Lambda = \sum_{i=1}^{J} \lambda_i$ is the overall arrival rate. We use the generalized Little's formula [23] that equates the time average values of the costs with the customer average values of the costs to obtain

$$\tilde{n}_j = \Lambda E[\mathbf{W}_j(V(T_\infty), n(T_\infty))], \qquad (9.101)$$

$$\tilde{V}_j = \Lambda E[\mathbf{G}_j(V(T_\infty), n(T_\infty))]. \qquad (9.102)$$

For the Poisson arrival, the fraction of time that the system is in any state is equal to the fraction of the arrivals when the system is in the state. This is the Poisson Arrivals See Time Averages property (PASTA) [24]. Then we have

$$\tilde{n}_j = E[n_j(T_\infty)], \qquad (9.103)$$

$$\tilde{V}_j = E[\overline{V}_j(T_\infty)]. \qquad (9.104)$$

From the equations between (9.1) and (9.6), we obtain

$$\tilde{n}_j = \sum_{i=1}^{J} \lambda_i \left\{ (\tilde{V}, \tilde{n}) \mathcal{W}_{ij} + w_{ij} \right\}, \quad j = 1, \ldots, J, \tag{9.105}$$

$$\tilde{V}_j = \sum_{i=1}^{J} \lambda_i \left\{ (\tilde{V}, \tilde{n}) \mathcal{G}_{ij} + g_{ij} \right\}, \quad j = 1, \ldots, J. \tag{9.106}$$

Define

$$\mathcal{S} = \sum_{i=1}^{J} \lambda_i (\mathcal{G}_{i1}, \ldots, \mathcal{G}_{iJ}, \mathcal{W}_{i1}, \ldots, \mathcal{W}_{iJ}), \tag{9.107}$$

$$s = \sum_{i=1}^{J} \lambda_i (g_{i1}, \ldots, g_{iJ}, w_{i1}, \ldots, w_{iJ}). \tag{9.108}$$

Then we arrive at the equation that determines the steady state expected value of the (aggregated) system state $(\tilde{V}, \tilde{n})$:

$$(\tilde{V}, \tilde{n}) = (\tilde{V}, \tilde{n}) \mathcal{S} + s. \tag{9.109}$$

Now we assume that the inverse matrix $(I - \mathcal{S})^{-1}$ exists. Then we have

$$(\tilde{V}, \tilde{n}) = s(I - \mathcal{S})^{-1}. \tag{9.110}$$

Finally, we can get the steady state values of the cost functions.

$$E[W_{ij}(V(T_\infty), n(T_\infty))] = s(I - \mathcal{S})^{-1} \mathcal{W}_{ij} + w_{ij}. \tag{9.111}$$

$$E[G_{ij}(V(T_\infty), n(T_\infty))] = s(I - \mathcal{S})^{-1} \mathcal{G}_{ij} + g_{ij}. \tag{9.112}$$

The first cost function denotes the (steady state) *mean sojourn time* that the customer arriving at the station i spends at the station j until his departure from the system.

These results are arranged in the following theorem.

Theorem 3. Assume that the multiclass M/G/1 system with feedback defined in Section 2 is in the steady state where the assumptions in Section 7 hold. Let $\tilde{n} = (\tilde{n}_1, \ldots, \tilde{n}_J)$ and $\tilde{V} = (\tilde{V}_1, \ldots, \tilde{V}_J)$ be the vectors of the *number of customers* and the *total amount of works* in the system, respectively. Further, let $\tilde{W}_{ij}$ $(1 \le i, j \le J)$ be the *mean sojourn time* of the customers, who initially arrive at the station i from the outside of the system, spend at the station j until their departure from the system. We put **Assumption 2.1** and **Assumption 6.1**. We assume that $(UQ)^m \to 0$ as $m \to \infty$ and that the inverse matrix $(I - \mathcal{S})^{-1}$ exists where UQ is the state transition matrix defined in Section 6 and where $\mathcal{S}$ is defined in (9.9). Then

$$(\tilde{V}, \tilde{n}) = s(I - \mathcal{S})^{-1}, \tag{9.113}$$

$$\tilde{W}_{ij} = s(I - \mathcal{S})^{-1} \mathcal{W}_{ij} + w_{ij}, \tag{9.114}$$

where s is defined in (9.10), and where $\mathcal{W}_{ij}$ and other constants are defined in Section 6. □

Of course, the total sojourn time $\tilde{W}_i$ of the customer, who initially arrives at the station i, spend from his arrival to his departure is

$$\tilde{W}_i = \sum_{j=1}^{J} \tilde{W}_{ij} = s(I - \mathcal{S})^{-1} \sum_{j=1}^{J} \mathcal{W}_{ij} + \sum_{j=1}^{J} w_{ij}. \tag{9.115}$$

Remark. We have not investigated the conditions of the existence of the inverse matrices and the existence of the steady state in detail. The elaboration in the direction will be required.

Numerical examples and the graphs.

Now we give numerical examples of the model. The number of the stations J is equal to 5. The system parameters are listed below.

