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Globally Determining a Minimum-Area Rectangle Enclosing the Projection of a Higher-Dimensional Set

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Abstract This paper addresses itself to methods for finding a rectangle of minimum area which encloses the projection of a given convex set in a higher dimensional space onto the plane of the rectangle. In case the given set is a polytope, a parametric simplex algorithm is proposed for obtaining a global solution, which needs the polynomial number of arithmetics on the average. In case the set is nonlinear convex, it is shown that a successive underestimation method generates an ϵ-global solution in finite time if $\epsilon > 0$.

Keywords nonconvex minimization, global minimization, parametric simplex method, successive underestimation method, computational geometry

1 Introduction

In this paper, we describes practical methods to determine a rectangle of minimum area which encloses the projection of a given convex set $D \in R^n$ onto the plane of the rectangle. This problem is a generalization of that introduced by Freeman and Shapiro [6] and can be applied in certain packing and optimum layout problems [9,14].

If D is a polytope and its vertices are known, we can solve the problem in $O(N \log N)$ time by using the techniques of computational geometry [3,7,18], where N represents the number of vertices (see Section 4). In more general cases, however, it is much more complicated to find a global solution because the problem has a highly nonconvex structure.

In Section 2, we propose a parametric simplex algorithm for obtaining a global solution of the problem, in which D is given by a system of linear inequalities. The average number of arithmetics needed for the algorithm are polynomial order of the

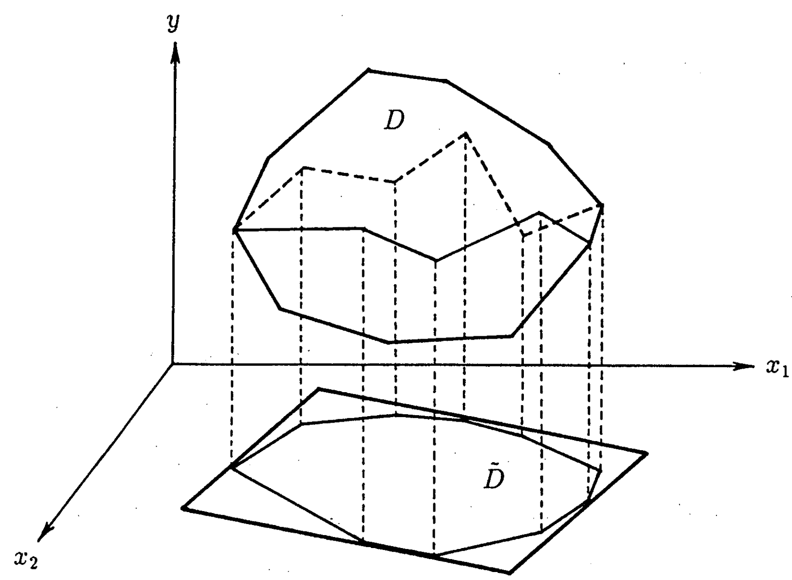

Figure 2.1: Illustration of the problem in R^3

size of the linear system. In Section 3, we consider the case where D is a nonlinear convex set. We define a function which underestimates the area of the encasing rectangle. By using the underestimating function we construct a successive underestimation algorithm for obtaining an ϵ-global solution. We discuss some remarks in Section 4.

2 Minimum-Area Rectangle Enclosing the Projection of a Polytope

2.1 Formulation of the problem

Let $D \in R^n$ be a given polytope defined as follows:

$$D = \{(x, y) \in R^n \mid A_1x + A_2y \leq b\}, \tag{2.1}$$

where $x \in R^2$ and $y \in R^{n-2}$ are vectors of variables and $A_1 \in R^{m\times 2}$, $A_2 \in R^{m\times(n-2)}$ and $b \in R^m$ are constants. We assume in the sequel that D has an interior point. Let us denote by $\tilde{D}$ the projection of D onto the plane of x, i.e.,

$$\tilde{D} = \{x \in R^2 \mid (\exists y \in R^{n-2})\ A_1x \leq b - A_2y\}. \tag{2.2}$$

Our problem is to find a minimum-area rectangle in the x-plane which encloses $\tilde{D}$ (see Figure 2.1). The set $\tilde{D}$ is a polytope because it is the image of a polytope under a linear transformation from R^n to R^2 (see Theorem 19.3 of [14]).

