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The Eigenvalue Problem for Infinite Compact Complex Symmetric Matrices with Application to the Numerical Computaition of Complex Zeros of $J_0(z)-iJ_1(z)$ and of Bessel Functions $J_m(z)$ of Any Real Order m

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THIS IS A REVISED AND EXPANDED VERSION OF THE EARLIER TECHNICAL REPORT ISE-TR-91-92 (SEPTEMBER 1, 1991). THE TITLE HAS BEEN CHANGED. THE REVISED VERSION INCLUDES, WITH THE HELP OF THREE ADDITIONAL COAUTHORS, NEW MATERIAL ON THE ZEROS OF BESSEL FUNCTIONS THAT IS NOT FOUND IN THE EARLIER VERSION. IT SUPERSEDES THE EARLIER VERSION.

The Eigenvalue Problem for Infinite Compact Complex Symmetric Matrices with Application to the Numerical Computation of Complex Zeros of $J_0(z)-iJ_1(z)$ and of Bessel Functions $J_m(z)$ of Any Real Order m

Running Title: The Eigenvalue Problem for Infinite Matrices

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Abstract.

Consider computing simple eigenvalues of a given compact infinite matrix regarded as operating in the complex Hilbert space l^2 by computing the eigenvalues of the truncated finite matrices and taking an obvious limiting process. In this paper we deal with a special case where the given matrix is compact, complex and symmetric (but not necessarily Hermitian). Two examples of application are studied. The first is concerned with the equation $J_0(z) - iJ_1(z) = 0$ appearing in the analysis of the solitary wave runup on a sloping beach, and the second with the zeros of the Bessel function $J_m(z)$ of any real order m. In each case, the problem is reformulated as an eigenvalue problem for a compact complex symmetric tridiagonal matrix operator in l^2 whose eigenvalues are all simple. A complete error analysis for the numerical solution by truncation is given based on the general theorems proved in this paper, where the usefulness of the seldom-used generalized Rayleigh quotient is demonstrated.

§1 Introduction and Summary.

This paper presents part of our investigation into the question of how much of the large existing body of knowledge on the so-called special functions of mathematical physics can be retold, reworked or reformulated in matrix language so that one might obtain new insight into the special functions or find a new class of algorithms for their computation. The work may also be regarded as an attempt of finding a new significant area of application of matrix theory. Our particular concern in this paper is the theory and application of the eigenvalue problem for compact complex symmetric matrix operators in the usual complex Hilbert space l^2 of all square-summable complex sequences. As concrete examples of application, we will consider the numerical computation of zeros of two functions ,namely, $J_0(z) - iJ_1(z)$ and $J_m(z)$ for any real m, where $i^2 = -1$ and $J_m(z)$ denotes the Bessel function of order m, each problem reformulated as an eigenvalue problem for an operator of the indicated type.

We first recall a few basic facts from the spectral theory of operators [10, Chap. XIII, §§3-4]. In the sequel, the generic symbol $B(X, Y)$ denotes the Banach space of all bounded linear operators from a Banach space X to a Banach space Y. We denote $B(X, X)$ simply by $B(X)$. A $T \in B(X, Y)$ is *compact* if for any bounded sequence $\{f_n\}$ in X, the image sequence $\{Tf_n\}$ in Y has a convergent subsequence. Given $T \in B(X)$, the set of all complex numbers λ for which $(T - \lambda I)^{-1} \in B(X)$ is known as the set $\rho(T)$ of *regular* values of T or the *resolvent* set. Its complement is the *spectrum* $\sigma(T)$ of T. In case T is compact and X is infinite-dimensional, 0 is always in $\sigma(T)$ and each nonzero $\lambda_0 \in \sigma(T)$ is an *eigenvalue* of T, namely, there is a corresponding *eigenvector* $x \in X$ such that $x \neq 0$ and $(T - \lambda_0 I)x = 0$, where I denotes the identity operator. For $\lambda \in \rho(T)$, the operator $(T - \lambda I)^{-1} \in B(X)$ is called the *resolvent* of T. For any $0 \neq \lambda_0 \in \sigma(T)$, $(T - \lambda_0 I)^{-1}$ is not well defined on the whole of X from the definition of $\sigma(T)$. However, $T - \lambda_0 I$ may have a bounded inverse on a smaller closed invariant subspace, say S, of T. A necessary and sufficient condition for this to be true is that λ_0 is not an eigenvalue of T restricted to S. Such an operator $(T - \lambda_0 I)^{-1}$ defined only on S will be denoted by $(T - \lambda_0 I)_S^{-1}$ in the sequel.

Our working hypothesis for the general theorems (Theorems 1.1-1.3) is given below and will be subsequently referred to as the hypothesis (H):

(H) We are given a sequence of compact complex symmetric (but not necessarily normal) matrix operators $\{A_n\}_1^\infty$ in the Hilbert space l^2, converging in operator norm to a compact complex symmetric matrix operator A also regarded as acting in l^2. We further assume that A has a nonzero eigenvalue λ that is *simple* in the sense that only one linearly independent eigenvector corresponds to λ and no generalized eigenvectors of rank 2 or more correspond to λ, namely, no vectors $y \in l^2$ exist such that $(A - \lambda I)^2 y = 0$ and $(A - \lambda I)y \neq 0$. Let x be an eigenvector of A corresponding to λ. We assume $x^T x \neq 0$, '^{T}' denoting transpose.

This situation occurs, for example, in the numerical solution of $J_0(z) - iJ_1(z) = 0$ and of $J_m(z) = 0$, where m is a given real number, as described in Theorems 1.4 - 1.8 below.

Our starting point is the following spectral convergence theorem, which is adapted from [12, p.272-274] in a specialized form suitable to our purpose:

Theorem 1.1. *Let A_n and A have the same meaning as defined in (H). We have:*

(a) For any eigenvalue $\lambda \neq 0$ of A, there is a sequence $\{\lambda_n\}_1^\infty$ of eigenvalues of A_n which converges to λ. Conversely, if a sequence of eigenvalues $\{\lambda_n\}_1^\infty$ of A_n converges to $\lambda \neq 0$, then λ is an eigenvalue of A [12, p.272, Theorem 18.1].

(b) If a sequence $\{\lambda_n\}_1^\infty$ of eigenvalues of A_n converges to a nonzero simple eigenvalue λ of A, then λ_n is simple for all sufficiently large n [12, p.273].

(c) If $\lambda \neq 0$ is a simple eigenvalue of A, x is an eigenvector of A corresponding to λ and a sequence $\{\lambda_n\}_1^\infty$ of eigenvalues of A_n converges to λ, then there is a sequence $\{x_n\}_1^\infty$ of eigenvectors of A_n corresponding to λ_n that converges to x [12, p.274, Theorem 18.3].

Theorem 1.2 below holds on the strength of the hypothesis (H).

Theorem 1.2. *Assume the hypothesis (H). Let $\lambda_n \to \lambda$ and $x_n \to x$, where λ_n is an eigenvalue of A_n and x_n is an eigenvector of A_n corresponding to λ_n. The existence of such λ_n and x_n is guaranteed by the last theorem. The λ_n are simple for all sufficiently large n, again by the last theorem. Let deflated subspaces S and S_n be defined as the orthogonal complement of $span\{x\}$ and $span\{x_n\}$ in the sense of transpose:*

$$S \equiv \{y \in l^2 : x^T y = 0\}, \quad S_n \equiv \{y \in l^2 : x_n^T y = 0\} \quad .$$

Note that S depends on λ only, since λ is simple. Similarly, S_n depends on λ_n only, for all n such that λ_n is simple. Clearly, S and S_n are closed subspaces of l^2 and $l^2 = span\{x\} \oplus S$ and $l^2 = span\{x_n\} \oplus S_n$ for all n such that λ_n is simple.

Let projections $Q : l^2 \to S$ and $Q_n : l^2 \to S_n$ be defined by:

$$Q \equiv I - \frac{xx^T}{x^T x}, \quad Q_n \equiv I - \frac{x_n x_n^T}{x_n^T x_n} \quad .$$

The Q_n are well-defined for all n such that λ_n is simple. One may easily verify that $Q^2 = Q$, $Q_n^2 = Q_n$ and $\| Q_n - Q \| \to 0$. Note further that Q and Q_n behave as identity when restricted to S and S_n, respectively.

We then have the following assertions:

(1) When restricted to S, A, $A - \lambda I$, $(A - \lambda I)_S^{-1} \in B(S)$

(2) For all n such that λ_n is simple and when restricted to S_n, A_n, $A_n - \lambda_n I$, $(A_n - \lambda_n I)_{S_n}^{-1} \in B(S_n)$

(3) $\| (A_n - \lambda_n I)_{S_n}^{-1} Q_n - (A - \lambda I)_S^{-1} Q \| \to 0$

(4) $\| (A_n - \lambda_n I)^{-1} \|_{S_n} \to \| (A - \lambda I)^{-1} \|_S$

Here the symbol $(A - \lambda I)_S^{-1}$ denotes, as stated earlier, the bounded inverse of $A - \lambda I$ restricted to S, and similarly for $(A_n - \lambda_n I)_{S_n}^{-1}$. The notation of the form $\| T \|_X$ denotes the operator norm of T whose domain is a subspace X.

The proof of Theorem 1.2 is given in §2. Here the proof for Parts (3) and (4), the core parts, is due to T. Ando, Associate Editor of this Journal and acting as the referee for this paper, and represents a considerable simplification over our original proof. Part (4) $\| (A_n - \lambda_n I)^{-1} \|_{S_n} \to \| (A - \lambda I)^{-1} \|_S$ is particularly useful for our subsequent work.

Remark. Let $B \equiv (A - \lambda I)_S^{-1} Q \in B(l^2, S)$ and $B_n \equiv (A_n - \lambda_n I)^{-1} Q_n \in B(l^2, S_n)$, then B and B_n are a generalized inverse of $A - \lambda I$ and $A_n - \lambda_n I$, respectively, as one can show by direct computation that

$$(1.1) \qquad \begin{cases} (A - \lambda I)B(A - \lambda I) = A - \lambda I \\ B(A - \lambda I)B = B \\ (A - \lambda I)B = B(A - \lambda I) = Q \end{cases}$$

and

$$(1.2) \qquad \begin{cases} (A_n - \lambda_n I)B_n(A_n - \lambda_n I) = A_n - \lambda_n I \\ B_n(A_n - \lambda_n I)B_n = B_n \\ (A_n - \lambda_n I)B_n = B_n(A_n - \lambda_n I) = Q_n \end{cases}.$$

Part (3) of this theorem then asserts the convergence of generalized inverses B_n of $A_n - \lambda_n I$ to the generalized inverse B of $A - \lambda I$, where clearly $A_n - \lambda_n I \to A - \lambda I$ in l^2. For a full, up-to-date treatment of generalized inverses in a variety of settings, we refer the reader to [15], a recent encyclopedic work on the subject including an extensive annotated bibliography of 1776 references. It should also be mentioned that the operator B (respectively B_n) is closely related to what is called the reduced resolvent of A (resp. A_n) for the eigenvalue λ (resp. λ_n) by T. Kato [11, p.180, (6.30)].

