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Yasuhiko Ikebe, Yasushi Kikuchi and Issei Fujishiro

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INSTITUTE
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UNIVERSITY OF TSUKUBA

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Yasuhiko Ikebe*, Yasushi Kikuchi* and Issei Fujishiro**

*Institute of Information Sciences and Electronics University of Tsukuba, Tukuba City, Ibaraki 305, Japan

**Department of Information Sciences, Faculty of Science,

Ochanomizu University, Otsuka 2-1-1, Bunkyo-ku, Tokyo 112, Japan

and Institute of Information Sciences and Electronics, University of Tsukuba, Tukuba City, Ibaraki 305, Japan

Abstract

Consider computing the eigenvalues of a given compact infinite matrix regarded as operating in the complex Hilbert space l^2 by computing the eigenvalues of the truncated finite matrices and taking an obvious limiting process. In this paper, we deal with a special case where the given matrix is complex and symmetric (but not necessarily Hermitian) and where each of its eigenvalues is simple. We give a complete error analysis where the resolvent functions defined only on a proper closed invariant subspace (the *singular resolvent*) play a critical role. In fact, it is proved that the sequence of norms of singular resolvents of the truncated matrices converges to the norm of the corresponding singular resolvent for the given exact matrix. As an application, the numerical solution of $J_0(z) - iJ_1(z) = 0$, which appears in the analysis of the solitary wave runup on a sloping beach, reformulated as an eigenvalue problem for a compact complex symmetric tridiagonal matrix is given together with a full error analysis. The corresponding results for the case where the given matrix is Hermitian is concisely presented.

Keywords

Compact Complex Symmetric Matrix Operator, Eigenvalue Problem, Resolvent, $J_0(z) - iJ_1(z) = 0$

§1 Introduction and Summary.

Our concern in this paper is a study of spectral and operator approximations for compact complex symmetric or Hermitian matrix operators in the usual complex Hilbert space l^2 of all square-summable complex sequences. In general, a bounded linear operator T from a Banach space X to a Banach space Y is *compact* if for any bounded sequence $\{f_n\}$ in X, the image sequence $\{Tf_n\}$ in Y has a convergent subsequence.

We first recall a few basic facts from the spectral theory of operators [6, Chap. XIII, §§3-4]. In the sequel, the generic symbol $B(X, Y)$ denotes the Banach space of all bounded linear operators from a Banach space X to a Banach space Y. We denote $B(X, X)$ simply by $B(X)$. Given $T \in B(X)$, the set of all complex numbers λ for whitch $(T - \lambda I)^{-1} \in B(X)$ is known as the set $\rho(T)$ of *regular* values of T or the *resolvent* set. Its complement is the *spectrum* $\sigma(T)$ of T. In case T is compact and X is infinite-dimensional, 0 is always in $\sigma(T)$ and each nonzero $\lambda_0 \in \sigma(T)$ is an *eigenvalue* of T, namely, there is a corresponding *eigenvector* $x \in X$ such that $x \neq 0$ and $(T - \lambda_0 I)x = 0$, where I denotes the identity operator. For $\lambda \in \rho(T)$, the operator $(T - \lambda I)^{-1} \in B(X)$ is called the *resolvent* of T. For any $0 \neq \lambda_0 \in \sigma(T)$, $(T - \lambda_0 I)^{-1}$ is not well defined on the whole of X from the definition of $\sigma(T)$. However, $T - \lambda_0 I$ may have a bounded inverse on a smaller closed invariant subspace, say S, of T. A necessary and sufficient condition for this to be true is that λ_0 is not an eigenvalue of T restricted to S. We will call such an operator $(T - \lambda_0 I)^{-1}$ defined only on S, a *singular resolvent* of T, and will be denoted by $(T - \lambda_0 I)_S^{-1}$ in the sequel. It is the theory of the singular resolvent of the said type and its applications that we wish to investigate in this paper. Our exact working hypotheses are given as (H1) below (the symmetric case) and (H2) in §7 (the Hermitian case).

(H1) We are given a sequence of compact complex symmetric (but not necessarily normal) matrix operators $\{A_n\}_1^\infty$ in the Hilbert space l^2, converging in operator norm to a compact complex symmetric matrix operator A in l^2. We further assume that A has a nonzero eigenvalue λ that is *simple* in the sense that only one linearly independent eigenvector corresponds to λ and no generalized eigenvectors of rank 2 or more correspond to λ, namely, no vectors $y \neq 0$ exist such that $(A-\lambda I)^2 y = 0$ and $(A-\lambda I)y \neq 0$. Let x be an eigenvector of A corresponding to λ. We assume $x^T x \neq 0$, '^{T}' denoting transpose.

An example of this situation occurs in the approximate solution of $J_0(z) - iJ_1(z) = 0$, as described in Theorems 1.4 and 1.5 below, where $J_m(z)$ represents the Bessel function of order m and $i^2 = -1$, and where A_n is taken to be that matrix which equals A at the upper-left $n \times n$ section and 0 elsewhere.

Our starting point is the following spectral convergence theorem, which is adapted from [7, p.272-274] in a specialized form suitable to our purpose:

Theorem 1.1. *Let A_n and A have the same meaning as defined in (H1). We have:*

(a) For any eigenvalue $\lambda \neq 0$ of A, there is a sequence of eigenvalues of A_n which converges to λ. Conversely, if a sequence of eigenvalues of A_n converges to $\lambda \neq 0$, then λ is an eigenvalue of A [7, p.272, Theorem 18.1].

(b) If a sequence of eigenvalues λ_n of A_n converges to a nonzero simple eigenvalue λ of A, then λ_n is simple for all sufficiently large n [7, p.273].

(c) If $\lambda \neq 0$ is a simple eigenvalue of A, x is an eigenvector of A corresponding to λ and a sequence of eigenvalues λ_n of A_n converges to λ, then there is a sequence of eigenvectors x_n of A_n corresponding to λ_n such that $x_n \to x$ [7, p.274, Theorem 18.3].

Our first main theorem, Theorem 1.2 below, holds on the strength of (H1) and deals with the inverse approximation of spectral operators restricted to appropriate subspaces (*singular resolvent approximation*):

Theorem 1.2. *(First Main Theorem) Assume the hypothesis (H1). Let $\lambda_n \to \lambda$ and $x_n \to x$, where λ_n is an eigenvalue of A_n and x_n is an eigenvector of A_n corresponding to λ_n. The existence of such λ_n and x_n is guaranteed by the last theorem. The λ_n are simple for all sufficiently large n, again by the last theorem. Let **deflated subspaces** S and S_n be defined as the orthogonal complement of $span\{x\}$ and $span\{x_n\}$ in the sense of transpose:*

$$S \equiv \{y \in l^2 : x^T y = 0\}, \quad S_n \equiv \{y \in l^2 : x_n^T y = 0\} \quad .$$

Note that S depends on λ only, since λ is simple. Similarly, S_n depends on λ_n only, for all n such that λ_n is simple. Clearly, S and S_n are closed subspaces of l^2 and $l^2 = span\{x\} \oplus S$ and $l^2 = span\{x_n\} \oplus S_n$ for all n such that λ_n is simple.