- $\lambda_j = 20.0$: the arrival rates ($j = 1, \ldots, 5$).
- The service time distributions are the 5 stage Erlang distributions with the means vary from 0.1 to 1.5.
- The feedback probabilities are as follows:

$$
\begin{aligned}
(p_{11}, p_{12}, p_{13}, p_{14}, p_{15}) &= (0.10, 0.10, 0.05, 0.05, 0.10),\\
(p_{21}, p_{22}, p_{23}, p_{24}, p_{25}) &= (0.10, 0.10, 0.15, 0.10, 0.10),\\
(p_{31}, p_{32}, p_{33}, p_{34}, p_{35}) &= (0.15, 0.10, 0.10, 0.10, 0.20),\\
(p_{41}, p_{42}, p_{43}, p_{44}, p_{45}) &= (0.15, 0.15, 0.15, 0.15, 0.15),\\
(p_{51}, p_{52}, p_{53}, p_{54}, p_{55}) &= (0.20, 0.20, 0.10, 0.10, 0.15).
\end{aligned}
$$

We calculate the values of the mean sojourn times of the systems where the service disciplines for all stations are either FCFS only or PR-LCFS only. We make the graphs (Figure 1. and Figure 2.) for the systems under these two disciplines in which the mean sojourn times $\tilde{W}_j$ of the customers who initially arrive at the station j ($j = 1, \ldots, 5$) is individually plotted.

10 Conclusions.

We have concerned with the multiclass M/G/1 system with feedback. First we define the cost functions $W_{ij}(V, n)$ and $G_{ij}(V, n)$ $(i, j = 1, \ldots, J; (V, n) \in \mathcal{E})$ which denote the *mean sojourn times* of the tagged customer and the *cumulative current works* of the tagged customer, respectively. We obtain the set of equations that are satisfied by the cost functions. It is shown in Section 6 that it can be solved explicitly under some assumptions that are satisfied by the FCFS and PR-LCFS disciplines. The important things to consider regarding the assumptions are

1. the *aggregated system state* $(\overline{V}, n)$ should be sufficient for estimating the *initial cost functions* $W_i(V, n)$, $G_i(V, n)$ and the aggregated system state $E[(\overline{V}^*, n^*)|(i), (V, n)]$, and
2. they should be the linear function of the aggregated system state $(\overline{V}, n)$.

Then the objective cost functions can be shown to have the same properties. Strictly speaking, these assumptions are the sufficient conditions that the objective cost functions $W_{ij}(V, n)$ and $G_{ij}(V, n)$ are the linear functions of $(\overline{V}, n)$. Finally, we evaluate the values of the functions of the system in the steady state. The generalized Little's formula ($H = \lambda G$) [9, 23] and the Poisson arrival see time averages (PASTA) property [25] are used. The *number of customers* and the expected *work* in each class are simultaneously obtained. It is worth noting that the equation (9.4) is a variation of the formula relating the expected time-stationary work, say $\tilde{V}$, in the general single class queue to the expected customer-stationary waiting time, say $W = V(T_\infty)$, i.e.,

$$\tilde{V} = \lambda\{E[SW] + E[S^2]/2\} \tag{10.116}$$

where λ denotes the arrival rate and S denotes the service time. Our method employed in the paper can be considered to be the *supplementary variable method* [3] where the supplementary variables are the (current) works in the system instead of the attained service times.

The special features of our method are summarized as follows.

1. We treat the system performance measures explicitly as the (cost) functions of the system state.
2. The analysis of the ordinary M/G/1 busy period processes has been applied.
3. The sufficient conditions that the objective cost functions are explicitly derived have been given (**Assumption 2.1.**, **Assumption 6.1.** and some assumptions regarding the existence of the inverse matrices and the steady states).
4. The generalized Little's formula has been applied as the generalizations of its usual use to the M/G/1 system (Pollaczek-Khinchin mean value formula [14]).
5. The algorithm that yields the values of the cost functions can be easily constructed.

The methodology given in the paper will be widely applicable to the analysis of the multiclass queueing systems.

Acknowledgement. The authors wish to express their appreciation to the professor K. Yamada and Mr. H. Usui for the valuable discussions. They also thank to the staffs of the Institute of Information Sciences and Electronics for their warm helpfulness.