The hypothesis (H) represents a useful special situation where an appropriately taken generalized Rayleigh quotient [21, p.179] well approximates, in the sense of Theorems 1.3 below, a given simple eigenvalue of a compact complex symmetric matrix operator in l^2.

Theorem 1.3. *Again assume the hypothesis (H) and suppose that we are given a sequence $\{v_n\}_1^\infty$ such that $v_n \to x$. Consider the generalized Rayleigh quotient $\mu_n = v_n^T A_n v_n / v_n^T v_n$ and take it as an approximation to λ_n where, as in Theorem 1.2, λ_n is an eigenvalue of A_n such that $\lambda_n \to \lambda$. Then we have the following error estimate for all n such that λ_n is simple:*

$$\begin{aligned} | \mu_n - \lambda_n | &\le \frac{1}{| v_n^T v_n |} \| (A_n - \lambda_n I)v_n \|^2 \| (A_n - \lambda_n I)^{-1} \|_{S_n} \\ &= \frac{1}{| x^T x |} \| (A_n - \mu_n I)v_n \|^2 \| (A - \lambda I)^{-1} \|_S \, (1 + o(1)) \quad \text{as } n \to \infty. \end{aligned}$$

The proof of Theorem 1.3 is given in §3. Note that, in the last theorem, the error $|\mu_n - \lambda_n|$ is bounded by a quantity of order $\| (A_n - \mu_n I)v_n \|^2$.

The theorem may typically be used in the following context: Suppose we are to estimate $\lambda - \lambda_n$. We write $\lambda - \lambda_n = (\lambda - \mu_n) + (\mu_n - \lambda_n)$. If it can be shown, as is the case in the proofs of Theorems 1.5 and 1.8, that $|\mu_n - \lambda_n| / |\lambda - \mu_n| \to 0$ as $n \to \infty$, then we can estimate $\lambda - \lambda_n$ as $\lambda - \lambda_n = (\lambda - \mu_n)(1 + o(1))$ as $n \to \infty$. The point is that $\lambda - \mu_n$ may be estimated accurately when one has detailed knowledge on an eigenvector corresponding to the exact eigenvalue λ, as is again the case in Theorems 1.5 and 1.8.

Before proceeding further to application, we give the following simple lemma on the similarity of two tridiagonal matrices that will be repeatedly used later without explicit reference: *Two complex tridiagonal matrices sharing a common main diagonal*

$$\begin{pmatrix} d_1 & e_2 & & 0 \\ f_2 & d_2 & \ddots & \\ & \ddots & \ddots & e_n \\ 0 & & f_n & d_n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} d_1 & e'_2 & & 0 \\ f'_2 & d_2 & \ddots & \\ & \ddots & \ddots & e'_n \\ 0 & & f'_n & d_n \end{pmatrix},$$

where none of the super- and sub-diagonal components vanish, are similar if the product of two corresponding super- and sub-diagonal components are equal, namely, if $e_k f_k = e'_k f'_k, \quad k = 2, \cdots, n$. Indeed one is transformed to another by a similarity transformation by an appropriate diagonal matrix.

As the first example of application of Theorems 1.2 and 1.3, we will consider the approximate solution of $J_0(z) - iJ_1(z) = 0$, where $J_m(z)$ denotes the Bessel function of the first kind of order m (see [4] or [20] for the general reference on the Bessel functions). The equation is of interest in the analysis of solitary wave runup on a beach with a constant slope ([5], [19]). It is known ([16], [18]) that infinitely many roots lie in the lower half complex plane (but none in the upper half plane or on the real axis), symmetrically about the imaginary axis. The first 30 roots with positive real part accurate up to 8 digits have been computed by Macdonald through the use of the following asymptotic expansion for the jth root in polar form, also obtained by him [13]:

$$(1.3)\qquad \begin{cases} z = r_j e^{i[(\pi/2)\pm\theta_j]}, \\ r_j = j\pi + \{1 - 4\alpha_j(1-\alpha_j)\}\dfrac{1}{8j\pi} \\ \quad + \dfrac{1}{384 j^3\pi^3}[-61 + 264\alpha_j - 360\alpha_j^2 + 256\alpha_j^3 - 48\alpha_j^4] + O(\dfrac{\alpha_j^2}{j^4\pi^4}), \\ \theta_j = -\dfrac{1}{2}\pi - \dfrac{\alpha_j}{j\pi} - \dfrac{1}{96 j^3\pi^3}[21 - 48\alpha_j + 72\alpha_j^2 - 32\alpha_j^3] + O(\dfrac{\alpha_j^2}{j^4\pi^4}), \\ \alpha_j = \dfrac{1}{2}\ln(4j\pi), \end{cases}$$

where the plus sign in the expression for z is taken for the fourth quadrant roots and the minus sign for the third quadrant roots.

It may be noted that, for large j, $r_j = j\pi(1 + O(j^{-1}))$ and $\theta_j = -\frac{\pi}{2} - \frac{\alpha_j}{j\pi}(1 + O(j^{-2}))$, indicating that the roots are approximately π apart. It may further be shown that the equation $J_0(z) - iJ_1(z) = 0$ has no roots on the imaginary axis; for, putting $z = -i\eta$, where η is real, we have $J_0(z) - iJ_1(z) = I_0(\eta) - I_1(\eta) > 0$ for all real η [14, p.151, 3.16.3]. Figure 1.1 below gives a plot of the first 6 roots of $J_0(z) - iJ_1(z) = 0$ in the fourth quadrant of the complex plane, based on the table values in Table 1.1 below.

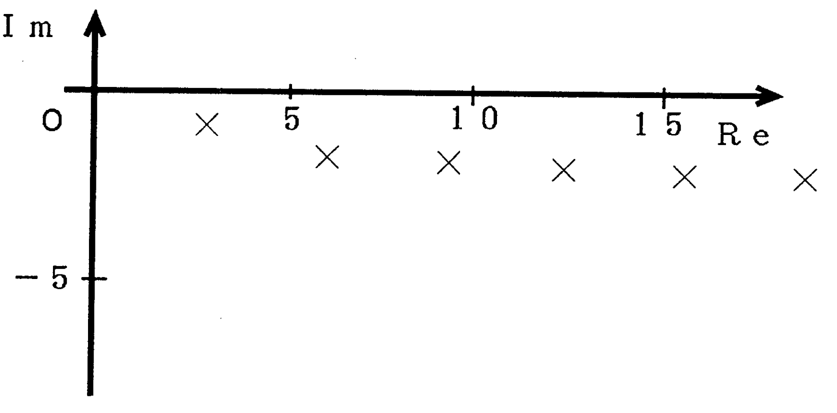

Figure 1.1 The first 6 roots of $J_0(z) - iJ_1(z) = 0$ in the fourth quadrant.

In order to apply Theorems 1.2 and 1.3 to $J_0(z) - iJ_1(z) = 0$, we first reformulate the equation as an eigenvalue problem for a compact complex symmetric matrix operator in l^2 :

Theorem 1.4.

(1) *A complex number z is a root of $J_0(z) - iJ_1(z) = 0$ if and only if $z \neq 0$ and $2/z$ is an eigenvalue of the compact complex symmetric matrix $A \in B(l^2)$ defined below.*

(2) *To the eigenvalue $2/z$ corresponds only one linearly independent eigenvector given by x defined below.*

(3) *Each eigenvalue of A is simple and nonzero.*

(4) *The eigenvalues of A distribute symmetrically about the imaginary axis.*

$$\begin{cases} Ax = \dfrac{2}{z}x \ , \\ A = \begin{pmatrix} i & f_2 & & 0 \\ f_2 & 0 & f_3 & \\ & f_3 & 0 & \ddots \\ 0 & & \ddots & \ddots \end{pmatrix}, \quad f_k = \dfrac{1}{\sqrt{(k-1)k}}, \ k = 2, 3, \cdots , \\ x = x(z) = [J_1(z), \sqrt{2}J_2(z), \sqrt{3}J_3(z), \cdots]^T \in l^2 \ . \end{cases}$$

For the proof of Theorem 1.4, see §4.

The theoretical basis for the numerical procedure for the approximate computation of the eigenvalues of the matrix A defined in Theorem 1.4 is given by the next theorem together with an accurate error estimate.

Theorem 1.5.

(1) *Let z be a root of $J_0(z) - iJ_1(z) = 0$. There exists a sequence $\{\lambda_n\}$ of eigenvalues of the $n \times n$ principal submatrix $\tilde{A}_n$ of A such that $\lambda_n \to \lambda \equiv 2/z$ and λ_n is simple and nonzero for all large n.*

(2) *Let $\{\lambda_n\}$ be any such sequence. Let $z_n = 2/\lambda_n$ be taken as an approximation to z. Then, for all large n such that λ_n is simple and nonzero, the relative error $(z_n - z)/z$ may be estimated by*

$$\frac{z_n - z}{z} = \frac{J_n(z)J_{n+1}(z)}{iJ_0^2(z)}(1 + o(1)) \quad (n \to \infty, \ z \text{ fixed})$$

with

$$J_0^2(z) = \mp i\frac{2}{\pi}(1 + o(1)) \quad \text{as } |z| \to \infty,$$

where the minus sign is to be taken for the roots z with positive real part and the plus sign for the roots z with negative real part.

The proof of Theorem 1.5 is given in §5.

The theorem is rather remarkable in the sense that the relative error $(z_n - z)/z$ is well approximated by a simple closed form such as one given above. The theorem also shows that the convergence $z_n \to z$ is eventually extremely rapid as the asymptotic expansion $J_n(z) \sim (z/2)^n/n!$ $(n \to \infty,\ z$ fixed) indicates [1, p.365, 9.3.1 or p.370, 9.5.10].

Consider now the numerical procedure for computing the λ_n. The $n \times n$ matrix $\tilde{A}_n$, the $n \times n$ principal submatrix of A defined in Theorem 1.4, is easily seen to be similar to

$$\hat{A}_n = i \begin{pmatrix} 1 & f_2 & & & 0 \\ -f_2 & 0 & f_3 & & \\ & -f_3 & 0 & \ddots & \\ & & \ddots & \ddots & f_n \\ 0 & & & -f_n & 0 \end{pmatrix} \equiv iB_n \ . \tag{1.4}$$

For a given n, we compute all eigenvalues of $\hat{A}_n$ (hence, of $\tilde{A}_n$ by similarity), which are given by i times the eigenvalues of B_n. The computation of the eigenvalues of B_n may be effected through the use of a QR algorithm for computing all eigenvalues of a real tridiagonal matrix; for example, the one implemented as the FORTRAN subroutine HQR in the EISPACK package [17] may be used. Since $\hat{A}_n$ is pure imaginary, it's eigenvalues distribute symmetrically about the imaginary axis just as the eigenvalues of A.