Let projections $Q : l^2 \to S$ and $R_n : l^2 \to S_n$ be defined by:

$$Q \equiv I - \frac{xx^T}{x^T x}, \quad R_n \equiv I - \frac{x_n x_n^T}{x_n^T x_n} \quad .$$

The R_n are well-defined for all n such that λ_n is simple. One may easily verify that $Q^2 = Q$ and $R_n^2 = R_n$. Note further that Q and R_n behave as identity when restricted to S and S_n, respectively.

We then have the following assertions:

(1) A, $A - \lambda I$, $(A - \lambda I)_S^{-1} \in B(S)$

(2) For all n such that λ_n is simple, A_n, $A_n - \lambda_n I$, $(A_n - \lambda_n I)_{S_n}^{-1} \in B(S_n)$

(3) $\| (A_n - \lambda_n I)_{S_n}^{-1} R_n - (A - \lambda I)_S^{-1} Q \|_{l^2} \to 0$

(4) $\| (A_n - \lambda_n I)^{-1} \|_{S_n} \to \| (A - \lambda I)^{-1} \|_S$

Here the symbol $(A - \lambda I)_S^{-1}$ denotes the bounded inverse of $A - \lambda I$ restricted to S, and similarly for $(A_n - \lambda_n I)_{S_n}^{-1}$. These two bounded inverses are a resolvent function of A restricted to S and of A_n restricted to S_n, respectively, each of which we call a singular resolvent in this paper,. The notation of the form $\| T \|_X$ denotes the operator norm of T whose domain is a subspace X.

The proof of Theorem 1.2 is given in §2. The fourth conclusion $\| (A_n - \lambda_n I)^{-1} \|_{S_n} \to \| (A - \lambda I)^{-1} \|_S$ is valuable for the subsequent applications as the proofs of Theorems 1.3-1.5 demonstrate.

Remark. Let $B \equiv (A - \lambda I)_S^{-1} Q \in B(l^2, S)$ and $B_n \equiv (A_n - \lambda_n I)^{-1} R_n \in B(l^2, S_n)$, then B and B_n are a generalized inverse of $A - \lambda I$ and $A_n - \lambda_n I$, respectively, as one can show by direct computation that

$$\begin{cases} (A - \lambda I)B(A - \lambda I) = A - \lambda I \\ B(A - \lambda I)B = B \\ (A - \lambda I)B = B(A - \lambda I) = Q \end{cases}$$

and

$$\begin{cases} (A_n - \lambda_n I)B_n(A_n - \lambda_n I) = A_n - \lambda_n I \\ B_n(A_n - \lambda_n I)B_n = B_n \\ (A_n - \lambda_n I)B_n = B_n(A_n - \lambda_n I) = R_n \end{cases} .$$

Part (3) of this theorem then asserts the convergence of generalized inverses B_n of $A_n - \lambda_n I$ to the generalized inverse B of $A - \lambda I$, where clearly $A_n - \lambda_n I \to A - \lambda I$ in l^2. For a full, up-to-date treatment of generalized inverses in a variety of settings, we refer the reader to [10], a recent encyclopedic work on the subject including an extensive annotated bibliography of 1776 references.

The hypothesis (H1) represents a useful special situation where an appropriately taken generalized Rayliegh quotient [16, p.179] well approximates, in the sense of Theorems 1.3 below, a given simple eigenvalue of a compact complex symmetric matrix operator in l^2.

Theorem 1.3. *(Second Main Theorem) Again assume the hypothesis (H1) and suppose that we are given a sequence $\{v_n\}_1^\infty$ such that $v_n \to x$. Consider the generalized Rayliegh quotient $\mu_n = v_n^T A_n v_n / v_n^T v_n$ and take it as an approximation to λ_n where, as in Theorem 1.2, λ_n is an eigenvalue of A_n such that $\lambda_n \to \lambda$. Then we have the following error estimate for all n such that λ_n is simple:*

$$\begin{aligned} | \mu_n - \lambda_n | &\leq \frac{1}{| v_n^T v_n |} \| (A_n - \lambda_n I)v_n \|^2 \| (A_n - \lambda_n I)^{-1} \|_{S_n} \\ &\cong \frac{1}{| x^T x |} \| (A_n - \mu_n I)v_n \|^2 \| (A - \lambda I)^{-1} \|_S \qquad \textit{for large } n. \end{aligned}$$

The proof of Theorem 1.3 is given in §3. Note that, in the last theorem, the error $| \mu_n - \lambda_n |$ is bounded by a quantity of order $\| (A_n - \lambda_n I)v_n \|^2$ (i.e. the norm of the *residual vector* $A_n v_n - \lambda_n v_n$ squared).

The theorem may typically be used in the following context: Suppose we are to estimate $\lambda - \lambda_n$. We write $\lambda - \lambda_n = (\lambda - \mu_n) + (\mu_n - \lambda_n)$. If it can be shown, as is the case in §5, that $| \lambda - \mu_n | \gg | \mu_n - \lambda_n |$ for all large n, then we can estimate $\lambda - \lambda_n$ accurately by $\lambda - \mu_n$ for all large n. The point is that $\lambda - \mu_n$ may be estimated accurately when one has a detailed knowledge on an eigenvector corresponding to the exact eigenvalue λ, as is the case in §5. For instance, if the components ξ_i of the eigenvector $x = [\xi_1, \xi_2, \cdots]^T$ corresponding to the eigenvalue λ are known, one may take $v_n = [\xi_1, \cdots, \xi_n, 0, 0, \cdots]^T$ $(n = 1, 2, \cdots)$ as is done in §5, where $\xi_n = \sqrt{n} J_n(z)$ with $z = 2/\lambda$.

Theorems 1.2 and 1.3 may be applied to the approximate solution of $J_0(z) - iJ_1(z) = 0$. The equation is of interest in the analysis of solitary wave runup on a beach with a constant slope [4][13]. It is known [11] [14] that the infinitely many roots lie in the lower half complex plane (but none in the upper half plane or on the real axis), symmetrically about the imaginary axis. The first 30 roots with positive real part accurate up to 8 digits have been computed by Macdonald through the use of the following asymptotic expansion for the jth root in polar form, also obtained by him [8]:

$$\begin{cases} z = r_A e^{i[(\pi/2)\pm\theta_A]}, \\ r_A = j\pi + \{1 - 4\alpha_A(1-\alpha_A)\}\dfrac{1}{8j\pi} \\ \quad + \dfrac{1}{384 j^3\pi^3}[-61 + 264\alpha_A - 360\alpha_A^2 + 256\alpha_A^3 - 48\alpha_A^4] + O(\dfrac{\alpha_A^2}{j^4\pi^4}), \\ \theta_A = -\dfrac{1}{2}\pi - \dfrac{\alpha_A}{j\pi} - \dfrac{1}{96 j^3\pi^3}[21 - 48\alpha_A + 72\alpha_A^2 - 32\alpha_A^3] + O(\dfrac{\alpha_A^2}{j^4\pi^4}), \\ \alpha_A = \dfrac{1}{2}\ln(4j\pi), \end{cases}$$

where the plus sign in the expression for z is taken for the fourth quadrant roots and the minus sign for the third quadrant roots.