For any fixed $\xi \in R^2$ let us consider the following two linear programming problems:

$$\mathrm{P}_1(\xi) \left| \begin{array}{ll} \text{maximize} & f_0(x, y; \xi) = \xi^t x \\ \text{subject to} & A_1 x + A_2 y \leq b, \end{array} \right. \tag{2.3}$$

$$\mathrm{P}_2(\xi) \left| \begin{array}{ll} \text{minimize} & f_0(x, y; \xi) = \xi^t x \\ \text{subject to} & A_1 x + A_2 y \leq b. \end{array} \right. \tag{2.4}$$

Since their common feasible set D is nonempty and bounded, $\mathrm{P}_1(\xi)$ and $\mathrm{P}_2(\xi)$ have optimal solutions $(x^1(\xi), y^1(\xi))$ and $(x^2(\xi), y^2(\xi))$, respectively. Let us define

$$f(\xi) = f_0(x^1(\xi), y^1(\xi); \xi) - f_0(x^2(\xi), y^2(\xi); \xi). \tag{2.5}$$

If $||\xi|| = 1$, $f(\xi)$ corresponds to the diameter of $\tilde{D}$ in the direction of ξ. Thus our problem can be formulated as follows:

$$\mathrm{P} \left| \begin{array}{ll} \text{minimize} & f(\xi_1) \cdot f(\xi_2) \\ \text{subject to} & ||\xi_1|| = ||\xi_2|| = 1, \quad \xi_1^t \xi_2 = 0, \end{array} \right. \tag{2.6}$$

where the objective function expresses the area of an encasing rectangle of $\tilde{D}$.

Theorem 2.1 *$f(\cdot)$ is a convex polyhedral function and satisfies the following:*

$$f(\alpha\xi) = \alpha f(\xi), \quad \forall \alpha \geq 0. \tag{2.7}$$

Proof Follows from the well-known results of linear programming [4,5] and the definition of $f(\cdot)$. □

For $\lambda \in [0, 1]$ let

$$\xi_1(\lambda) = \begin{pmatrix} \lambda \\ 1 - \lambda \end{pmatrix}, \quad \xi_2(\lambda) = \begin{pmatrix} \lambda - 1 \\ \lambda \end{pmatrix} \tag{2.8}$$

and let us define

$$F(\lambda) = \frac{f(\xi_1(\lambda)) \cdot f(\xi_2(\lambda))}{\lambda^2 + (1-\lambda)^2}. \tag{2.9}$$

Then we have

$$F(\lambda) = f(\frac{\xi_1(\lambda)}{||\xi_1(\lambda)||}) \cdot f(\frac{\xi_2(\lambda)}{||\xi_2(\lambda)||}).$$

by noting (2.7) and

$$[\xi_1(\lambda)]^t[\xi_2(\lambda)] = 0$$

for every $\lambda \in [0,1]$. Hence solving P amounts to locate a global minimum point λ^* of $F(\lambda)$ over the interval $[0,1]$.

2.2 Parametric simplex method for solving P

Let us proceed to the algorithm to find a global minimum point λ^* of $F(\lambda)$ over the interval $[0,1]$. Since the projection $\tilde{D}$ of D is a polytope in R^2, the following theorem [6] is useful to construct the algorithm.