Let $z^{(k)} \equiv 2/\lambda^{(k)}$ denote the kth fourth quadrant root of $J_0(z) - iJ_1(z) = 0$. Theorem 1.5 guarantees the existence of a sequence $\{\lambda_n\}$ such that $\lambda_n \to \lambda^{(k)}$, where λ_n is an eigenvalue of $\tilde{A}_n$ (hence, of $\hat{A}_n$). The inspection of the computed eigenvalues of $\tilde{A}_n$ easily reveals which eigenvalue of $\tilde{A}_n$ is to be taken as the λ_n. Indeed, $\lambda_n = \lambda_n^{(k)}$, the kth largest (in modulus) first quadrant eigenvalue of $\tilde{A}_n$. The similar fact holds for the third quadrant roots of $J_0(z) - iJ_1(z) = 0$.

All computations were performed in the quadruple-precision floating-point arithmetic (28-digits in hexadecimal) on the FACOM M-780/20 system at University of Tsukuba.

Table 1.1 below gives the first 10 roots of $J_0(z) - iJ_1(z) = 0$ in the fourth quadrant, computed from $\tilde{A}_n$ with $n = n_{min}$, the smallest positive integer n for which the approximate root z_n computed from $\tilde{A}_n$ has 15 correct significant figures of real and of imaginary parts.

	Real	Imaginary	n_{min}
1	.29803 82414 79049 x 10^1	−.12796 02540 29915 x 10^1	14
2	.61751 53070 95484 x 10^1	−.16187 17384 47149 x 10^1	19
3	.93419 60983 46134 x 10^1	−.18188 72787 77295 x 10^1	25
4	.12498 50706 39585 x 10^2	−.19614 59538 01999 x 10^1	28
5	.15650 10438 53098 x 10^2	−.20723 09817 83076 x 10^1	33
6	.18798 91168 36963 x 10^2	−.21630 10983 27459 x 10^1	37
7	.21945 97998 43811 x 10^2	−.22397 72492 27609 x 10^1	41
8	.25091 88576 39076 x 10^2	−.23063 12806 67550 x 10^1	44
9	.28236 97314 53980 x 10^2	−.23650 36120 66197 x 10^1	48
10	.31381 46098 96480 x 10^2	−.24175 86986 36241 x 10^1	51

Table 1.1 The first 10 roots of $J_0(z) - iJ_1(z) = 0$ in the fourth quadrant, computed from $\tilde{A}_n$ with $n = n_{min}$, the smallest positive integer n for which the approximate root z_n computed from $\tilde{A}_n$ has 15 correct significant figures of real and of imaginary parts.

Table 1.2 compares the observed relative error $(z_n - z)/z$ with its estimate $(\pi/2)J_n(z)J_{n+1}(z)$ for the first and second roots z in the fourth quadrant and $n = 4, 8, 12, 10, 16, 20$ and for the fifth root and $n = 12, 16, 20, 24, 28$. It may be seen from the table that two quantities agree to about one digit except for low values of n , even for the first root, the fact that is quite satisfactory for practical purpose of estimating the correct number of digits of a given approximate root.

For the first root:

	$(z_n - z)/z$		$(\pi/2)J_n(z)J_{n+1}(z)$	
n	Real	Imaginary	Real	Imaginary
4	-0.181×10^{-01}	-0.385×10^{-02}	-0.213×10^{-01}	-0.498×10^{-02}
8	$+0.262 \times 10^{-06}$	-0.867×10^{-07}	$+0.267 \times 10^{-06}$	-0.543×10^{-07}
12	-0.620×10^{-13}	$+0.393 \times 10^{-13}$	-0.651×10^{-13}	$+0.297 \times 10^{-13}$
16	$+0.111 \times 10^{-20}$	-0.101×10^{-20}	$+0.121 \times 10^{-20}$	-0.820×10^{-21}
20	-0.245×10^{-29}	$+0.366 \times 10^{-29}$	-0.320×10^{-29}	$+0.293 \times 10^{-29}$

For the second root:

4	$-0.216 \times 10^{+00}$	$-0.561 \times 10^{+00}$	$+0.584 \times 10^{+00}$	$+0.163 \times 10^{+00}$
8	-0.482×10^{-02}	-0.147×10^{-03}	-0.576×10^{-02}	$+0.288 \times 10^{-03}$
12	$+0.428 \times 10^{-06}$	$+0.305 \times 10^{-06}$	$+0.439 \times 10^{-06}$	$+0.335 \times 10^{-06}$
16	-0.234×10^{-12}	-0.318×10^{-11}	-0.584×10^{-13}	-0.326×10^{-11}
20	-0.197×10^{-17}	$+0.158 \times 10^{-17}$	-0.209×10^{-17}	$+0.147 \times 10^{-17}$

For the fifth root:

12	$+0.116 \times 10^{-00}$	-0.360×10^{-00}	$+0.177 \times 10^{-00}$	$+0.306 \times 10^{-00}$
16	-0.741×10^{-02}	-0.219×10^{-01}	-0.151×10^{-01}	-0.392×10^{-01}
20	-0.146×10^{-03}	$+0.648 \times 10^{-04}$	-0.173×10^{-03}	$+0.941 \times 10^{-04}$
24	$+0.290 \times 10^{-07}$	$+0.908 \times 10^{-07}$	$+0.350 \times 10^{-07}$	$+0.103 \times 10^{-06}$
28	$+0.110 \times 10^{-10}$	-0.409×10^{-12}	$+0.120 \times 10^{-10}$	-0.436×10^{-12}

Table 1.2 The relative error $(z_n - z)/z$ and its estimate $(\pi/2)J_n(z)J_{n+1}(z)$ for the first, second and fifth roots z in the fourth quadrant, where $n = 4, 8, 12, 16, 20$ for the first and second roots and $n = 12, 16, 20, 24, 28$ for the fifth root.

As the second example, we consider computing the roots of $J_m(z) = 0$ for any real order m. Since $J_{-m}(z) = (-1)^m J_m(z)$ for any integer m [4, p.87, (6.4)] we assume $m \neq -1, -2, \cdots$ in the sequel. The next theorem, reformulating the problem of computing the nonzero roots of $J_m(z) = 0$ as an eigenvalue problem, is essentially a restatement of what is already known (see [7] and [9]):

Theorem 1.6.

(1) Let m be a real number $\neq -1, -2, \cdots$, then $z \neq 0$ is a root of $J_m(z) = 0$ if and only if $4/z^2$ is an eigenvalue of the compact complex symmetric tridiagonal matrix A defined below with a corresponding eigenvector x also defined below:

$$Ax = \frac{4}{z^2}x \ ,$$

where

$$\left\{ \begin{array}{l} A = \begin{pmatrix} d_1 & f_2 & & 0 \\ f_2 & d_2 & f_3 & \\ & f_3 & d_3 & \ddots \\ 0 & & \ddots & \ddots \end{pmatrix}, \quad \begin{array}{l} d_k = \dfrac{2}{(\alpha_k - 1)(\alpha_k + 1)}, \quad k = 1, 2, \cdots \\ f_k = \dfrac{1}{(\alpha_k - 1)\sqrt{(\alpha_k - 2)}\sqrt{\alpha_k}}, \quad k = 2, 3, \cdots \\ \alpha_k = m + 2k, \quad k = 1, 2, \cdots \end{array} \\ \\ x = x(z) = [\sqrt{m+2}J_{m+2}(z), \sqrt{m+4}J_{m+4}(z), \cdots]^T \ , \end{array} \right.$$

where for $a < 0$, $\sqrt{a} = i\sqrt{|a|}$. The matrix A is real symmetric for $m > -2$. In particular, for $m > -1$, it is positive-definite (hence, it's eigenvalues are all positive). For $m < -2$, the components of A are *real except for the single component f_p, which is pure imaginary, where p denotes the smallest positive integer exceeding $\mid m \mid /2$:*

$$f_p = \frac{1}{(m+2p-1)\cdot i \cdot \sqrt{\mid (m+2p-2)(m+2p) \mid}}$$

(2) For $m < -2$, A is similar to $\hat{A}$, the real matrix obtained from A by replacing the off-diagonal pair (f_p, f_p) (the first f_p on the super-diagonal, the second on the sub-diagonal) by the pair $(\mid f_p \mid, - \mid f_p \mid)$.

The proof is omitted, save for a brief remark that Part (1) is a matrix equation obtained from the three-term recurrence relation

$$(1.5) \qquad \frac{J_k(z)}{(k+1)(k+2)} + \frac{2J_{k+2}(z)}{(k+1)(k+3)} + \frac{J_{k+4}(z)}{(k+2)(k+3)} = \frac{4}{z^2}J_{k+2}(z), \qquad k \neq -1, -2, \cdots ,$$

which itself derives from the well-known recurrence formula

$$(1.6) \qquad J_{k-1}(z) - (2k/z)J_k(z) + J_{k+1}(z) = 0 \qquad (k \text{ arbitrary}) .$$

Example 1.1 The case $m = -1.5$: A is real symmetric and the d's and f's are given by

$$d_1 = \frac{2}{(-0.5)(1.5)}, \quad d_2 = \frac{2}{(1.5)(3.5)}, \quad d_3 = \frac{2}{(3.5)(5.5)}, \quad \cdots,$$

and

$$f_2 = \frac{1}{1.5\sqrt{0.5}\sqrt{2.5}}, \quad f_3 = \frac{1}{3.5\sqrt{2.5}\sqrt{4.5}}, \quad \cdots .$$

The d's are all positive except d_1, which is negative, and the f's are all positive.

Example 1.2 The case $m = -7.5$: A is complex symmetric and

$$d_1 = \frac{2}{(-6.5)(-4.5)}, \quad d_2 = \frac{2}{(-4.5)(-2.5)}, \quad d_3 = \frac{2}{(-2.5)(-0.5)}, \quad d_4 = \frac{2}{(-0.5)(1.5)}, \quad d_5 = \frac{2}{(1.5)(3.5)}, \quad \cdots,$$

and

$$f_2 = \frac{1}{(-4.5)\sqrt{-5.5}\sqrt{-3.5}}, \quad f_3 = \frac{1}{(-2.5)\sqrt{-3.5}\sqrt{-1.5}}, \quad f_4 = \frac{1}{(-0.5)\sqrt{-1.5}\sqrt{0.5}}, \quad f_5 = \frac{1}{(1.5)\sqrt{0.5}\sqrt{2.5}}, \quad \cdots .$$

The d's are all positive except d_4, which is negative, and the f's are all positive except f_4, which is pure imaginary. In this case, the matrix A is similar to the real nonsymmetric matrix $\hat{A}$, i.e.

$$\begin{pmatrix} d_1 & f_2 & & 0 \\ f_2 & d_2 & f_3 & \\ & f_3 & d_3 & \ddots \\ 0 & & \ddots & \ddots \end{pmatrix} \text{ is similar to } \begin{pmatrix} d_1 & f_2 & & & & & 0 \\ f_2 & d_2 & f_3 & & & & \\ & f_3 & d_3 & |f_4| & & & \\ & & -|f_4| & d_4 & f_5 & & \\ & & & f_5 & d_5 & \ddots & \\ 0 & & & & \ddots & \ddots & \end{pmatrix}.$$

It follows from Theorem 1.6 Parts (1) and (2) the well-known fact (see [20, p.40, (8)]) that, given any real m, if $z \neq 0$ is a root of $J_m(z) = 0$ so are $-z$, $\bar{z}$ and $-\bar{z}$. For $m > -1$, the roots of $J_m(z) = 0$ are known to be all real [20, p.483]. But this may again be easily deduced from Theorem 1.6. What cannot be deduced directly from Theorem 1.6 is the following fact known as the *theorem of Hurwitz* [20, p.483]: If $m < -1$ and $m \neq -1, -2, \cdots$, $J_m(z) = 0$ has an infinity of roots, of which only $2[|\, m \,|]$ roots are complex and the rest real; of the $2[|\, m \,|]$ complex roots, precisely 2 are pure imaginary provided that $[|\, m \,|]$ is odd.