It may be noted that, for large j, $r_A \cong j\pi$ and $\theta_A \cong -\frac{\pi}{2} - \frac{\alpha_A}{j\pi}$, indicating that the roots are approximately π apart. It may further be shown *a priori* that the equation $J_0(z) - iJ_1(z) = 0$ has no roots on the imaginary axis; for, putting $z = -i\eta$, where η is real and positive, we have $J_0(z) - iJ_1(z) = I_0(\eta) - I_1(\eta) > 0$, since $I_0(\eta) > I_1(\eta)$ for all $\eta > 0$ [9, p.151].

In order to apply Theorems 1.2 and 1.3 to the approximate solution of $J_0(z) - iJ_1(z) = 0$, we first reformulate the equation as an eigenvalue problem for a compact complex symmetric matrix operator in $B(l^2)$ whose eigenvalues are all simple, as done in Theorem 1.4 below.

Theorem 1.4. *(Third main theorem) A complex number z is a root of $J_0(z) - iJ_1(z) = 0$ if and only if $z \neq 0$ and $2/z$ is an eigenvalue of the compact complex symmetric matrix $A \in B(l^2)$ defined below. Each eigenvalue of A is simple and nonzero. An eigenvector corresponding to the eigenvalue $2/z$ is given by x*

defined below.

$$\begin{cases} Ax = \dfrac{2}{z}x \ , \\ A = \begin{pmatrix} i & f_2 & & 0 \\ f_2 & 0 & f_3 & \\ & f_3 & 0 & \ddots \\ 0 & & \ddots & \ddots \end{pmatrix}, \quad f_k = \dfrac{1}{\sqrt{k(k-1)}}, \ k = 2, 3, \cdots \ , \\ x = [J_1(z), \sqrt{2}J_2(z), \sqrt{3}J_3(z), \cdots]^T \in l^2 \ . \end{cases}$$

For the proof of Theorem 1.4, see §4.

The theoretical basis for the actual numerical procedure for the approximate computation of the eigenvalues of the matrix A defined in Theorem 1.4 is given by the next theorem, Theorem 1.5, together with an accurate estimate for the relative error associated with the approximate root computed from the approximate eigenvalue.

Theorem 1.5. *(Fourth main theorem) Let $z \neq 0$ be a root of $J_0(z) - iJ_1(z) = 0$ and let λ_n be an eigenvalue of the $n \times n$ principal submatrix $\tilde{A}_n$ of A such that $\lambda_n \to \lambda = 2/z$. Let $z_n = 2/\lambda_n$ be taken as an approximation to z. Then, for all large n such that λ_n is simple and nonzero, the relative error $(z_n - z)/z$, where $| z | >> 1$, may be estimated by*

$$\frac{z_n - z}{z} \cong \pm\frac{\pi}{2}J_n(z)J_{n+1}(z) \ ,$$

where the plus sign is for the roots z with positive real part and the minus sign is for the roots z with negative real part.

The proof of Theorem 1.5 is given in §5.

The theorem is rather remarkable in the sense that the relative error $(z_n - z)/z$ is well approximated by a simple closed form as given above. Numerical evidence for this will be given in §6 for a selected set of values of n and z, together with further discussion on the implication of the theoretical error estimate given by Thorem 1.5.

Consider now the actual numerical procedure for computing the λ_n. The $n \times n$ matrix $\tilde{A}_n$, the $n \times n$

principal submatrix of A defined in Theorem 1.4, is easily seen to be similar to

$$\hat{A}_n = i \begin{pmatrix} 1 & f_2 & & & 0 \\ -f_2 & 0 & f_3 & & \\ & -f_3 & 0 & \ddots & \\ & & \ddots & \ddots & f_n \\ 0 & & & -f_n & 0 \end{pmatrix}.$$

Indeed, $\hat{A}_n = D_n^{-1} \tilde{A}_n D_n$, where

$$D_n = \begin{pmatrix} 1 & & & & & 0 \\ & i & & & & \\ & & i^2 & & & \\ & & & i^3 & & \\ & & & & \ddots & \\ 0 & & & & & i^n \end{pmatrix}.$$

Hence, the computation of the λ_n and $z_n = 2/\lambda_n$ may be effected through the use of a QR algorithm for computing all eigenvalues of a real tridiagonal matrix, for example, the one implemented as the FORTRAN subroutine HQR in the EISPACK package [12].

Finally, we briefly consider the *Hermitian* case, i.e., the case where $A^H = A$ and $A_n^H = A_n$ $(n = 1, 2, \cdots)$ in (H1), '^{H}' denoting conjugate transpose. With the definitions of S, S_n, Q and R_n $(n = 1, 2, \cdots)$ in Theorem 1.2 modified by replacing '^{T}' by '^{H}', Theorems 1.2 and 1.3 hold exactly as they stand, where in the latter theorem, '^{T}' should be replaced by '^{H}'. The proofs of these two theorems run in parallel to those of Theorems 1.2 and 1.3 with minor modifications and are omitted. It only remains for us to mention the validity of the following two relations which one can prove by exploiting the well-known facts that a compact Hermitian matrix operator in l^2, of which 0 is not an eigenvalue, has a complete orthonormal system of eigenvectors (see, for example, [6, p.256]) and that 0 is the only possible accumulation point of the spectrum of the compact operator (see, for example, [6, p.376]):

$$\begin{cases} \| (A - \lambda I)^{-1} \|_S = \dfrac{1}{\min\limits_{\lambda' \in \sigma(A),\ \lambda' \neq \lambda} | \lambda' - \lambda |} \\ \| (A_n - \lambda_n I)^{-1} \|_{S_n} = \dfrac{1}{\min\limits_{\lambda' \in \sigma(A_n),\ \lambda' \neq \lambda_n} | \lambda' - \lambda_n |} \end{cases}$$

For the sake of comparison, we should mention the following result which is valid for a *normal* complex matrix B $(BB^H = B^H B)$ of *finite* dimension, say, m:

$$| \mu - \lambda | \leq \frac{\| (B - \lambda I) v \|^2}{\| v \|^2} \| (B - \lambda I)^{-1} \|_M = \frac{\| (B - \lambda I) v \|^2}{\| v \|^2} \frac{1}{\min\limits_{\lambda' \in \sigma(B),\ \lambda' \neq \lambda} | \lambda' - \lambda |},$$

where $\mu = v^H Bv/v^H v$, $0 \neq v \in \mathbf{C}^m$, λ is a simple eigenvalue of B and M denotes the orthogonal complement of the eigenvector corresponding to λ. The proof of the inequality is similar to that of Theorem 1.3. For the proof of the equality, we use the fact that an orthonormal basis for $\mathbf{C}^m$ exists that consists solely of eigenvectors of a given normal matrix. In fact, the normal matrices are precisely those which can be diagonalized by a unitary similarity transformation, the fact that is not necessarily valid for matrices of infinite dimension.

We should also mention the well-known inequality closely related to the last one that hold under the same setting [16, p.173]:

$$| \mu - \lambda | \leq \frac{\epsilon^2}{a} / (1 - \frac{\epsilon^2}{a^2})$$

where $a = \min_{\lambda' \in \sigma(B),\ \lambda' \neq \lambda} | \lambda' - \mu |$ and $\epsilon = \| (B - \mu I)v \| / \| v \|$, provided that μ is close enough to λ.