Theorem 2.2 *A minimum-area rectangle enclosing a polytope in R^2 has a side collinear with one of the facet of the polytope.*

Proof See Theorem 2 of [6]. □

Corollary 2.3 *At least one of $P_j(\xi_k(\lambda^*))$, $j = 1,2$; $k = 1,2$, has multiple optimal solutions.*

Proof Immediately follows from Theorem 2.2. □

Each of the linear programs $P_j(\xi_k(\lambda))$'s can be solved parametrically by increasing the value of λ from zero to one. By barring degeneracy, for $P_j(\xi_k(\lambda))$ we obtain a sequence of intervals $[0, \lambda_1^{jk}], [\lambda_1^{jk}, \lambda_2^{jk}], \ldots, [\lambda_{p_{jk}}^{jk}, 1]$ such that $\lambda_l^{jk} < \lambda_{l+1}^{jk}$ and the associated sequence of bases $B_0^{jk}, B_1^{jk}, \ldots B_{p_{jk}}^{jk}$ such that B_l^{jk} is optimal for all $\lambda \in [\lambda_l^{jk}, \lambda_{l+1}^{jk}]$. Since $P_j(\xi_k(\lambda_l^{jk}))$ has different basic optimal solutions corresponding to the bases B_{l-1}^{jk} and B_l^{jk}, respectively, Corollary 2.3 can be rewritten as follows:

Corollary 2.4 *Among* λ_l^{jk}, $k = l, \ldots, p_{jk}$; $j = 1, 2$; $k = 1, 2$, *is a global minimum point* λ^* *of* $F(\lambda)$ *over the interval* $[0, 1]$. □

Thus we obtain the following parametric simplex algorithm for solving the problem P:

Algorithm A

Step 1 Solve the linear programs $\mathrm{P}_j(\xi_k(0))$, $j = 1, 2$; $k = 1, 2$. Let B_0^{jk}, $j = 1, 2$; $k = 1, 2$, be their respective optimal bases.

Step 2 Solve each of $\mathrm{P}_j(\xi_k(\lambda))$, $j = 1, 2$; $k = 1, 2$, parametrically by increasing $\lambda \in [0, 1]$. Let $[0, \lambda_1^{jk}], [\lambda_1^{jk}, \lambda_2^{jk}], \ldots, [\lambda_{p_{jk}}^{jk}]$ be a sequence of intervals generated in the course of computation and $B_0^{jk}, B_1^{jk}, \ldots, B_{p_{jk}}^{jk}$ be the associated sequences of bases such that B_l^{jk} is an optimal basis of $\mathrm{P}_j(\xi_k(\lambda))$ for all $\lambda \in [\lambda_l^{jk}, \lambda_{l+1}^{jk}]$.

Step 3 Let

$$\lambda^* \in \operatorname{argmin}\{F(\lambda_l^{jk}) \mid l = 1, \ldots, p_{jk};\ j = 1, 2;\ k = 1, 2\}.$$

and let

$$\xi_1^* = \frac{\xi_1(\lambda^*)}{||\xi_1(\lambda^*)||}, \qquad \xi_2^* = \frac{\xi_2(\lambda^*)}{||\xi_2(\lambda^*)||}.$$

□

After finitely many iterations we obtain a globally optimal solution (ξ_1^*, ξ_2^*) of P by barring degeneracy.

2.3 Average performance of the algorithm

Adler and Haimovich showed in [1,8] that the average number of simplex pivots needed for solving a parametric linear program of the form:

$$\left|\begin{array}{ll} \text{minimize} & \lambda c^t x + (1 - \lambda) d^t x \\ \text{subject to} & Ax \leq b, \end{array}\right. \tag{2.10}$$

which is generated randomly, is $O(\min\{m,n\})$, where m and n represent the size of A. Since each $\mathrm{P}_j(\xi_k(\lambda))$ solved at Step 2 of Algorithm A is just the same form as (2.10), the expected number of intervals $[\lambda_l^{jk}, \lambda_{l+1}^{jk}]$'s will be no more than $O(\min\{m,n\})$. On the other hand, the problems solved at Step 1 are standard linear programs, which can be solved in $O((\min\{m,n\})^2)$ steps on the average by using the algorithm developed by Todd [16] or Adler and Megiddo [2]. Hence the average number of arithmetics needed for Algorithm A is a lower order polynomial functions of the size of the matrices A_1 and A_2.