In order to illustrate the fact stated in the last paragraph for the case $m < -1$, we show below in Figure 1.2 a sketch of the 18 zeros of $J_{-7.5}(z)$ closest to the origin (they may be computed according to the computational procedure to be stated after Theorem 1.8 below). As the theorem of Hurwitz asserts, there are only $14(=2[|\, m \,|])$ complex zeros, of which 2 are pure imaginary, $[|\, m \,|] = 7$ being odd. For their numerical values correct to 15 digits, see Table 1.3 following Theorem 1.8.

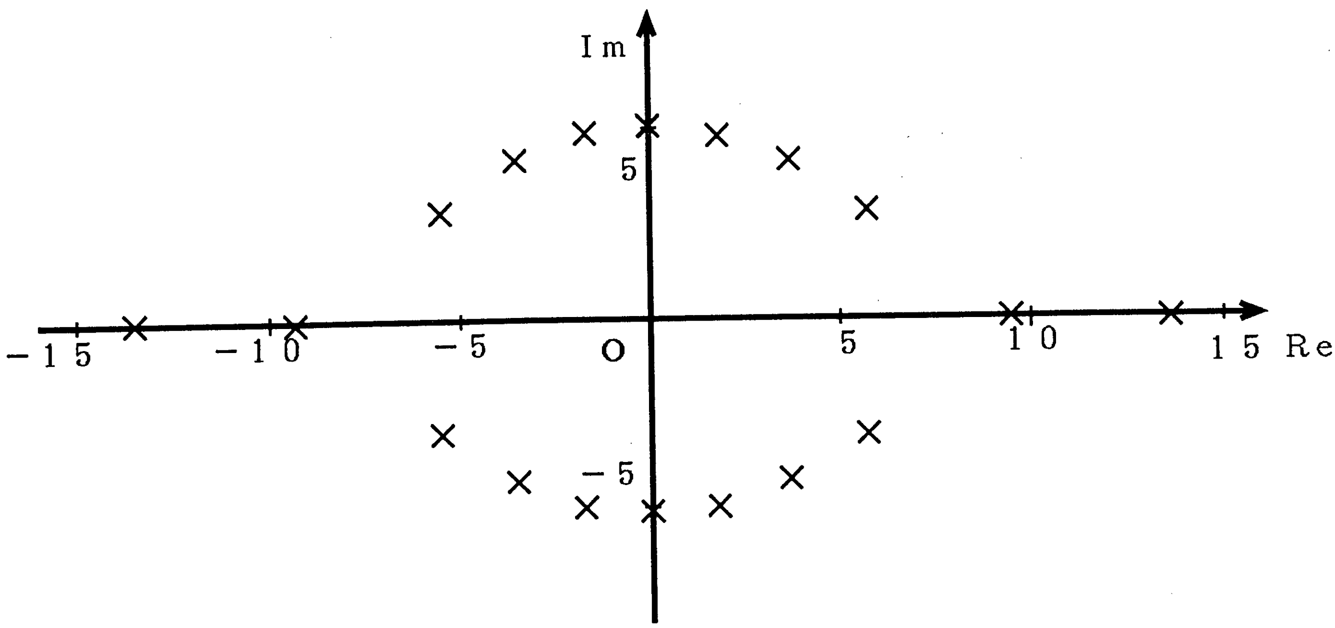

Figure 1.2 The 18 zeros of $J_{-7.5}(z)$ closest to the origin in the complex plane.

Our next theorem, when combined with the last theorem, guarantees that the matrix A under consideration is of the same type as the matrix operator, also called A, in the hypothesis (H).

Theorem 1.7. *Let m be a real number$\neq -1, -2, \cdots$. Then 0 is not an eigenvalue of A and every eigenvalue of A is simple in the sense of the hypothesis (H), where A is the matrix defined in Theorem 1.6.*

For the proof see §6.

Although asymptotic expansion formulas for large zeros of $J_m(z)$ for any real m are known (e.g., [20, the last formula on p.506]), an algorithm for computing a given number of generally complex roots of $J_m(z)$ closest to the origin with a given accuracy appears unknown, especially for the case $m < -1$ and $m \neq -1, -2, \cdots$. A matrix algorithm for this computational problem may be constructed based on our next theorem:

Theorem 1.8. *Let m be a real number $\neq -1, -2, \cdots$. Let a complex number $z \neq 0$ be a root of $J_m(z) = 0$.*

(1) Then there exists a sequence $\{\lambda_n\}$ of eigenvalues of the $n \times n$ principal submatrix $\tilde{A}_n$ of A such that $\lambda_n \to 4/z^2 \equiv \lambda$, where A means the matrix A defined in Theorem 1.6, and such that λ_n is simple and nonzero for all large n.

(2) Let $\{\lambda_n\}$ be any such sequence and take z_n that satisfies $4/z_n^2 = \lambda_n$ (or $z_n = \pm 2/\sqrt{\lambda_n}$) and $z_n \to z$. Then, for all large n such that λ_n is simple and nonzero, the relative error $(z_n - z)/z$ may be estimated by

$$\frac{z_n - z}{z} = \frac{J_{m+2n}(z)J_{m+2n+2}(z)}{2J_{m+1}^2(z)(m+2n+1)}(1 + o(1)) \qquad (\, n \to \infty,\ z \text{ fixed}\,)\ ,$$

where

$$J_{m+1}^2(z) = \frac{2}{\pi z}(1 + o(1)) \qquad (z \text{ real},\ | z | \to \infty,\ m \text{ fixed})\ .$$

The proof is given in §7. It is remarkable that the same form of error estimate holds for any real $m \neq -1, -2, \cdots$, not just for $m > -1$ as assumed in [9].

For the actual numerical procedure based on the last theorem, we only consider the case $m < -1$ since the case $m > -1$ has been discussed in [9]. Let $\tilde{A}_n$ denote the $n \times n$ principal submatrix of A ($n = 1, 2, \cdots$). For $m < -2$, let $\hat{A}_n$ denote the $n \times n$ real matrix obtained from $\tilde{A}_n$ by replacing the f_p in the super-diagonal by $| f_p |$ and the f_p in the sub-diagonal by $- | f_p |$, where we take $n \geq p$. The matrix $\tilde{A}_n$ and $\hat{A}_n$ are similar. Hence, the eigenvalues of $\tilde{A}_n$ are the same as those of $\hat{A}_n$. Therefor the problem of computing all eigenvalues of any given principal submatrix of A may be reduced to that of computing all eigenvalues of a real matrix. For the special case $-2 < m < -1$, the definition of $\hat{A}_n$ obviously does not apply and we directly deal with $\tilde{A}_n$, which is real symmetric as noted before.

The rest of the numerical procedure for computing the zeros of $J_m(z)$ is exactly in parallel with the previous problem, namely, the problem of computing the roots of $J_0(z) - iJ_1(z) = 0$, and will not be repeated here.

Table 1.3 below gives the first 9 zeros of $J_{-7.5}(z)$ in the first quadrant, correct to 15 significant figures both of real and imaginary parts. The first 4 zeros are complex, of which the first is pure imaginary. The last column, n_{min}, indicates the smallest positive integer n for which the given approximate root to be computed from $\tilde{A}_n$ has the indicated accuracy. Hence, from computing all 25 eigenvalues of $\tilde{A}_n$ ($n = 25$), we would obtain 50 approximate zeros of $J_{-7.5}(z)$, of which 24 zeros closest to the origin (14 complex zeros, of which 2 are pure imaginary, plus 10 real zeros distributed symmetrically about the real and imaginary axes) would have accuracy of 15 correct significant figures or better.

	Real	Imaginary	n_{min}
1	0	.51656 06291 45118 × 10^1	12
2	.17869 37489 75208 × 10^1	.49542 27658 75524 × 10^1	12
3	.36237 53314 34266 × 10^1	.42745 29632 26164 × 10^1	12
4	.56304 67230 18165 × 10^1	.29198 38074 41747 × 10^1	13
5	.94578 82431 67948 × 10^1	0	16
6	.13600 62896 81802 × 10^2	0	18
7	.17197 77667 14518 × 10^2	0	20
8	.20619 61246 38732 × 10^2	0	22
9	.23955 26725 49266 × 10^2	0	25

Table 1.3 The first 9 approximate zeros of $J_{-7.5}(z)$ in the first quadrant correct to 15 significant figures.

As in the first example, we will compare numerically two estimates to be called Estimate1 and Estimate2 (for z real) derived from Theorem 1.8 against the observed relative error:

$$\text{(1.7)} \qquad \text{Estimate1} = \frac{J_{m+2n}(z)J_{m+2n+2}(z)}{2J_{m+1}^2(z)(m+2n+1)}, \quad \text{Estimate2} = \frac{\pi z J_{m+2n}(z)J_{m+2n+2}(z)}{4(m+2n+1)}$$

Table 1.4 below gives an example of such comparison for 4 zeros of $J_{-7.5}(z)$ in the first quadrant, where the values of Estimate2 are computed for the real zeros only.