§2 Proof of Theorem 1.2.

We use the notation already established in Theorem 1.2.

Proof of Part (1), Theorem 1.2. To prove $A \in B(S)$, it is enough to show that $A = A^T$, $Ax = \lambda x$ and $x^T y = 0$ implies $x^T A y = 0$. Indeed, $x^T A y = x^T A^T y = (Ax)^T y = \lambda x^T y = 0$. From $A \in B(S)$, $A - \lambda I \in B(S)$ follows easily.

To prove the last assertion of Part (1), $(A - \lambda I)_S^{-1} \in B(S)$, it is enough to prove by [6, p.375, Theorem I] that λ is not an eigenvalue of the restriction A_S of A to the closed subspace S. To prove this, it suffices to show that $Ay = \lambda y$ and $x^T y = 0$ implies $y = 0$. Suppose $y \neq 0$. By the simplicity hypothesis for λ, we have $y = ax$ for some scalar $a \neq 0$. Multiplying x^T from left, we have $0 = x^T y = a x^T x$, hence $a = 0$ since $x^T x \neq 0$ from (H1), and $y = 0$, a contradiction.

Part (2) of Theorem 1.2 may be similarly proved.

Proof of Part (3), Theorem 1.2. The proof will be done in 6 lemmas, Lemma 2.1-2.6 below.

Lemma 2.1. *The sequence* $\{\| (A_n - \lambda_n I)^{-1} \|_{S_n}\}$ *is bounded, where* $n \geq n_0$ *(say) and* λ *is simple for all* $n \geq n_0$.

Proof. Proof by contradiction. Assume the contrary and we would have a sequence $n_0 \leq n_1 \leq n_2 \leq \cdots$ of positive integers such that $\| (A_{n_k} - \lambda_{n_k} I)^{-1} \|_{S_{n_k}} > k$, $k = 1, 2, \cdots$. This implies the existence of $u_{n_k} \in S_{n_k}$ with $\| u_{n_k} \| = 1$ and $\| (A_{n_k} - \lambda_{n_k} I)^{-1} u_{n_k} \|_{S_{n_k}} > k$, $k = 1, 2, \cdots$. Let $z_{n_k} = (A_{n_k} - \lambda_{n_k} I)_{S_{n_k}}^{-1} u_{n_k}$ and we have

$$(A_{n_k} - \lambda_{n_k} I) \frac{z_{n_k}}{\| z_{n_k} \|} = \frac{u_{n_k}}{\| z_{n_k} \|} \tag{1}$$

Solving this for $z_{n_k} / \| z_{n_k} \|$,

$$\frac{z_{n_k}}{\| z_{n_k} \|} = \frac{1}{\lambda_{k_n}} \{ A \frac{z_{n_k}}{\| z_{n_k} \|} - \frac{u_{n_k}}{\| z_{n_k} \|} \} + \frac{1}{\lambda_{k_n}} (A_{n_k} - A) \frac{z_{n_k}}{\| z_{n_k} \|}.$$

By compactness of A, we may assume, by extracting a subsequence of the sequence $\{z_{n_k}\}$ if necessary, that the sequence $\{A z_{n_k} / \| z_{n_k} \|\}$ converges, where $\| z_{n_k} \| > k$ remains valid. Letting $k \to \infty$ in the last equation, we find $z_{n_k} / \| z_{n_k} \| \to w$ (say), since $\| A_{n_k} - A \| \to 0$ and $\lambda_{n_k} \to \lambda \neq 0$. Clearly $\| w \| = 1$.

Again letting $k \to \infty$ in (1), we have

$$(2) \qquad (A - \lambda I)w = 0.$$

On the other hand, $x_{n_k}^T(z_{n_k}/ \| z_{n_k} \|) = 0$, since $z_{n_k} \in S_{n_k}$. Letting $k \to \infty$ again, we have

$$(3) \qquad x^T w = 0,$$

since $x_{n_k} \to x$. The two relations (2) and (3) are incompatible, since the first implies that w would be a nonzero scalar multiple of x, say, $w = ax$, $a \neq 0$, by the simplicity of λ, and thus the latter gives $ax^T x = 0$ despite the assumption $x^T x \neq 0$ made in the hypothesis (H1). Thus we arrived at a desired contradiction. ∎

Lemma 2.2. *For all large n, $(Q_n R_n)_S^{-1} \in B(S)$, where the domain of R_n is understood to be restricted to S, and $(R_n Q_n)_{S_n}^{-1} \in B(S_n)$, where Q_n denotes the restriction of Q to S_n (for the definition of Q and R_n, refer to Theorem 1.2).*

Proof. With the convention just made for Q_n and R_n in the above, it is clear that $Q_n R_n \in B(S)$ and $R_n Q_n \in B(S_n)$.

We first prove $(Q_n R_n)_S^{-1} \in B(S)$. To this end, compute for any given $y \in S$ (i.e., $x^T y = 0$),

$$(1) \qquad Q_n R_n y = (I - \frac{xx^T}{x^T x})(I - \frac{x_n x_n^T}{x_n^T x_n})y = y - \frac{(\Delta x_n)^T y}{x_n^T x_n}(\Delta x_n - \frac{x^T \Delta x_n}{x^T x} x), \qquad \Delta x_n = x_n - x \quad ,$$

whence,

$$(2) \qquad \| Q_n R_n - I_S \|_S = \sup_{\|y\|=1, y \in S} \| (Q_n R_n - I_S)y \| \leq \frac{\| \Delta x_n \|^2}{| x_n^T x_n |}(1 + \frac{\| x \|^2}{| x^T x |}) \ .$$

Letting $n \to \infty$, we find $\| Q_n R_n - I_S \|_S \to 0$, since $\Delta x_n \to 0$ and $x_n \to x$. Then, by [7, p.210, lemma 15.2], we conclude that $(Q_n R_n)_S^{-1} \in B(S)$ for all sufficiently large n.

To prove $(R_n Q_n)_{S_n}^{-1} \in B(S_n)$, we proceed similarly. Indeed, for any given $y_n \in S_n$ (i.e., $x_n^T y_n = 0$),

$$R_n Q_n y_n = (I - \frac{x_n x_n^T}{x_n^T x_n})(I - \frac{xx^T}{x^T x})y_n = y_n - \frac{(\Delta x_n)^T y}{x^T x}(\Delta x_n - \frac{x_n^T \Delta x_n}{x_n^T x_n} x_n)$$

and

$$\| R_n Q_n - I_{S_n} \|_{S_n} \leq \frac{\| \Delta x_n \|^2}{| x^T x |}(1 + \frac{\| x_n \|^2}{| x_n^T x_n |}),$$

hence, $\| R_n Q_n - I_{S_n} \|_{S_n} \to 0$, and [7, p.210, lemma 15.2] again allows us to conclude $(R_n Q_n)_{S_n}^{-1} \in B(S_n)$ for all sufficiently large n. ∎

Lemma 2.3. *For all large n, $R_n^{-1} \in B(S_n, S)$, where R_n is restricted to S as in the last lemma. We also have $\| R_n - I_S \|_S \to 0$, $\| R_n^{-1} - I_{S_n} \|_{S_n} \to 0$, $\| R_n \|_S \to 1$ and $\| R_n^{-1} \|_{S_n} \to 1$.*

Proof. We will prove $R_n^{-1} \in B(S_n, S)$ by showing the existence of a left inverse $X_L \in B(S_n, S)$ of R_n and a right inverse $X_R \in B(S_n, S)$ of R_n. Indeed, letting $X_L = (Q_n R_n)_S^{-1} Q_n \in B(S_n, S)$ and $X_R = Q_n (R_n Q_n)_{S_n}^{-1} \in B(S_n, S)$, where, by the last lemma, $(Q_n R_n)_S^{-1} \in B(S)$ and $(R_n Q_n)_{S_n}^{-1} \in B(S_n)$ for all sufficiently large n, we can easily see $X_L R_n = I_S$ and $R_n X_R = I_{S_n}$ for all sufficiently large n. Hence, $X_L = X_R = R_n^{-1}$.