3 Minimum Rectangle Enclosing the Projection of a Convex Set

3.1 Formulation of the problem

Now let us consider a more general case of the problem stated in the previous section, which finds a minimum-area rectangle in R^2 enclosing the projection of a given non-linear convex set in R^n onto the plane of the rectangle. Let D be a given convex set defined by

$$D = \{(x,y) \in R^n \mid g_i(x,y) \leq 0, \quad i = 1,\ldots,m\}, \tag{3.1}$$

where $x \in R^2$ and $y \in R^{n-2}$ and $g_i : R^n \to R$, $i = 1,\ldots,m$, are nonlinear convex functions. We assume that D is compact and has an interior point. The problem is formulated as follows:

$$\mathrm{Q} \left| \begin{array}{ll} \text{minimize} & g(\xi_1)\cdot g(\xi_2) \\ \text{subject to} & ||\xi_1|| = ||\xi_2|| = 1, \quad \xi_1^t \xi_2 = 0, \end{array} \right. \tag{3.2}$$

where ξ_1, $\xi_2 \in R^2$ are variables and $g(\xi)$ represents the difference of the optimal values of the following two convex programming problems:

$$\mathrm{Q}_1(\xi) \left| \begin{array}{ll} \text{maximize} & g_0(x,y;\xi) = \xi^t x \\ \text{subject to} & g_i(x,y) \leq 0, \quad i = 1,\ldots,m, \end{array} \right. \tag{3.3}$$

$$\mathrm{Q}_2(\xi) \left| \begin{array}{ll} \text{minimize} & g_0(x,y;\xi) = \xi^t x \\ \text{subject to} & g_i(x,y) \leq 0, \quad i = 1,\ldots,m. \end{array} \right. \tag{3.4}$$

Thus we have

$$g(\xi) = g_0(x^1(\xi), y^1(\xi); \xi) - g_0(x^2(\xi), y^2(\xi); \xi), \tag{3.5}$$

where $(x^j(\xi), y^j(\xi))$ is an optimal solution of $Q_j(\xi)$. We obtain the following theorem in the similar way to Theorem 2.1:

Theorem 3.1 *$g(\cdot)$ is a convex function and satisfies that*

$$g(\alpha\xi) = \alpha g(\xi), \quad \forall \alpha \geq 0. \tag{3.6}$$

□

As before let

$$\xi_1(\lambda) = \begin{pmatrix} \lambda \\ 1-\lambda \end{pmatrix}, \qquad \xi_2(\lambda) = \begin{pmatrix} \lambda - 1 \\ \lambda \end{pmatrix} \tag{3.7}$$

for $\lambda \in [0, 1]$. Then we need to locate a global minimum point λ^* of a function:

$$G(\lambda) = \frac{g(\xi_1(\lambda)) \cdot g(\xi_2(\lambda))}{\lambda^2 + (1-\lambda)^2}. \tag{3.8}$$

over the interval $[0, 1]$.

3.2 Underestimating function of G

Since the denominator $\lambda^2 + (1-\lambda)^2$ of $G(\lambda)$ is positive for all λ, the slope of $G(\cdot)$ has the same sign as that of the numerator:

$$\tilde{G}(\lambda) = g(\xi_1(\lambda)) \cdot g(\xi_2(\lambda)). \tag{3.9}$$

Hence, it is sufficient to enumerate every local minimum of $\tilde{G}(\lambda)$ over the interval $[0, 1]$.