For the first root (pure imaginary)= $i \cdot (5.1656 \cdots)$:

	$(z_n - z)/z$		Estimate1	
n	Real	Imaginary	Real	Imaginary
8	+0.200 × 10^{-04}	0	+0.201 × 10^{-04}	0
10	+0.494 × 10^{-10}	0	+0.495 × 10^{-10}	0
12	+0.150 × 10^{-16}	0	+0.150 × 10^{-16}	0
14	+0.852 × 10^{-24}	0	+0.852 × 10^{-24}	0

For the second root (complex)= $1.7869\cdots + i\cdot(4.9542\cdots)$:

n	$(z_n - z)/z$		Estimate1	
	Real	Imaginary	Real	Imaginary
8	-0.276×10^{-05}	-0.242×10^{-04}	-0.286×10^{-05}	-0.242×10^{-04}
10	-0.337×10^{-10}	$+0.662 \times 10^{-10}$	-0.337×10^{-10}	$+0.663 \times 10^{-10}$
12	$+0.225 \times 10^{-16}$	-0.155×10^{-16}	$+0.225 \times 10^{-16}$	-0.155×10^{-16}
14	-0.183×10^{-23}	$+0.272 \times 10^{-24}$	-0.183×10^{-23}	$+0.272 \times 10^{-24}$

For the fifth root (real)= $9.4578\cdots$:

n	$(z_n - z)/z$	Estimate1	Estimate2
10	0.907×10^{-04}	0.933×10^{-04}	0.597×10^{-04}
12	0.934×10^{-08}	0.941×10^{-08}	0.602×10^{-08}
14	0.122×10^{-12}	0.123×10^{-12}	0.785×10^{-13}
16	0.328×10^{-18}	0.328×10^{-18}	0.210×10^{-18}
18	0.240×10^{-24}	0.240×10^{-24}	0.153×10^{-24}

For the ninth root (real)= $23.955\cdots$:

n	$(z_n - z)/z$	Estimate1	Estimate2
18	0.361×10^{-04}	0.388×10^{-04}	0.369×10^{-04}
20	0.526×10^{-07}	0.543×10^{-07}	0.516×10^{-07}
22	0.205×10^{-10}	0.209×10^{-10}	0.198×10^{-10}
24	0.274×10^{-14}	0.277×10^{-14}	0.263×10^{-14}
26	0.146×10^{-18}	0.147×10^{-18}	0.140×10^{-18}

Table 1.4 The observed relative error and its two estimates for the zeros of $J_{-7.5}(z)$ in the first quadrant.

One may see from Table 1.4 that the values of Estimate1 agree with the observed relative errors to 1 significant figure or better. The less accurate Estimate2 is also found accurate to about one significant figure for zeros with larger moduli.

Remark. It may be proved that $2n$ approximate zeros of $J_m(z)$ given as the roots of $\det(\tilde{A}_n - \frac{4}{z^2}I) = 0$ (i.e. those z such that $4/z^2$ are an eigenvalue of $\tilde{A}_n$, the $n \times n$ principal submatrix of A in Theorem 1.6) are precisely the zeros of what is known as the Lommel's polynomial $R_{2n+1,m+1}(z)$ [20, §§9.6-9.7]. In fact, the following relation may be derived from [20, §§9.6-9.7]:

$$R_{2n+1,m+1}(z) = \frac{2}{z}(-1)^n(m+1)(m+2)\cdots(m+2n+1)\det(\tilde{A}_n - \frac{4}{z^2}I). \tag{1.8}$$

The proof is omitted.

We conclude this section with a remark on the applicability of the matrix technique expounded in this paper. Indeed, the matrix technique turns out to be applicable to other classes of special functions which represent a minimal solution of a three-term recurrence relation. For example, the eigenvalue problem of Mathieu's equation with a complex parameter q, namely,

$$\frac{d^2w}{dz^2} + (a - 2q\cos 2z)w = 0 \tag{1.9}$$

[3, p.26] is amenable to the technique described in this paper. The result will be reported elsewhere.

For another example of the matrix technique, see [8], where the numerical computation of the zeros of regular Coulomb wave functions and of their derivatives are studied.

§2 Proof of Theorem 1.2.

We use the notation already established in Theorem 1.2. As stated in §1, the improved proof below for Parts (3) and (4) is due to T. Ando.

Proof of Part (1). To prove $A \in B(S)$, it is enough to show that $x^T y = 0$ implies $x^T Ay = 0$ under the given hypothesis $A = A^T$, $Ax = \lambda x$. Indeed, $x^T Ay = x^T A^T y = (Ax)^T y = \lambda x^T y = 0$. From $A \in B(S)$, $A - \lambda I \in B(S)$ follows easily.

To prove $(A - \lambda I)_S^{-1} \in B(S)$, it is enough to prove by [10, p.375, Theorem 1] that λ is not an eigenvalue of the restriction A_S of A to the closed subspace S. To prove this, it suffices to show that $Ay = \lambda y$ and $x^T y = 0$ implies $y = 0$. Suppose $y \neq 0$. By the simplicity hypothesis for λ, we have $y = ax$ for some scalar $a \neq 0$. Multiplying x^T from left, we have $0 = x^T y = ax^T x$, hence $a = 0$ since $x^T x \neq 0$ from the hypothesis (H). This is a contradiction.

Part (2) of Theorem 1.2 may be proved similarly.

Proof of Part (3). Choose an $\alpha > \sup_n \| A_n \|$ and let $\hat{A} = \alpha P + AQ$ and $\hat{A}_n = \alpha P_n + A_n Q_n$, where $P = xx^T/x^T x$ and $P_n = x_n x_n^T / x_n^T x_n$ so that $P + Q = I = P_n + Q_n$. Since $\| x_n - x \| \rightarrow 0$, we have $\| \hat{A}_n - \hat{A} \| \rightarrow 0$. By compactness of P, P_n, AQ and $A_n Q_n$, $\hat{A}$ and $\hat{A}_n$ are also compact. We will show that $(\hat{A} - \lambda I)^{-1} \in B(l^2)$ and $(\hat{A}_n - \lambda_n I)^{-1} \in B(l^2)$ for all large n such that λ_n is simple. To prove the first, it is enough to show $(\hat{A} - \lambda I)y = 0$ implies $y = 0$ by virtue of compactness of $\hat{A}$ [10, p.375, Theorem 1]. Indeed, from $(\hat{A} - \lambda I)y = 0$ follow

$$0 = P(\hat{A} - \lambda I)y = (\alpha - \lambda)Py$$

and

$$0 = Q(\hat{A} - \lambda I)y = (A - \lambda I)Qy \quad ,$$

using $P^2 = P$, $Q^2 = Q$, $PQ = QP = 0$, $PAQ = 0$ and $QAQ = AQ$. From the former follows $Py = 0$ since

$$\alpha > \sup_n \| A_n \| \geq \| A \| \geq | \lambda | \quad .$$

From the latter it follows that Qy is a scalar multiple of x, since λ is a simple eigenvalue of A by the hypothesis (H). Hence, $PQy = Qy$ since P is a projection onto the $span\{x\}$. But $PQ = 0$, hence $Qy = 0$. We have now proved $Py = Qy = 0$, hence $y = Py + Qy = 0$.

The other assertion $(\hat{A}_n - \lambda_n I)^{-1} \in B(l^2)$ for all large n such that λ_n is simple may be proved similarly.

The main utility of $\hat{A}$ and $\hat{A}_n$ is found in the following identities:

$$(2.1) \qquad (A - \lambda I)_S^{-1} = (\hat{A} - \lambda I)^{-1} Q \mid_S$$

and

$$(2.2) \qquad (A_n - \lambda_n I)_{S_n}^{-1} = (\hat{A}_n - \lambda_n I)^{-1} Q_n \mid_{S_n}$$

To prove (2.1), it is enough to prove $(\hat{A} - \lambda I)_S = Q(A - \lambda I)_S$ or $(\hat{A} - \lambda I)Q = Q(A - \lambda I)Q$, since Q is a projection onto S. This last relation reads $AQ = QAQ$ since $\hat{A} = \alpha P + AQ$, $PQ = 0$ and $Q^2 = Q$. This is clearly true since $A \in B(S)$ by Part (1). The identity (2.2) may be proved similarly.

Part (3) now follows from (2.1), (2.2) and from the fact that $Q^2 = Q$, $Q_n^2 = Q_n$, $\| \hat{A}_n - \hat{A} \| \to 0$ and $\| Q_n - Q \| \to 0$.

Proof of Part (4). For simplicity, let $B \equiv (A - \lambda I)_S^{-1}$, $B_n \equiv (A_n - \lambda_n I)_{S_n}^{-1}$, $\hat{B} \equiv (\hat{A} - \lambda I)^{-1}$, $\hat{B}_n \equiv (\hat{A}_n - \lambda_n I)^{-1}$, and $\Delta Q_n = Q_n - Q$. We will prove

$$(2.3) \qquad \| B \|_S \leq \| B_n \|_{S_n} (1 + \| \Delta Q_n \|) + o(1)$$

and

$$(2.4) \qquad \| B_n \|_{S_n} \leq \| B \|_S (1 + \| \Delta Q_n \|) + o(1)$$

Part(4), namely, $\| B_n \|_{S_n} \to \| B \|_S$, clearly follows from these two inequalities.

To prove (2.3), take any $y \in S$ such that $\| y \| = 1$. Then

$$\| By \| = \| (\hat{A} - \lambda I)^{-1} Qy \| \qquad \text{(by (2.1))}$$

$$= \| \hat{B} Qy \| \qquad \text{(by definition)}$$

$$= \| (\hat{B}Q - \hat{B}_n Q_n) y + \hat{B}_n Q_n y \|$$

$$\leq \| \hat{B}Q - \hat{B}_n Q_n \| + \| \hat{B}_n Q_n y \| \qquad \text{(by } \| y \| = 1\text{)}$$

$$= o(1) + \| \hat{B}_n Q_n Q_n y \| \qquad \text{(by } \| \hat{B}_n - \hat{B} \| \to 0, \ \| Q_n - Q \| \to 0 \text{ and } Q_n^2 = Q_n\text{)}$$

$$= o(1) + \| B_n Q_n y \| \qquad \text{(by (2.2): } \hat{B}_n Q_n = B_n\text{)}$$

$$\leq o(1) + \| B_n \|_{S_n} \| Q_n y \|$$

$$\leq o(1) + \| B_n \|_{S_n} \{ \| (Q_n - Q) y \| + \| Qy \| \}$$

$$\leq o(1) + \| B_n \|_{S_n} (\| \Delta Q_n \| + 1) \qquad \text{(by } Qy = y\text{)}$$

The last member is independent of any particular $y \in S$ such that $\| y \| = 1$. Hence (2.3) follows. The inequality (2.4) may be proved similarly.

The proof of Theorem 1.2 is now complete.

§3 Proof of Theorem 1.3.

We adhere to the notation established in Theorem 1.3.

To prove the first inequality in the theorem, we compute for all sufficiently large n

$$\mu_n - \lambda_n = \frac{v_n^T(A_n - \lambda_n I)v_n}{v_n^T v_n}$$

$$= \frac{1}{v_n^T v_n} v_n^T (A_n - \lambda_n I)(A_n - \lambda_n I)_{S_n}^{-1}(A_n - \lambda_n I)v_n$$

(by Theorem 1.2 Part(2) and the fact $(A_n - \lambda_n I)v_n \in S_n$)

$$= \frac{r_n^T(A_n - \lambda_n I)_{S_n}^{-1} r_n}{v_n^T v_n} \qquad (\text{by } A_n^T = A_n \text{ }),$$

where

$$r_n \equiv (A_n - \lambda_n I)v_n \in S_n \ . \tag{3.1}$$

Taking absolute value and applying the Cauchy-Schwarz inequality, we obtain the first part of the theorem, as required.