To prove $\| R_n - I_S \|_S \to 0$, where $R_n = I_S - \frac{x_n x_n^T}{x_n^T x_n} \in B(S, S_n)$, compute, for every $y \in S$ (i.e., $x^T y = 0$),

$$(R_n - I_S)y = -\frac{x_n x_n^T}{x_n^T x_n} y = -\frac{(\Delta x_n)^T y}{x_n^T x_n} x_n \quad ,$$

where $\Delta x_n = x_n - x$ as in the last lemma. The conclusion follows from this, since $\Delta x_n \to 0$.

To prove $\| R_n^{-1} - I_{S_n} \|_{S_n} \to 0$, let $w = R_n^{-1} y \in S$, where $y \in S_n$ (i.e., $x_n^T y = 0$). We will show

$$w - y = (R_n^{-1} - I_{S_n})y = \frac{(\Delta x_n)^T y}{x^T x_n} x_n \quad .$$

Indeed, using the definition of R_n and w, we have $y = R_n w = w - \alpha x_n$, where $\alpha = \frac{x_n^T w}{x_n^T x_n}$. Multiplying x^T from left, we find $x^T y = x^T w - \alpha x^T x_n$. But $w \in S$, so $x^T w = 0$. Hence, $x^T y = -\alpha x^T x_n$. Substituting $x = x_n - \Delta x_n$ into the left-hand side and noting $x_n^T y = 0$, we have $(\Delta x_n)^T y = \alpha x^T x_n$. Then

$$w - y = \alpha x_n = \frac{(\Delta x_n)^T y}{x^T x_n} x_n \quad ,$$

as required. It is now clear that

$$\| R_n^{-1} - I_{S_n} \|_{S_n} = \sup_{\|y\|=1,\ y \in S_n} \| R_n^{-1} y - y \| \leq \frac{\| \Delta x_n \|}{| x^T x_n |} \| x_n \| \quad ,$$

whence the conclusion follows since again $\Delta x_n \to 0$ and $x_n \to x$ (we recall $x^T x \neq 0$ from the hypothesis (H1) in §1).

The fact that $\| R_n \|_S \to 1$ and $\| R_n^{-1} \|_{S_n} \to 1$ now follows easily from the assertions $\| R_n - I_S \|_S \to 0$ and $\| R_n^{-1} - I_{S_n} \|_{S_n} \to 0$. ∎

Lemma 2.4. *For all sufficiently large n such that λ_n is simple, $R_n(A_n - \lambda_n I) = (A_n - \lambda_n I)R_n = A_n - \lambda_n I \in B(l^2, S)$.*

Proof. Using the definition $R_n = I - \dfrac{x_n x_n^T}{x_n^T x_n} \in B(l^2, S)$, $A_n x_n = \lambda_n x_n$ and $A_n^T = A_n$, the lemma follows immediately. ∎

Lemma 2.5. $\| R_n(A - \lambda I)_S^{-1} - (A - \lambda I)_S^{-1} \|_S \to 0$ *and* $\| R_n Q - R_n \|_{l^2} \to 0$.

Proof. The first convergence is obvious from the fact that $\| R_n - I_S \|_S \to 0$ (Lemma 2.3).

To prove the second, compute $R_n Q - R_n = R_n(Q - I) = -\dfrac{xx^T}{x^T x} + \dfrac{x_n x_n^T}{x_n^T x_n} \dfrac{xx^T}{x^T x}$, which converges to 0 since $x_n \to x$ as $n \to \infty$. ∎

Lemma 2.6. $\| (A_n - \lambda_n I)_{S_n}^{-1} R_n Q - R_n (A - \lambda I)_S^{-1} Q \|_{l^2} \to 0$, *where the existence of $(A_n - \lambda_n I)_{S_n}^{-1}$ for all large n and of $(A - \lambda I)_S^{-1}$ is guaranteed from the already proved Parts (1) and (2) of Theorem 1.2.*

Proof. Compute

$$(A_n - \lambda_n I)_{S_n}^{-1} R_n Q - R_n (A - \lambda I)_S^{-1} Q = (A_n - \lambda_n I)_{S_n}^{-1} \{R_n(A - \lambda I) - (A_n - \lambda_n I) R_n\}_S (A - \lambda I)_S^{-1} Q_{l^2} \quad .$$

Taking norm, we have

$$\begin{aligned} &\| (A_n - \lambda_n I)_{S_n}^{-1} R_n Q - R_n (A - \lambda I)_S^{-1} Q \|_{l^2} \\ &\leq \| (A_n - \lambda_n I)^{-1} \|_{S_n} \| R_n (A - \lambda I) - (A_n - \lambda_n I) R_n \|_S \| (A - \lambda I)^{-1} \|_S \| Q \|_{l^2} \quad . \end{aligned}$$

The first term on the right is bounded by Lemma 2.1. The second term converges to 0 by Lemmas 2.4 and 2.5 and by the fact that $\| A_n - A \|_{l^2} \to 0$ and $\lambda_n \to \lambda$ by the hypothesis (H1). The last two terms are independent of n. The lemma now follows. ∎

Part (3) of Theorem 1.2, $\| (A_n - \lambda_n I)_{S_n}^{-1} R_n - (A - \lambda I)_S^{-1} Q \|_{l^2} \to 0$, now follows from the last two lemmas.

Proof of Part (4). This will be done with the aid of two lemmas below, Lemmas 2.7 and 2.8.

Lemma 2.7. $| \| (A_n - \lambda_n I)_{S_n}^{-1} R_n \|_S - \| (A - \lambda I)^{-1} \|_S | \to 0$

Proof. It is clear from Part (3) and from the definition of the operator norm that $\| (A_n - \lambda_n I)_{S_n}^{-1} R_n - (A - \lambda I)_S^{-1} Q \|_S \to 0$. The lemma follows from this. ∎

Lemma 2.8. $| \| (A_n - \lambda_n I)_{S_n}^{-1} R_n \|_S - \| (A_n - \lambda_n I)^{-1} \|_{S_n} | \to 0$.