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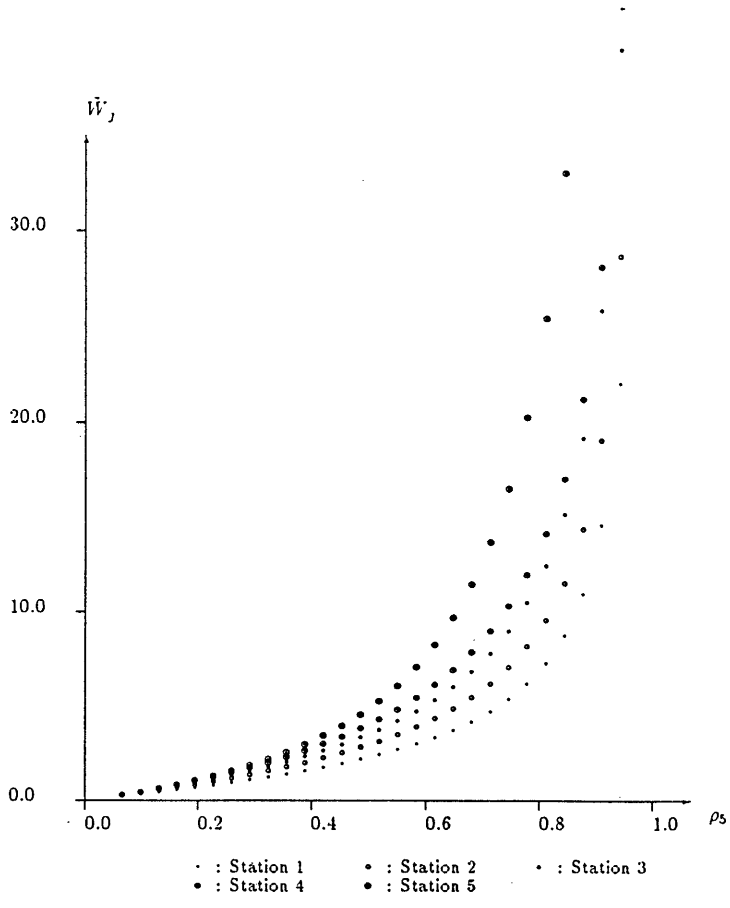

Figure 1. The mean sojourn times for the FCFS discipline.

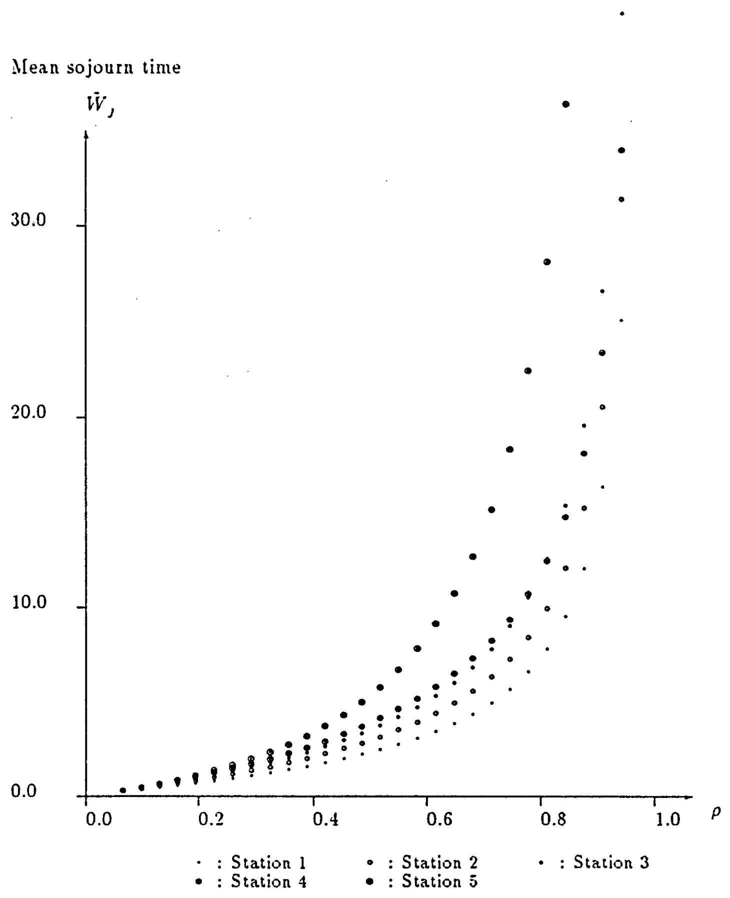

Figure 2. The mean sojourn times for the PR-LCFS discipline.

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REPORT DOCUMENTATION PAGE	REPORT NUMBER ISE-TR-93-102

TITLE

Analysis of Multiclass M/G/1 Queues with Feedback

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REPORT DATE	NUMBER OF PAGES
August 17, 1993	25

MAIN CATEGORY	CR CATEGORIES
Queueing Theory	

KEY WORDS

Multiclass M/G/1 queues, feedbacks of customers, mean sojourn times, scheduling disciplines and Little's formula.

ABSTRACT

We consider the multiclass M/G/1 queues with feedback. J classes of customers arrive at the stations from the outside of the system according to a Poisson process. The service time distribution for each class is arbitrary and the service discipline at each station is either the FCFS or the preemptive LCFS. After receiving a service, the customer at the station i either departs from the system with probability p_{i0}, or feeds back to the system and proceeds to the station k with probability p_{ik} $(i, k = 1, \cdots, J)$. We consider the cost functions naturally stem from the analysis of the system. They include the mean sojourn times for customers arriving at every station. First we consider the system at the arbitrary states to obtain the explicit formulae of the cost functions and then derive the steady state solutions by using the generalized Little's formula and PASTA. We also attempt to generalize the results to the other service disciplines.

SUPPLEMENTARY NOTES