To prove the second part, it is enough to prove

$$\| r_n \|^2 = \| (A_n - \mu_n I)v_n \|^2 \left(1 + o(1)\right), \tag{3.2}$$

since $v_n \to x$ by the hypothesis of the theorem and since $\| (A_n - \lambda_n I)^{-1} \|_{S_n} \to \| (A - \lambda I)^{-1} \|_S$ by Theorem 1.2 Part (4) (it is here that Theorem 1.2 Part (4) is useful). To prove this, write

$$(A_n - \mu_n I)v_n = r_n + q_n \ . \tag{3.3}$$

It is enough to prove $\| q_n \| = O(\| r_n \|^2)$. To prove this, compute

$$q_n = (A_n - \mu_n I)v_n - (A_n - \lambda_n I)v_n \qquad (\text{ by the definition (3.1) of } r_n \text{ })$$

$$= (\lambda_n - \mu_n)v_n \ .$$

Taking norm,

$$\| q_n \| = | \lambda_n - \mu_n | \| v_n \|$$

$$\le \frac{1}{| v_n^T v_n |} \| r_n \|^2 \| (A_n - \lambda_n I)^{-1} \|_{S_n} \| v_n \| \qquad (\text{by the first part of this theorem}) \ .$$

Again the same fact as we used above, namely, the fact that $v_n \to x$ and $\| (A_n - \lambda_n I)^{-1} \|_{S_n} \to \| (A - \lambda I)^{-1} \|_S$, allows us to conclude from the last inequality that $\| q_n \| = O(\| r_n \|^2)$.

This completes the proof of Theorem 1.3.

§4 Proof of Theorem 1.4.

We use the same notation as in Theorem 1.4.

Proof of Parts (1) and (2). The proof will be done in 4 lemmas, Lemmas 4.1–4.1.

Lemma 4.1. *For any $z \neq 0$, the following matrix equation in l^2 holds:*

$$\begin{bmatrix} i & f_2 & & 0 \\ f_2 & 0 & f_3 & \\ & f_3 & 0 & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} J_1(z) \\ \sqrt{2}J_2(z) \\ \sqrt{3}J_3(z) \\ \vdots \end{bmatrix} = \frac{2}{z} \begin{bmatrix} J_1(z) \\ \sqrt{2}J_2(z) \\ \sqrt{3}J_3(z) \\ \vdots \end{bmatrix} - \begin{bmatrix} J_0(z) - iJ_1(z) \\ 0 \\ 0 \\ \vdots \end{bmatrix} \tag{4.1}$$

or

$$Ax = \frac{2}{z}x - [J_0(z) - iJ_1(z), 0, 0, \cdots]^T \quad ,$$

where A is a compact complex symmetric operator in $B(l^2)$ and $0 \neq x \in l^2$.

Proof. The relation (4.1) may be verified directly by using the well-known three-term recurrence relations [4, p.93]:

$$J_{k-1}(z) + J_{k+1}(z) = \frac{2k}{z}J_k(z), \qquad k = 1, 2, \cdots \quad , \tag{4.2}$$

hence,

$$f_k y_{k-1} + f_{k+1} y_{k+1} = \frac{2}{z} y_k, \quad k = 2, 3, \cdots, \quad \text{where} \quad y_k = \sqrt{k} J_k(z), \quad k = 1, 2, \cdots \quad . \tag{4.3}$$

The matrix A is obviously complex (i.e., non-real due to the presence of i as the first diagonal element) and symmetric. Compactness of A follows from the fact that a band matrix $B = [b_{ij}]$ (i.e., $b_{ij} = 0$ for all i and j such that $| i - j |> r$ for some fixed positive integer r) is compact, if and only if $\lim_{i,j \to \infty} b_{ij} = 0$ [2, p.59]. The fact that $x \in l^2$ can be deduced from the well-known continued fraction expression of $J_k(z)/J_{k-1}(z)$ [1, p.363, 9.1.73]:

$$\frac{J_k(z)}{J_{k-1}(z)} = \frac{z}{2k}\left(1 + O\left(\frac{1}{k^2}\right)\right) \qquad (k \to \infty, \ z \text{ fixed}) \tag{4.4}$$

Since no two consecutive J's, i.e. $J_k(z)$ and $J_{k+1}(z)$ for $k = 1, 2, \cdots$, vanish simultaneously at any $z \neq 0$ [4, p.105], it is clear that $x \neq 0$ for any $z \neq 0$. ∎

Remark. By direct computation, one can show that $AA^H \neq A^H A$, i.e., that A is not normal.

Lemma 4.2. *If $J_0(z) - iJ_1(z) = 0$, then $z \neq 0$, and $2/z$ is an eigenvalue of A with a corresponding eigenvector $x = [J_1(z), \sqrt{2}J_2(z), \cdots]^T \in l^2$.*

Proof. $J_0(0) - iJ_1(0) = 1 - i \cdot 0 = 1$. Hence, if $J_0(z) - iJ_1(z) = 0$, then $z \neq 0$ and Lemma 4.1 implies that $2/z$ is an eigenvalue of A with a corresponding eigenvector $x = [J_1(z), \sqrt{2}J_2(z), \cdots]^T \in l^2$. ∎

Lemma 4.3. *For a given complex number $z \neq 0$, an arbitrary solution of the three-term recurrence relation*

$$f_k y_{k-1} + f_{k+1} y_{k+1} = \frac{2}{z} y_k, \quad k = 2, 3, \cdots, \tag{4.5}$$

satisfying the condition $y_k \to 0$, has the form $y_k = c\sqrt{k}J_k(z)$, $k = 1, 2, \cdots$, for some constant c.

Proof. From (4.3), $y_k = \sqrt{k}J_k(z)$, $k = 1, 2, \cdots$, obviously satisfy the recurrence relation (4.5). The fact that $y_k \to 0$ was noted there also.

Conversely, if y_k $(k = 1, 2, \cdots)$ satisfies (4.5) and $y_k \to 0$, then the y_k's represent a *minimal* solution of (4.5), i.e., a second solution w_k of (4.5) exists such that $y_k/w_k \to 0$ (e.g., $w_k = \sqrt{k}Y_k(z)$, where $Y_k(z)$ is the Bessel function of the second kind of order k) [6, p.25]. Since the minimal solution is unique up to scalar multiplication [6, p.25], the lemma clearly holds. ∎

Lemma 4.4. *If λ is an eigenvalue of A, then $\lambda \neq 0$,and only one linearly independent eigenvector x corresponds to λ, where x is as defined in Theorem 1.4. Moreover, $z = 2/\lambda$ is a root of $J_0(z) - iJ_1(z) = 0$.*

Proof. To prove that 0 is not an eigenvalue of A, suppose the contrary and let $Ay = 0 \cdot y = 0$ for some $y = [y_1, y_2, \cdots]^T \in l^2$, where $y \neq 0$. Expanding $Ay = 0$, we find $y_k = (-i)^{k-1}\sqrt{k}y_1$, $k = 2, 3, \cdots$. Since $y \neq 0$, we conclude $y_1 \neq 0$. But then, $| y_k | \to \infty$ as $k \to \infty$, a contradiction of the fact that $y \in l^2$.

Let λ be an eigenvalue of A. To prove that only one linearly independent eigenvector corresponds to λ, let $Ay = \lambda y$, $0 \neq y = [y_1, y_2, \cdots]^T \in l^2$. Suffices to show that $y = cx$ for some $c \neq 0$, where $z = 2/\lambda$, a well-defined number, since $\lambda \neq 0$ as proved above. Expanding $Ay = \lambda y$, we obtain the same relation as (4.5), where $y_k \to 0$ since $y \in l^2$. Then, Lemma 4.3 applies, and $y_k = c\sqrt{k}J_k(z)$, $k = 1, 2, \cdots$ for some constant c, namely, $y = cx$.

Again, let λ be an eigenvalue of A and y be a corresponding eigenvector, then $y = c[J_1(z), \sqrt{2}J_2(z), \cdots]^T$ for some constant $c \neq 0$, as proved above, where $z = 2/\lambda$. Then, substitution of $z = \lambda/2$ into (4.1) gives $J_0(z) - iJ_1(z) = 0$, as required. ∎

Lemmas 4.1–4.4 prove Parts (1) and (2).

Proof of Part (3). In view of Lemma 4.4, it suffices to prove that A has no generalized eigenvectors of rank 2. We will do this in two lemmas below.

Lemma 4.5. *The function $f(z) = J_0(z) - iJ_1(z)$ has no multiple zeros, namely, if $f(z) = 0$, then $f'(z) \neq 0$.*

Proof. We prove that $f(z) = f'(z) = 0$ leads to a contradiction. Indeed, $f(z) = 0$ gives $J_0(z) = iJ_1(z)$ and $z \neq 0$. Then the assumption $f'(z) = 0$ gives

$$\begin{aligned}(4.6)\quad 0 = f'(z) &= J_0'(z) - iJ_1'(z)\\ &= -J_1(z) - i\{J_0(z) - \frac{1}{z}J_1(z)\} \qquad (\text{by } J_k'(z) = J_{k-1}(z) - \frac{k}{z}J_k(z)\ , k = 1, 2, \cdots \quad [4, \text{p.}93])\\ &= \frac{i}{z}J_1(z) \qquad (\text{using } J_0(z) = iJ_1(z)).\end{aligned}$$

Hence, $J_1(z) = 0$. But then, $f(z) = 0$ gives $J_0(z) = 0$. This is a contradiction, since $J_0(z)$ and $J_1(z)$ never vanish simultaneously [4, p.105]. ∎

Lemma 4.6. *The matrix A has no generalized eigenvectors of rank 2.*

Proof. Suppose the contrary and let w be a generalized eigenvector of rank 2 corresponding to an eigenvalue λ of A, i.e., let

$$(4.7)\qquad \begin{cases} (A - \lambda I)w \equiv u \neq 0 \\ (A - \lambda I)^2 w = (A - \lambda I)u = 0\ , \end{cases}$$

hold for $u, w \in l^2$. We will derive a contradiction. To this end, consider again the identity (4.1) in Lemma 4.1, which holds for any $z \neq 0$. For convenience, we rewrite it in the following form:

$$
(4.8) \qquad \begin{cases} (A - \frac{2}{z}I)x(z) = f(z)[-1,0,0,\cdots]^T, \\ f(z) = J_0(z) - iJ_1(z), \\ x(z) = [J_1(z), \sqrt{2}J_2(z), \cdots]^T, \end{cases}
$$

where the vector denoted previously by x is written as $x(z)$ to emphasize its dependence on z. Differentiation gives

$$
(4.9) \qquad \frac{2}{z^2}x(z) + (A - \frac{2}{z}I)x'(z) = f'(z)[-1,0,0,\cdots]^T,
$$

where

$$
x'(z) = [J_1'(z), \sqrt{2}J_2'(z), \cdots]^T = \frac{1}{2}\{[J_0(z), \sqrt{2}J_1(z), \cdots]^T - [J_2(z), \sqrt{2}J_3(z), \cdots]^T\} \in l^2,
$$

since $J_k'(z) = (1/2)\{J_{k-1}(z) - J_{k+1}(z)\}$, $k = 1, 2, \cdots$ [4, p.93].