Proof. Let $X_n = (A_n - \lambda_n)_{S_n}^{-1} \in B(S_n)$ and $Y_n = (A_n - \lambda_n I)_{S_n}^{-1} R_n \in B(S, S_n)$ for all sufficiently large n. We must prove $\| Y_n \|_S - \| X_n \|_{S_n} \to 0$. For all sufficiently large n, $R_n^{-1} \in B(S_n, S)$ exists by Lemma 2.3, so $X_n = Y_n R_n^{-1} \in B(S_n)$. Taking norm, we find

$$\| X_n \|_{S_n} \leq \| Y_n \|_S \| R_n^{-1} \|_{S_n} \leq \| X_n \|_{S_n} \| R_n \|_S \| R_n^{-1} \|_{S_n} ,$$

whence

$$0 \leq \| Y_n \|_S \| R_n^{-1} \|_{S_n} - \| X_n \|_{S_n} \leq \| X_n \|_{S_n} (\| R_n \|_S \| R_n^{-1} \|_{S_n} - 1)) .$$

But, by Lemma 2.1, $\{\| X_n \|_{S_n}\}$ is bounded, and, by Lemma 2.3, $\| R_n \|_S \to 1$ and $\| R_n^{-1} \|_{S_n} \to 1$. Using these in the last inequality, we conclude $\| Y_n \|_S - \| X_n \|_{S_n} \to 0$. ∎

The proof of Part (4) is obtained by combining Lemmas 2.7 and 2.8. The proof of Theorem 1.2 is now complete.

§3 Proof of Theorem 1.3.

We use the same notation as in Theorem 1.3.

We will prove the first inequality in the theorem, i.e.,

$$(1) \qquad | \mu_n - \lambda_n | \leq \frac{1}{| v_n^T v_n |} \| (A_n - \lambda_n I) v_n \|^2 \| (A_n - \lambda_n I)^{-1} \|_{S_n} \ ,$$

for all sufficiently large n. Indeed, using the definition of μ_n,

$$\mu_n - \lambda_n = \frac{v_n^T (A_n - \lambda_n I) v_n}{v_n^T v_n}$$

By Theorem 1.2 (2), $(A_n - \lambda_n I)_{S_n}^{-1} \in B(S_n)$ for all sufficiently large n, whence

$$\begin{aligned} \mu_n - \lambda_n &= \frac{1}{v_n^T v_n} v_n^T (A_n - \lambda_n I)(A_n - \lambda_n I)_{S_n}^{-1} (A_n - \lambda_n I) v_n \\ &= \frac{1}{v_n^T v_n} [(A_n - \lambda_n I) v_n]^T (A_n - \lambda_n I)_{S_n}^{-1} [(A_n - \lambda_n I) v_n] \quad (\text{by } A_n^T = A_n) \\ &= \frac{r_n^T (A_n - \lambda_n I)_{S_n}^{-1} r_n}{v_n^T v_n}, \quad \text{where } r_n = (A_n - \lambda_n I) v_n \in S_n \ . \end{aligned}$$

Application of the well known Cauchy-Schwarz inequality gives the inequality (1).

We will next show that $(A_n - \mu_n I) v_n = r_n + q_n$ with $\| q_n \| = O(\| r_n \|^2)$, where $\| r_n \| \to 0$. For, $q_n = (A_n - \mu_n I) v_n - r_n = (A_n - \mu_n I) v_n - (A_n - \lambda_n I) v_n = (\lambda_n - \mu_n) v_n$. Taking norm,

$$\begin{aligned} \| q_n \| = | \lambda_n - \mu_n | \, \| v_n \| &\leq \frac{1}{| v_n^T v_n |} \| (A_n - \lambda_n I) v_n \|^2 \| (A_n - \lambda_n I)^{-1} \|_{S_n} \| v_n \| \quad (\text{using Part (1) }) \\ &\cong \frac{1}{| x^T x |} \| r_n \|^2 \| (A - \lambda I)^{-1} \|_S \| x \| \quad (\text{for all larg } n, \text{ by Theorem 1.2 }) \\ &= O(\| r_n \|^2) \ . \end{aligned}$$

Hence, for all large n, we may be justified to replace v_n by x, $(A_n - \lambda_n I) v_n$ by $(A_n - \mu_n I) v_n$ and $\| (A_n - \lambda_n I)^{-1} \|_{S_n}$ by $\| (A - \lambda I)^{-1} \|_S$ to obtain the second estimate for $| \mu_n - \lambda_n |$ in the theorem.

§4 Proof of Theorem 1.4.

We use the same notation as in Theorem 1.4. We will prove Theorem 1.4 using a series of lemmas.

Lemma 4.1. *For any $z \neq 0$, the following matrix equation in l^2 holds:*

$$
(1) \qquad \begin{bmatrix} i & f_2 & & 0 \\ f_2 & 0 & f_3 & \\ & f_3 & 0 & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} J_1(z) \\ \sqrt{2}J_2(z) \\ \sqrt{3}J_3(z) \\ \vdots \end{bmatrix} = \frac{2}{z} \begin{bmatrix} J_1(z) \\ \sqrt{2}J_2(z) \\ \sqrt{3}J_3(z) \\ \vdots \end{bmatrix} - \begin{bmatrix} J_0(z) - iJ_1(z) \\ 0 \\ 0 \\ \vdots \end{bmatrix},
$$

$$
\text{where } f_k = \frac{1}{\sqrt{k(k-1)}}, \quad k = 1, 2, \cdots, \quad (recap.),
$$

or

$$
Ax = \frac{2}{z}x - [J_0(z) - iJ_1(z), 0, 0, \cdots]^T \ ,
$$

where A is a compact complex symmetric operator in $B(l^2)$ and $0 \neq x \in l^2$.

Proof. The relation (1) may be verified directly by using the well-known three-term recurrence relations [3, p.93]:

$$
(2) \qquad J_{k-1}(z) + J_{k+1}(z) = \frac{2k}{z}J_k(z), \quad k = 1, 2, \cdots \ ,
$$

or rewriting,

$$
(3) \qquad f_k y_{k-1} + f_{k+1} y_{k+1} = \frac{2}{z} y_k, \quad k = 2, 3, \cdots, \quad \text{where } y_k = \sqrt{k}J_k(z), \quad k = 1, 2, \cdots \ .
$$

The matrix A is obviously complex (i.e., non-real due to the presence of i as the first diagonal element) and symmetric. Compactness of A follows from the fact that a band matrix $B = [b_{ij}]$ (i.e., $b_{ij} = 0$ for all i and j such that $| i - j | > r$ for some fixed positive integer r) is compact, if and only if $\lim_{i,j \to \infty} a_{ij} = 0$ [2, p.59]. The fact that $x \in l^2$ can be seen from the well-known behavior of the J_k's: $(2k/z)(J_k(z)/J_{k-1}(z)) \to 1$ $(k \to \infty)$. Since no two consecutive J's, i.e. $J_k(z)$ and $J_{k+1}(z)$ for $k = 1, 2, \cdots$, vanish at any $z \neq 0$ [3, p.105], it is clear that $x \neq 0$ for any $z \neq 0$. ∎

Remark. By direct computation, one can show that $AA^H \neq A^H A$, i.e., that A is not normal.