Let $\mathcal{H}$ be a family of functions $h(\cdot\,; p)$ which has the following form:

$$h(\lambda; p) = [p_1\lambda + p_2(1-\lambda)] \cdot [p_3(\lambda - 1) + p_4\lambda], \tag{3.10}$$

where $p = (p_1, p_2, p_3, p_4)^t$ satisfies that

$$p_1 = x_1^1 - x_1^2; \;\; p_2 = x_2^1 - x_2^2; \;\; p_3 = x_1^3 - x_1^4; \;\; p_4 = x_2^3 - x_2^4 \tag{3.11}$$

for some x^1, x^2, x^3 and x^4 in the projection $\tilde{D}$ of D. Then $\tilde{G}(\lambda)$ is the pointwise maximum of functions belonging to $\mathcal{H}$ over the interval $[0,1]$. Let

$$\overline{x}_j = \max\{x_j \mid g_i(x,y) \leq 0, \ i=1,\ldots,m\}, \qquad j=1,2, \tag{3.12}$$

$$\underline{x}_j = \min\{x_j \mid g_i(x,y) \leq 0, \ i=1,\ldots,m\}, \qquad j=1,2. \tag{3.13}$$

Lemma 3.2 *Every function $h(\cdot\,;p) \in \mathcal{H}$ is Lipschitz continuous over the interval $[0,1]$ with a Lipschitz constant:*

$$L = (\overline{x}_1 - \underline{x}_1 + \overline{x}_2 - \underline{x}_2)^2. \tag{3.14}$$

Proof We have

$$\frac{\partial h(\lambda;p)}{\partial \lambda} = 2(p_1-p_2)(p_3+p_4)\lambda - (p_1-p_2)p_3 + p_2(p_3+p_4),$$

and

$$\left|\frac{\partial h(\lambda;p)}{\partial \lambda}\right| \leq \max\left\{\left|\frac{\partial h(0;p)}{\partial \lambda}\right|, \left|\frac{\partial h(1;p)}{\partial \lambda}\right|\right\}$$

for any $\lambda \in [0,1]$. It follows from (3.11) $\sim$ (3.13) that

$$\begin{aligned}\left|\frac{\partial h(0;p)}{\partial \lambda}\right| &= |-p_1p_3 + 2p_2p_3 + p_2p_4| \\ &\leq |p_1||p_3| + 2|p_2||p_3| + |p_2||p_4| \\ &\leq (\overline{x}_1 - \underline{x}_1 + \overline{x}_2 - \underline{x}_2)^2.\end{aligned}$$

Similarly, we have $\left|\frac{\partial h(1;p)}{\partial \lambda}\right| \leq (\overline{x}_1 - \underline{x}_1 + \overline{x}_2 - \underline{x}_2)^2$. □

Let us define a piecewise linear function:

$$U(\lambda;\lambda_s,\lambda_t) = \max\{-L\lambda + L\lambda_s + \tilde{G}(\lambda_s), \ L\lambda - L\lambda_t + \tilde{G}(\lambda_t)\}. \tag{3.15}$$

Theorem 3.3 *For any $\lambda \in [\lambda_s, \lambda_t] \subset [0,1]$*

$$U(\lambda;\lambda_s,\lambda_t) \leq \tilde{G}(\lambda). \tag{3.16}$$

Proof Assume the contrary. Then there exists $\lambda' \in [\lambda_s, \lambda_t]$ such that

$$U(\lambda'; \lambda_s, \lambda_t) > \tilde{G}(\lambda'). \tag{3.17}$$

Let

$$\tilde{G}(\lambda_t) = h(\lambda_t; p') \tag{3.18}$$

and assume without loss of generality that

$$U(\lambda'; \lambda_s, \lambda_t) = L\lambda' - L\lambda_t + \tilde{G}(\lambda_t). \tag{3.19}$$