From the second equation of (4.7), u is an eigenvector corresponding to the eigenvalue λ. Lemma 4.4 shows that $\lambda \neq 0$ and $u = cx(z_1)$, where $z_1 = 2/\lambda$, $f(z_1) = 0$ and c is a nonzero constant. Then, the first equation of (4.7) leads to

$$
(4.10) \qquad -cx(z_1) + (A - \lambda I)w = 0.
$$

Eliminating $x(z_1)$ from (4.9) (with $z = z_1$) and (4.10), we obtain

$$
(4.11) \qquad (A - \frac{2}{z_1}I)(x'(z_1) + w_1) = f'(z_1)[-1,0,0,\cdots]^T, \qquad \text{where } w_1 = \frac{2w}{cz_1^2}.
$$

Write $x'(z_1) + w_1 = \varphi \equiv [\varphi_1, \varphi_2, \cdots]^T \in l^2$. Expanding (4.11), we find the kth component ($k = 2, 3, \cdots$) given by

$$
f_k\varphi_{k-1} + f_{k+1}\varphi_{k+1} = \frac{2}{z_1}\varphi_k, \quad k = 2, 3, \cdots,
$$

where $\varphi_k \to 0$, since $\varphi \in l^2$. Then, Lemma 4.3 applies and we conclude $\varphi = c'x(z_1)$ for some constant c'. Since $u = cx(z_1)$ with $c \neq 0$, we see that φ is a scalar multiple of u. Then, the second relation of

(4.7) gives $(A - (2/z_1)I)\varphi = 0$. Using this in (4.11), where $x'(z_1) + w_1 = \varphi$ as defined earlier, we find $0 = f'(z_1)[-1, 0, 0, \cdots]^T$. Hence, $f'(z) = 0$.

On the other hand, $f(z_1) = 0$ as has been shown. Hence, $f(z_1) = f'(z_1) = 0$, a contradiction of Lemma 4.5. ■

This completes the proof of Part (3).

Proof of Part (4). The routine computation $\overline{J_0(z) - iJ_1(z)} = J_0(-\bar{z}) - iJ_1(-\bar{z})$ shows that the roots of $J_0(z) - iJ_1(z) = 0$ appear in pairs of z and $-\bar{z}$, giving the required proof via Part (1).

A more direct proof is given by the similarity transformation

$$D^{-1}AD = i \begin{pmatrix} 1 & f_2 & & 0 \\ -f_2 & 0 & f_3 & \\ & -f_3 & 0 & \cdots \\ 0 & & \cdots & \cdots \end{pmatrix} = \text{a pure imaginary matrix,}$$

where $D = diag[1, i, i^2, \cdots] \in B(l^2)$, a diagonal matrix, and $D^{-1} \in B(l^2)$. The eigenvalues of A are precisely those of $D^{-1}AD$, whose eigenvalues obviously appear in pairs of, say, μ and $-\bar{\mu}$, since it is a pure imaginary matrix.

The proof of Theorem 1.4 is now complete.

§5 Proof of Theorem 1.5.

We keep the same notation as defined in Theorem 1.5.

Proof of Part (1). By Theorem 1.4, we have $z \neq 0$ and $\lambda = 2/z$ is a simple eigenvalue of the compact complex symmetric matrix $A \in B(l^2)$ defined there, with the corresponding eigenvector $x = [J_1(z), \sqrt{2}J_2(z), \cdots]^T \in l^2$. Let the infinite matrix A_n be defined by

$$(5.1) \qquad A_n = \left(\begin{array}{ccccc|c} i & f_2 & & & 0 & \\ f_2 & 0 & f_3 & & & 0 \\ & f_3 & 0 & \ddots & f_n & \\ 0 & & \ddots & f_n & 0 & \\ \hline & & 0 & & & 0 \end{array}\right) = \begin{pmatrix} \tilde{A}_n & 0 \\ 0 & 0 \end{pmatrix}, \qquad n = 1, 2, \cdots .$$

Lemma 5.1. *The hypothesis (H) in §1 holds for this particular choice of* $\{A_n\}_1^\infty$, A ,λ *and* x *defined above.*

Proof. From what we stated above, it only remains to verify $\| A_n - A \| \to 0$ as $n \to \infty$ and $x^T x \neq 0$. The first is clear from the inequality

$$(5.2) \qquad \| A_n - A \|^2 \leq 2(f_{n+1}^2 + f_{n+2}^2 + \cdots) = 2(\frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \cdots) = \frac{2}{n} .$$

For the proof of $x^T x \neq 0$, we may use the following remarkable summation formula [20, p.152]:

$$(5.3) \qquad \sum_{k=0}^{\infty}(m+2k)J_{m+2k}^2(z) = \frac{1}{4}z^2\{J_{m-1}^2(z) - J_{m-2}(z)J_m(z)\} \quad \text{for any } m, \text{ real or complex.}$$

Hence we find

$$(5.4) \qquad \begin{aligned} x^T x &= J_1^2(z) + 2J_2^2(z) + \cdots \\ &= \{J_1^2(z) + 3J_3^2(z) + \cdots\} + \{2J_2^2(z) + 4J_4^2(z) + \cdots\} \\ &= (z^2/4)\{J_0^2(z) - J_{-1}(z)J_1(z)\} + (z^2/4)\{J_1^2(z) - J_0(z)J_2(z)\} \\ &= (z/2) \cdot iJ_0^2(z) \ , \end{aligned}$$

where in the final equality, $J_0(z) - iJ_1(z) = 0$, $J_{-1}(z) = -J_1(z)$ and the recurrence relation $J_2(z) = (2/z)J_1(z) - J_0(z)$ are used. Now, $z \neq 0$ as noted above; also $J_0(z) \neq 0$, for otherwise $J_0(z) - iJ_1(z) = 0$

would imply $J_1(z) = 0$, a contradiction of the fact that $J_0(z)$ and $J_1(z)$ do not vanish simultaneously [4, p.105]. It follows that $(z/2) \cdot iJ_0^2(z) \neq 0$, i.e., $x^T x \neq 0$. ∎

Theorem 1.1 now applies. In particular, Theorem 1.1 (a) and (b) guarantee the existence of a sequence $\{\lambda_n\}$ of eigenvalues of A_n such that $\lambda_n \to \lambda(\neq 0)$ and such that λ_n is simple and nonzero for all large n. These nonzero eigenvalues are precisely those of $\tilde{A}_n$. This proves Part (1).

Remark. The computation

$$det\tilde{A}_n = \begin{cases} (-f_n^2)(-f_{n-2}^2)\cdots(-f_2^2), & n : \text{even} \\ (-f_n^2)(-f_{n-2}^2)\cdots(-f_3^2)\cdot i, & n : \text{odd} \end{cases}$$

shows that the λ_n's are nonzero for all $n = 1, 2, \cdots$.

Proof of Part (2). Using the relation $z = 2/\lambda$ and $z_n = 2/\lambda_n$, and noting $z_n \to z$, we have

$$\frac{z_n - z}{z} = \frac{z_n}{2}(\lambda - \lambda_n) = \frac{z}{2}(\lambda - \lambda_n)(1 + o(1)) \quad \text{as } n \to \infty \ . \tag{5.5}$$

One is thus to estimate $\lambda - \lambda_n$. For $n = 1, 2, \cdots$, let v_n denote the n^{th} truncation of x, i.e.

$$v_n = [J_1(z), \sqrt{2}J_2(z), \cdots, \sqrt{n}J_n(z), 0, 0, \cdots]^T \in l^2 \tag{5.6}$$

and let μ_n denote the generalized Rayleigh quotient

$$\mu_n = \frac{v_n^T A_n v_n}{v_n^T v_n} \ , \tag{5.7}$$

where A_n is defined by (5.1). Clearly $v_n \to x$.

We decompose $\lambda - \lambda_n$ as

$$\lambda - \lambda_n = (\lambda - \mu_n) + (\mu_n - \lambda_n) \tag{5.8}$$

and will show that the first term is dominant, i.e. $(\mu_n - \lambda_n)/(\lambda - \mu_n) \to 0$ ($n \to \infty$, z fixed), so that it suffices to estimate $\lambda - \mu_n$ in stead of $\lambda - \lambda_n$. (It is here that μ_n makes itself useful.)

We estimate the first term $\lambda - \mu_n$ first. By the definition of μ_n,

$$\lambda - \mu_n = v_n^T(\lambda v_n - A_n v_n)/v_n^T v_n \ . \tag{5.9}$$

But from the relation $Ax = \lambda x$ and by the definition of A_n and v_n, we deduce

$$\lambda v_n - A_n v_n = [\underbrace{0, \cdots, 0}_{n-1 \text{ zeros}}, J_{n+1}(z)/\sqrt{n}, 0, \cdots]^T \quad . \tag{5.10}$$

Using this in (5.9), we find

$$\begin{aligned} \lambda - \mu_n &= J_n(z)J_{n+1}(z)/v_n^T v_n \qquad \text{(by the definition of } v_n) \\ &= J_n(z)J_{n+1}(z)/\{x^T x(1+o(1))\} \qquad (\text{by } v_n \to x) \\ &= \frac{J_n(z)J_{n+1}(z)}{(z/2)\, i\, J_0^2(z)}(1+o(1)) \qquad (\text{by } (5.4)) \\ &= O\big(J_n(z)J_{n+1}(z)\big) \qquad (n \to \infty, \quad z \text{ fixed}) \end{aligned} \tag{5.11}$$

We next estimate $\mu_n - \lambda_n$. By Lemma 5.1 and by the fact that $v_n \to x$, the hypothesis of Theorem 1.3 is seen to be satisfied. The conclusion of the theorem then gives

$$| \mu_n - \lambda_n | \le \frac{1}{| x^T x |} \| (A_n - \mu_n I)v_n \|^2 \| (A - \lambda I)^{-1} \|_S \; (1 + o(1)) \qquad (n \to \infty, \quad z \text{ fixed}) \quad . \tag{5.12}$$

Now,

$$\begin{aligned} (A_n - \mu_n I)v_n &= (A_n v_n - \lambda v_n) + (\lambda - \mu_n)v_n \\ &= -[0, \cdots, 0, J_{n+1}(z)/\sqrt{n}, \, 0, \cdots]^T + \frac{J_n(z)J_{n+1}(z)}{(z/2)\, i\, J_0^2(z)} v_n(1+o(1)) \qquad (n \to \infty, \quad z \text{ fixed}) \quad , \end{aligned}$$

using (5.10) and (5.11). Hence,

$$\| (A_n - \mu_n I)v_n \| = \Big(| J_{n+1}(z) | / \sqrt{n} \Big)(1 + o(1)) \qquad (n \to \infty, \quad z \text{ fixed}) \quad . \tag{5.13}$$

Substituting this into the inequality (5.12), we find

$$\mu_n - \lambda_n = O\Big(J_{n+1}^2(z)/n\Big) \qquad (n \to \infty, \quad z \text{ fixed}) \quad . \tag{5.14}$$

From (5.11) and (5.14) follows

$$\frac{\mu_n - \lambda_n}{\lambda - \mu_n} = O\Big(\frac{J_{n+1}(z)}{n J_n(z)}\Big) \;\to 0 \qquad (n \to \infty, \; z \text{ fixed}) \;\; (\text{by } (4.4)) \quad , \tag{5.15}$$

proving the claim $(\mu_n - \lambda_n)/(\lambda - \mu_n) \to 0$.