Lemma 4.2. *If $J_0(z) - iJ_1(z) = 0$, then $z \neq 0$, and $2/z$ is an eigenvalue of A with a corresponding eigenvector $[J_1(z), \sqrt{2}J_2(z), \cdots]^T \in l^2$.*

Proof. Since $J_0(0) = 1$ and $J_1(0) = 0$, $z = 0$ is clearly not a root of $J_0(z) - iJ_1(z) = 0$. Hence, if $J_0(z) - iJ_1(z) = 0$, then Lemma 4.1 implies that $2/z$ is an eigenvalue of A with a corresponding eigenvector $[J_1(z), \sqrt{2}J_2(z), \cdots]^T \in l^2$. ∎

Lemma 4.3. *For a given complex number $z \neq 0$, an arbitrary solution of the three-term recurrence relation*

$$(1) \qquad f_k y_{k-1} + f_{k+1} y_{k+1} = \frac{2}{z} y_k, \quad k = 2, 3, \cdots,$$

satisfying the condition $y_k \to 0$, has the form $y_k = c\sqrt{k}J_k(z)$, $k = 1, 2, \cdots$, for some constant c.

Proof. From (3) in the last proof, $y_k = \sqrt{k}J_k(z)$, $k = 1, 2, \cdots$, obviously satisfy the recurrence relation (1). The fact that $y_k \to 0$ was noted there also.

Conversely, if y_k $(k = 1, 2, \cdots)$ satisfies (1) and $y_k \to 0$, then the y_k's represent a *minimal* solution of (1), i.e., a second solution w_k of (1) exists such that $y_k/w_k \to 0$ (e.g., $w_k = \sqrt{k}Y_k(z)$, where $Y_k(z)$ is the Bessel function of second kind of order k) [5, p.25]. Since the minimal solution is unique up to scalar multiplication [5, p.25], the lemma clearly holds. ∎

Lemma 4.4. *If λ is an eigenvalue of A, then $\lambda \neq 0$,and only one linearly independent eigenvector cx corresponds to λ, where c is a nonzero constant and x is as defined in Theorem 1.4. Moreover, $z = 2/\lambda$ is a root of $J_0(z) - iJ_1(z) = 0$.*

Proof. To prove that 0 is not an eigenvalue of A, suppose the contraly and let $Ay = 0 \cdot y = 0$ for some $y = [y_1, y_2, \cdots]^T \in l^2$, where $y \neq 0$. Expanding $Ay = 0$, we can derive $y_k = (-i)^{k-1}\sqrt{k}y_1$, $k = 2, 3, \cdots$. Since $y \neq 0$, we conclude $y_1 \neq 0$. But then, $| y_k | \to \infty$ as $k \to \infty$, a contradicton to the fact that $y \in l^2$.

Let λ be an eigenvalue of A. To prove that only one linearly independent eigenvector corresponds to λ, let $Ay = \lambda y$, $0 \neq y = [y_1, y_2, \cdots]^T \in l^2$. Suffices to show that $y = cx$ for some $c \neq 0$, where $z = 2/\lambda$, a well-defined number, since $\lambda \neq 0$ as proved above. Expanding $Ay = \lambda y$, we obtain the same relation as (1) in the last lemma, where $y_k \to 0$ since $y \in l^2$. Then, Lemma 4.3 applies, and $y_k = c\sqrt{k}J_k(z)$, $k = 1, 2, \cdots$ for some constant c, namely, $y = cx$.

Again, let λ be an eigenvalue of A and y be a corresponding eigenvector, then $y = c[J_1(z), \sqrt{2}J_2(z), \cdots]^T$ for some constant $c \neq 0$, as proved above, where $z = 2/\lambda$. Then, consideration of the identity (1) for this particular z in Lemma 4.1 gives $J_0(z) - iJ_1(z) = 0$, as required. ∎

Lemmas 4.2 and 4.4 prove the first and third assertions of Theorem 1.4. It only remains to prove the second assertion. In view of Lemma 4.4, it suffices to prove that A has no generalized eigenvectors of rank 2. We will do this in two lemmas below.

Lemma 4.5. *The function $f(z) = J_0(z) - iJ_1(z)$ has no multiple zeros, namely, if $f(z) = 0$, then $f'(z) \neq 0$.*

Proof. We prove that $f(z) = f'(z) = 0$ leads to a contradiction. The derivative $f'(z)$ at $z \neq 0$ is given by

$$\begin{aligned} f'(z) &= J_0'(z) - iJ_1'(z) \\ &= -J_1(z) - i\{J_0(z) - \frac{1}{z}J_1(z)\}, \quad \text{by } J_k'(z) = J_{k-1}(z) - \frac{k}{z}J_k(z)\ , k = 1, 2, \cdots \quad [3, \text{p.93}] \\ &= (-i)\{J_0(z) - iJ_1(z)\} + \frac{i}{z}J_1(z) = -if(z) + \frac{1}{z}J_1(z)\ . \end{aligned} \tag{1}$$

If $f(z) = f'(z) = 0$, then clearly $z \neq 0$ and (1) implies $J_1(z) = 0$. But then, $f(z) = 0$ gives $J_0(z) = 0$. This is a contradiction, since $J_0(z)$ and $J_1(z)$ do not vanish simultaneously [3, p.105]. ∎

Lemma 4.6. *The matrix A has no generalized eigenvectors of rank 2.*

Proof. Suppose the contrary and let w be a generalized eigenvector of rank 2 corresponding to an eigenvalue λ of A, i.e., let

$$\begin{cases} (A - \lambda I)w \equiv u \neq 0 \\ (A - \lambda I)^2 w = (A - \lambda I)u = 0\ , \end{cases} \tag{1}$$

hold for $u, w \in l^2$. We will derive a contradiction. To this end, consider again the identity (1) in Lemma 4.1, which holds for any $z \neq 0$. For convenience, we rewrite it in the following form:

$$(2) \qquad \begin{cases} (A - \frac{2}{z}I)x(z) = f(z)[-1, 0, 0, \cdots]^T, \\ f(z) = J_0(z) - iJ_1(z), \\ x(z) = [J_1(z), \sqrt{2}J_2(z), \cdots]^T, \end{cases}$$

where the vector denoted previously by x is written as $x(z)$ to emphasize its dependence on z. Differentiation gives

$$(3) \qquad \frac{2}{z^2}x(z) + (A - \frac{2}{z}I)x'(z) = f'(z)[-1, 0, 0, \cdots]^T,$$

where

$$x'(z) = [J_1'(z), \sqrt{2}J_2'(z), \cdots]^T = \frac{1}{2}\{[J_0(z), \sqrt{2}J_1(z), \cdots]^T - [J_2(z), \sqrt{2}J_3(z), \cdots]^T\} \in l^2,$$

since $J_k'(z) = (1/2)\{J_{k-1}(z) - J_{k+1}(z)\}$, $k = 1, 2, \cdots$ [3, p.93].