Since $\tilde{G}(\cdot)$ is the pointwise maximum of functions of $\mathcal{H}$, we have

$$\tilde{G}(\lambda') \geq h(\lambda'; p'). \tag{3.20}$$

It follows from (3.17) $\sim$ (3.20) that

$$|\frac{h(\lambda'; p') - h(\lambda_t; p')}{\lambda' - \lambda_t}| > L,$$

which contradicts Lemma 3.2. □

For the minimum point $\lambda_0 = 1/2$ of $U(\lambda; 0, 1)$ let us define

$$U_1(\lambda) = \min\{U(\lambda, 0, \lambda_0),\ U(\lambda, \lambda_0, 1)\}. \tag{3.21}$$

Then $U_1(\lambda)$ underestimates $\tilde{G}(\lambda)$ for all $\lambda \in [0, 1]$ as well as $U(\lambda\,; 0, 1)$. In addition, it is a better underestimating function than $U(\cdot\,; 0, 1)$, i.e.,

$$U(\lambda, 0, 1) \leq U_1(\lambda) \leq \tilde{G}(\lambda), \qquad \forall\lambda \in [0, 1].$$

Another underestimating function $U_2(\cdot)$ of $\tilde{G}(\cdot)$ over $[0, 1]$ would be generated by applying the same operation to (3.21) to either $U(\lambda, 0, \lambda_0)$ or $U(\lambda, \lambda_0, 1)$ (see Figure 3.1). In this way, we would obtain a sequence of underestimating functions $U_l(\cdot)$'s of $\tilde{G}(\cdot)$ as follows:

$$U_1(\lambda) \leq U_2(\lambda) \leq \cdots \leq U_l(\lambda) \leq \cdots \leq \tilde{G}(\lambda), \qquad \forall\lambda \in [0, 1]. \tag{3.22}$$

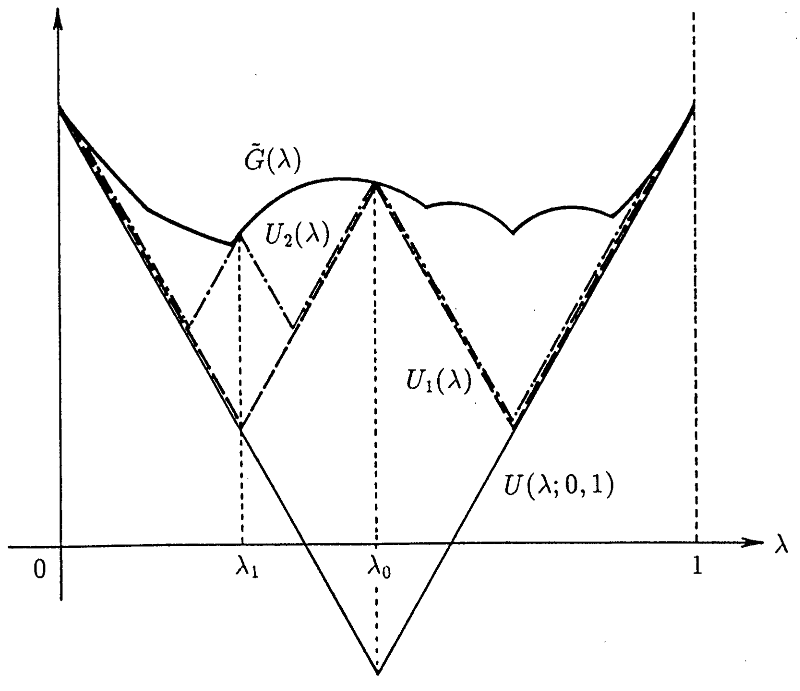

Figure 3.1: Underestimating functions of $\tilde{G}(\cdot)$

3.3 Successive underestimation method for solving Q

By exploiting the property of $U(\cdot\,; \lambda_s, \lambda_t)$ the following recursive procedure $\mathrm{B}(l, z, \lambda_s, \lambda_t)$ can be constructed, which generate an ϵ-minimum point λ^* of $G(\lambda)$ over the subinterval $[\lambda_s, \lambda_t]$ such that $G(\lambda^*) < z$ if it exists:

Procedure B$(l, z, \lambda_s, \lambda_t)$

1° Generate the underestimating function $U(\lambda; \lambda_s, \lambda_t)$ of $\tilde{G}(\lambda)$ over $[\lambda_s, \lambda_t]$ by using the Lipschitz constant L and both the values of $\tilde{G}(\lambda_s)$ and $\tilde{G}(\lambda_t)$.