It now follows from (5.8) and (5.15) that $\lambda - \lambda_n = (\lambda - \mu_n)(1 + o(1))$. Substituting this into (5.5) and using (5.11) for $\lambda - \mu_n$, one finally obtains

$$\frac{z_n - z}{z} = \frac{J_n(z)J_{n+1}(z)}{iJ_0^2(z)}(1 + o(1)) \qquad (n \to \infty,\ z \text{ fixed}), \tag{5.16}$$

proving the first relation in Part (2) of Theorem 1.5

It only remains to prove the second relation in Part (2). We will first prove for the fourth quadrant roots z that

$$J_0^2(z) = -(2i/\pi)(1 + o(1)) \qquad \text{as } |z| \to \infty. \tag{5.17}$$

Thus let z be the j^{th} root of $J_0(z) - iJ_1(z) = 0$ in the fourth quadrant. Then by the Macdonald's formula (1.3) in §1,

$$\begin{cases} z = z_j = r_j e^{i[(\pi/2)+\theta_j]}, & r_j = j\pi(1 + o(1)) \qquad (j \to \infty) \\ \theta_j = -(\pi/2) - \{\alpha_j/(j\pi)\}(1 + o(1)) \quad (j \to \infty), & \alpha_j = \{\ln(4j\pi)\}/2 \end{cases} \tag{5.18}$$

On the other hand, the asymptotic expansion of $J_0(z)$ as $|z| \to \infty$ is given from [1, p.364, 9.2.1] by

$$J_0(z) = \sqrt{\frac{2}{\pi z}}\{\cos(z - \frac{1}{4}\pi) + e^{|Im(z)|}O(|z|^{-1})\}, \qquad |arg\ z| < \pi \tag{5.19}$$

From (5.18) and (5.19) one obtains, after some computation, the desired asymptotic expansion (5.17).

An arbitrary third quadrant root z', being the reflection of some fourth quadrant root z about the imaginary axis, has the form $z' = -\bar{z}$. Hence,

$$J_0^2(z') = J_0^2(-\bar{z}) = \overline{J_0^2(z)} = (2i/\pi)(1 + o(1)) \qquad \text{as } |z'| \to \infty \tag{5.20}$$

Theorem 1.5 is now fully proved.

§6 Proof of Theorem 1.7.

We inherit the notation established in Theorem 1.6.

In order to obviate the difficulty of directly proving the first half, we let

$$(6.1) \qquad B = \begin{pmatrix} 0 & b_1 & & 0 \\ b_1 & 0 & b_2 & \\ & b_2 & 0 & \ddots \\ 0 & & \ddots & \ddots \end{pmatrix} , \qquad b_k = \frac{1}{\sqrt{m+k}\sqrt{m+k+1}} , \quad k = 1, 2, \cdots ,$$

which is compact, symmetric and tridiagonal. Then by direct computation, one sees that B^2 is a symmetric band matrix with its $(i,j)^{th}$ component $= 0$ for all i and j such that $i + j =$ odd and that the matrix A is obtained from B^2 by deleting the odd-numbered rows and columns. It follows that every eigenvalue of A is an eigenvalue of B^2. Hence, it is enough to prove that 0 is not an eigenvalue of B^2, hence, of B. It may be easily verified that $y = 0$ is the only solution in l^2 for $By = 0$. This proves the first half of Theorem 1.7.

The proof of the last half of Theorem 1.7 may be carried out exactly in parallel with the proof of Theorem 1.4 Part (3), and is omitted.

§7 Proof of Theorem 1.8.

We use the same notation as established in Theorem 1.8.

Proof of Part(1). The proof runs exactly in parallel with the proof for Theorem 1.5 Part(1). For $n = 1, 2, \cdots$, we let A_n denote the n^{th} truncation of A, i.e. let

$$A_n = \begin{pmatrix} \tilde{A}_n & 0 \\ 0 & 0 \end{pmatrix} . \tag{7.1}$$

Then $\| A_n - A \| \to 0$. Indeed, for all sufficiently large n we have

$$\begin{aligned} \| A_n - A \|^2 &\le (d_{n+1}^2 + d_{n+2}^2 + \cdots) + 2(f_{n+1}^2 + f_{n+2}^2 + \cdots) \\ &< (\frac{4}{n^4} + \frac{4}{(n+1)^4} + \cdots) + 2(\frac{1}{n^4} + \frac{1}{(n+1)^4} + \cdots) \\ &< \frac{2}{(n-1)^3} \to 0 . \end{aligned} \tag{7.2}$$

We also find $x^T x \neq 0$. In fact,

$$\begin{aligned} x^T x &= (m+2)J_{m+2}^2(z) + (m+4)J_{m+4}^2(z) + \cdots \\ &= \frac{z^2}{4}\{J_{m+1}^2(z) - J_m(z)J_{m+2}(z)\} \qquad \text{(by (5.3))} \\ &= \frac{z^2}{4}J_{m+1}^2(z) \qquad (\text{by } J_m(z) = 0) \\ &\neq 0 \qquad (\text{since } J_m(z) \text{ and } J_{m+1}(z) \text{ do not vanish simultaneously}) . \end{aligned} \tag{7.3}$$

The nonzero eigenvalues of A_n are obviously those of $\tilde{A}_n$. The eigenvalue λ under consideration is known to be simple from Theorem 1.7. Theorem 1.1 now applies and finishes the proof of Part (1).

Proof of Part (2). The proof may be carried out again in parallel with the proof of Theorem 1.5 Part (2). First, we find

$$\frac{z_n - z}{z} = \frac{z^2}{8}(\lambda - \lambda_n)(1 + o(1)) \qquad (n \to \infty,\ z \text{ fixed}) . \tag{7.4}$$

For $n = 1, 2, \cdots$ let v_n denote the n^{th} truncation of the exact eigenvector corresponding to the exact eigenvalue λ, i.e. let

$$v_n = [\sqrt{m+2}J_{m+2}(z), \sqrt{m+4}J_{m+4}(z), \cdots, \sqrt{m+2n}J_{m+2n}(z), 0, \cdots]^T \tag{7.5}$$

and let μ_n denote the generalized Rayleigh quotient

$$\mu_n = (v_n^T A_n v_n)/(v_n^T v_n) \ . \tag{7.6}$$

Then one obtains after some computation

$$\begin{aligned}
\lambda - \mu_n &= v_n^T(\lambda_n v_n - A_n v_n)/v_n^T v_n \\
&= \frac{1}{v_n^T v_n}\frac{J_{m+2n}(z)J_{m+2n+2}(z)}{m+2n+1} \\
&= \frac{1}{x^T x}\frac{J_{m+2n}(z)J_{m+2n+2}(z)}{m+2n+1}(1+o(1)) \quad (n\to\infty,\ z \text{ fixed}) \quad (\text{by } v_n \to x) \\
&= \frac{J_{m+2n}(z)J_{m+2n+2}(z)}{(z^2/4)J_{m+1}^2(z)(m+2n+1)}(1+o(1)) \quad (n\to\infty,\ z \text{ fixed}) \quad (\text{by (7.3)})
\end{aligned} \tag{7.7}$$

and by Theorem 1.3

$$\begin{aligned}
|\mu_n - \lambda_n| &\le \frac{1}{|x^T x|} \| (A_n - \mu_n I)v_n \|^2 \| (A-\lambda I)^{-1} \|_S (1+o(1)) \quad (n\to\infty,\ z \text{ fixed}) \\
&= \frac{1}{|x^T x|} \left| \frac{J_{m+2n+2}(z)}{(m+2n+1)\sqrt{m+2n}} \right|^2 \| (A-\lambda I)^{-1} \|_S (1+o(1)) \\
&= O\Big(\frac{J_{m+2n+2}^2(z)}{n^3}\Big) \quad (n\to\infty,\ z \text{ fixed}) \ .
\end{aligned} \tag{7.8}$$

It follows that $(\mu_n - \lambda_n)/(\lambda - \mu_n) \to 0$ and

$$\begin{aligned}
\lambda - \lambda_n &= (\lambda - \mu_n)(1+o(1)) \\
&= \frac{J_{m+2n}(z)J_{m+2n+2}(z)}{(z^2/4)J_{m+1}^2(z)(m+2n+1)}(1+o(1)) \quad (n\to\infty,\ z \text{ fixed}) \quad (\text{by (7.7)}) \ .
\end{aligned} \tag{7.9}$$

Substitution of the last relation into (7.4) gives the first equality in Part (2).

It only remains to prove the second equality of Part (2). By [1, p.364, 9.2.1], we have the following expression for $J_{m+1}(z)$:

$$J_{m+1}(z) = \sqrt{\frac{2}{\pi z}}\{\cos(z - \frac{1}{2}(m+1) - \frac{1}{4}\pi) + e^{|Im(z)|}O(|z^{-1}|)\} \quad (\arg|z| < \pi,\ |z| \to \infty,\ m \text{ fixed}) \ . \tag{7.10}$$

On the other hand, the following asymptotic expansion for the large zeros of $J_m(z) = 0$ is known [20, p.506]:

$$z = (k + \frac{1}{2}m - \frac{1}{4})\pi + O(z^{-1}) \quad (k\to\infty,\ m \text{ fixed}) \ . \tag{7.11}$$

Substitution of (7.11) into (7.10) gives

$$J_{m+1}(z) = \sqrt{\frac{2}{\pi z}}(-1)^{k-1}(1+o(1)) \quad (|z| \to \infty,\ J_m(z) = 0,\ m \text{ fixed}) \ , \tag{7.12}$$

giving the second relation in Part (2).

This completes the proof of Theorem 1.8.

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TITLE

The Eigenvalue Problem for Infinite Compact Complex Symmetric Matrices with Application to the Numerical Computation of Complex Zeros of $J_0(z)-iJ_1(z)$ and of Bessel Functions $J_m(z)$ of Any Real Order m

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ABSTRACT

Consider computing simple eigenvalues of a given compact infinite matrix regarded as operating in the complex Hilbert space l^2 by computing the eigenvalues of the truncated finite matrices and taking an obvious limiting process. In this paper we deal with a special case where the given matrix is compact, complex and symmetric (but not necessarily Hermitian). Two examples of application are studied. The first is concerned with the equation $J_0(z) - iJ_1(z) = 0$ appearing in the analysis of the solitary wave runup on a sloping beach, and the second with the zeros of the Bessel function $J_m(z)$ of any real order m. In each case, the problem is reformulated as an eigenvalue problem for a compact complex symmetric tridiagonal matrix operator in l^2 whose eigenvalues are all simple. A complete error analysis for the numerical solution by truncation is given based on the general theorems proved in this paper, where the usefulness of the seldom-used generalized Rayleigh quotient is demonstrated.

SUPPLEMENTARY NOTES