2° Let $\lambda_l \in \mathrm{argmin}\{U(\lambda; \lambda_s, \lambda_t) \mid \lambda \in [\lambda_s, \lambda_t]\}$. If $U(\lambda_l; \lambda_s, \lambda_t) \geq [\lambda_l^2 + (1 - \lambda_l)^2]z$, then return.

3° Compute $\tilde{G}(\lambda_l)$ by solving the convex programs $\mathrm{Q}_j(\xi_k(\lambda_l))$, $j = 1, 2$; $k = 1, 2$. If

$$\tilde{G}(\lambda_l) - U(\lambda_l; \lambda_s, \lambda_t) < [\lambda_l^2 + (1 - \lambda_l)^2]\epsilon, \tag{3.23}$$

then let $\lambda^* = \lambda_l$ and $z = \tilde{G}(\lambda_l)/[\lambda_l^2 + (1 - \lambda_l)^2]$.

4° Call Procedure B$(l+1, z, \lambda_s, \lambda_l)$ and Procedure B$(l+1, z, \lambda_l, \lambda_t)$. □

Choosing an appropriate $\epsilon > 0$, we obtain an globally ϵ-optimal solution (ξ_1^*, ξ_2^*) of Q by the following algorithm:

Algorithm C

Step 1 Compute the Lipschitz constant L by solving the convex programs (3.12) and (3.13). Compute $\tilde{G}(0)$ and $\tilde{G}(1)$ by solving the convex programs $\mathrm{Q}_j(\xi_k(0))$ and $\mathrm{Q}_j(\xi_k(1))$ $(j = 1, 2;\ k = 1, 2)$, respectively.

Step 2 Call Procedure B$(0, +\infty, 0, 1)$.

Step 3 For the output λ^* of Procedure B$(0, +\infty, 0, 1)$ let

$$\xi_1^* = \frac{\xi_1(\lambda^*)}{||\xi_1(\lambda^*)||}, \qquad \xi_2^* = \frac{\xi_2(\lambda^*)}{||\xi_2(\lambda^*)||}.$$

□

Theorem 3.4 *Algorithm* C *is terminate after finitely many iterations if* $\epsilon > 0$.

Proof Assume that Algorithm C is infinite for some $\epsilon > 0$. Then there exists a convergent subsequence $\{\lambda_{l_q}\}$ of $\{\lambda_l\}$ such that for every q

$$\tilde{G}(\lambda_{l_q}) - U(\lambda_{l_q}; \lambda_{s_q}, \lambda_{t_q}) \geq [\lambda_{l_q}^2 + (1 - \lambda_{l_q})^2]\epsilon, \tag{3.24}$$

where either λ_{s_q} of λ_{t_q} is equal to λ_{l_q-1}. By noting that $U(\lambda_{l_q-1}; \lambda_{s_q}, \lambda_{t_q}) = \tilde{G}(\lambda_{l_q-1})$, we have

$$\begin{aligned}
&|\tilde{G}(\lambda_{l_q}) - U(\lambda_{l_q}; \lambda_{s_q}, \lambda_{t_q})| \\
&= |\tilde{G}(\lambda_{l_q}) - U(\lambda_{l_q-1}; \lambda_{s_q}, \lambda_{t_q}) + U(\lambda_{l_q-1}; \lambda_{s_q}, \lambda_{t_q}) - U(\lambda_{l_q}; \lambda_{s_q}, \lambda_{t_q})| \\
&\leq |\tilde{G}(\lambda_{l_q}) - \tilde{G}(\lambda_{l_q-1})| + |U(\lambda_{l_q}; \lambda_{s_q}, \lambda_{t_q}) - U(\lambda_{l_q-1}; \lambda_{s_q}, \lambda_{t_q})|.
\end{aligned}$$

Let $\tilde{G}(\lambda_{l_q}) = h(\lambda_{l_q}; p')$. Since $h(\lambda_{l_q-1}; p') \leq \tilde{G}(\lambda_{l_q-1})$, we have

$$\begin{aligned}
&|\tilde{G}(\lambda_{l_q}) - U(\lambda_{l_q}; \lambda_{s_q}, \lambda_{t_q})| \\
&\leq |h(\lambda_{l_q}; p') - h(\lambda_{l_q-1}; p')| + |U(\lambda_{l_q}; \lambda_{s_q}, \lambda_{t_q}) - U(\lambda_{l_q-1}; \lambda_{s_q}, \lambda_{t_q})| \\
&\leq 2L|\lambda_{l_q} - \lambda_{l_q-1}|.
\end{aligned}$$

This contradict (3.24) because $\lim_{q\to\infty} |\lambda_{l_q} - \lambda_{l_q-1}| = 0$. □

4 Remarks

In case D is a polytope given by its vertices (x^1, y^1), (x^2, y^2), ..., (x^N, y^N), we can solve the problem P in Section 2 with the tools of computational geometry. Let X be the projection of the set of the vertices onto the x-plane, i.e., $X = \{x^j \mid j = 1, \ldots, N\}$. We can compute the convex hull $\overline{X}$ of X in $O(N \log N)$ time [3,7]. Based on Theorem 2.2, the minimum-area rectangle enclosing $\overline{X}$ can be obtained in $O(N)$ time by using the caliper method [17]. Therefore, the total number of arithmetics needed for this case is $O(N \log N)$ at worst.

The problem Q can be solved by exploiting the following parametric representation of $\tilde{G}(\cdot)$ as well:

$$G'(\lambda; \eta) = \eta g(\xi_1(\lambda)) + \frac{1}{\eta} g(\xi_2(\lambda)). \tag{4.1}$$

By noting that $g(\xi_k(\lambda)) > 0$ for any $\lambda \in [0, 1]$, we have

$$G'(\lambda; \eta) \leq 2\sqrt{\tilde{G}(\lambda)}$$

for $\eta > 0$, where the equality holds at $\eta = \sqrt{g(\xi_2(\lambda))/g(\xi_1(\lambda))}$. Therefore, to find a minimum point of $G(\lambda)$ over $[0, 1]$ we need to enumerate every local minimum of a function:

$$G''(\eta) = \min\{G'(\lambda; \eta) \mid \lambda \in [0, 1]\} \tag{4.2}$$

over $\eta > 0$. Since $G''(\eta)$ is the pointwise minimum of functions of the form $a\eta + b/\eta$, we can utilize a good underestimating function of $G''(\cdot)$ proposed in Kuno and Konno [12] and Suzuki, et al.[15] for multiplicative programming. However, it is not easy to compute the right-hand side of (4.2) because $G'(\cdot\,; \eta)$ is a nonconvex function. A similar approach is developed in [11] for another problem in the plane.

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TITLE

Globally Determining a Minimum-Area Rectangle Enclosing the Projection of a Higher-Dimensional Set

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ABSTRACT

This paper addresses itself to methods for finding a rectangle of minimum area which encloses the projection of a given convex set in a higher dimensional space onto the plane of the rectangle. In case the given set is a polytope, a parametric simplex algorithm is proposed for obtaining a global solution, which needs the polynomial number of arithmetics on the average. In case the set is nonlinear convex, it is shown that a successive underestimation method generates an ϵ-global solution in finite time if $\epsilon > 0$.

SUPPLEMENTARY NOTES