From the second equation of (1), u is an eigenvector corresponding to the eigenvalue λ. Lemma 4.4 shows that $\lambda \neq 0$ and $u = cx(z_1)$, where $z_1 = 2/\lambda$, $f(z_1) = 0$ and c is a nonzero constant. Then, the first equation of (1) leads to

$$(4) \qquad -cx(z_1) + (A - \lambda I)w = 0.$$

Eliminating $x(z_1)$ from (3) with $z = z_1$, and (4), we obtain

$$(5) \qquad (A - \frac{2}{z_1}I)(x'(z_1) + w_1) = f'(z_1)[-1, 0, 0, \cdots]^T, \qquad \text{where } w_1 = \frac{2w}{cz_1^2}.$$

Write $x'(z_1) + w_1 = \varphi = [\varphi_1, \varphi_2, \cdots]^T \in l^2$. Expanding (5), we find the kth component ($k = 2, 3, \cdots$) given by

$$f_k\varphi_{k-1} + f_{k+1}\varphi_{k+1} = \frac{2}{z_1}\varphi_k, \quad k = 2, 3, \cdots,$$

where $\varphi_k \to 0$, since $\varphi \in l^2$. Then, Lemma 4.3 applies, and we conclude $\varphi = c'x(z_1)$ for some nonzero constant c'. Since $u = cx(z_1)$, we see that φ is a scalar multiple of u. Then, the second relation of (1), i.e.,

$(A - \lambda I)u = (A - \frac{2}{z_1}I)u = 0$, gives $(A - \frac{2}{z_1}I)\varphi = 0$. Using this in (5), where $x'(z_1) + w_1 = \varphi$ as defined earlier, we find $0 = f'(z_1)[-1, 0, 0, \cdots]^T$. Hence, $f'(z) = 0$.

On the other hand, $f(z_1) = 0$ as has been shown. Hence, $f(z_1) = f'(z_1) = 0$, a contradiction of Lemma 4.5. ∎

This complets the proof of Theorem 1.4.

§5 Proof of Theorem 1.5.

Let $z \neq 0$ be a root of $J_0(z) - iJ_1(z) = 0$. Then, by Theorem 1.4, $\lambda = 2/z$ is a simple eigenvalue of the compact complex symmetric matrix $A \in B(l^2)$ defined in Theorem 1.4 with the corresponding eigenvector $x = [J_1(z), \sqrt{2}J_2(z), \cdots]^T \in l^2$. Let the infinite matrix A_n be defined by

$$(1) \qquad A_n = \left(\begin{array}{ccccc|c} i & f_2 & & & 0 & \\ f_2 & 0 & f_3 & & & 0 \\ & f_3 & 0 & \ddots & f_n & \\ 0 & & \ddots & f_n & 0 & \\ \hline & & 0 & & & 0 \end{array}\right) \equiv \begin{pmatrix} \tilde{A}_n & 0 \\ 0 & 0 \end{pmatrix}, \qquad n = 1, 2, \cdots ,$$

whose $n \times n$ principal submatrix equals the $n \times n$ principal submatrix $\tilde{A}_n$ of A, and whose components are zero elsewhere.

Lemma 5.1. *The hypothesis (H1) in §1 holds for the particular choice of* $\{A_n\}_1^\infty$, A ,λ *and* x *defined above.*

Proof. What we have already stated in this section, it only remains to prove that $\| A_n - A \|_{l^2} \to 0$ as $n \to \infty$ and $x^T x \neq 0$. The first is clear from the inequality

$$(2) \qquad \| A_n - A \|_{l^2}^2 \leq 2(f_{n+1}^2 + f_{n+2}^2 + \cdots) = 2(\frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \cdots) = \frac{2}{n} .$$

For the proof of $x^T x \neq 0$, we may use the following summation formula [15, p.152]:

$$(3) \qquad \sum_{k=0}^{\infty} (m + 2k) J_{m+2k}^2(z) = \frac{1}{4} z^2 \{ J_{m-1}^2(z) - J_{m-2}(z) J_m(z) \} \quad \text{for any } m, \text{ real or complex.}$$

Hence we find

$$(4) \qquad \begin{aligned} x^T x &= J_1^2(z) + 2J_2^2(z) + \cdots \\ &= \{J_1^2(z) + 3J_3^2(z) + \cdots\} + \{2J_2^2(z) + 4J_4^2(z) + \cdots\} \\ &= \frac{1}{4} z^2 \{J_0^2(z) - J_{-1}(z) J_1(z)\} + \frac{1}{4} z^2 \{J_1^2(z) - J_0(z) J_2(z)\} \\ &= \frac{z}{2} \cdot i J_0^2(z) \ , \end{aligned}$$

where in the final equality, $J_0(z) - iJ_1(z) = 0$, $J_{-1}(z) = -J_1(z)$ and the three-term recurrence relation $J_2(z) = (2/z)J_1(z) - J_0(z)$ are used. Now, $z \neq 0$ from Theorem 1.4; also $J_0(z) \neq 0$, for otherwise

$J_0(z) - iJ_1(z) = 0$ would imply $J_1(z) = 0$, a contradiction of the fact that $J_0(z)$ and $J_1(z)$ do not vanish simultaneously [3, p.105]. It follows that $(z/2) \cdot iJ_0^2(z) \neq 0$, i.e., $x^T x \neq 0$. ∎

Theorems 1.1 and 1.2 now apply. In particular, Theorem 1.1 (a) and (b) guarantee the existence of a sequence $\{\lambda_n\}$ of eigenvalues of A_n such that $\lambda_n \to \lambda (\neq 0)$ and such that λ_n is simple and nonzero for all large n. These nonzero eigenvalues are clearly those of $\tilde{A}_n$. In fact, $\tilde{A}_n$ is nonsingular, as the next lemma shows.

Lemma 5.2. *For each $n = 1, 2, \cdots$, 0 is not an eigenvalue of $\tilde{A}_n$.*

Proof. For each n, $det\tilde{A}_n \neq 0$ since

$$det\tilde{A}_n = \begin{cases} (-f_n^2)(-f_{n-2}^2)\cdots(-f_2^2), & n : \text{even} \\ (-f_n^2)(-f_{n-2}^2)\cdots(-f_3^2)\cdot i, & n : \text{odd} \end{cases} \quad ∎$$

Let $z_n = 2/\lambda_n$. We must prove

$$\frac{z_n - z}{z} \cong \pm\frac{\pi}{2} J_n(z) J_{n+1}(z) \tag{5}$$

for all sufficiently large n, with the compound sign $\pm$ chosen as indicated in Theorem 1.5.

The proof may be effected with the catalytic aid of the generalized Rayliegh quatient

$$\mu_n = \frac{v_n^T A_n v_n}{v_n^T v_n}, \qquad n = 1, 2, \cdots \tag{6}$$

where

$$v_n = [J_1(z), \sqrt{2}J_2(z), \cdots, \sqrt{n}J_n(z), 0, 0, \cdots]^T \in l^2 \tag{7}$$

It is clear that $\| v_n - x \| \to 0$ $(n \to \infty)$. It may now be seen that the hypothesis for Theorem 1.3 is fully satisfied. And hence, the following estimate for $| \mu_n - \lambda_n |$ holds asymptotically as $n \to \infty$:

$$| \mu_n - \lambda_n | \leq \frac{1}{x^T x} \| (A_n - \mu_n I) v_n \|^2 \beta \quad \text{for large } n \ , \tag{8}$$

where $\beta = \| (A - \lambda I)^{-1} \|_S$, a constant depending only on A and λ.

Consider now $\lambda - \lambda_n$. We decompose this as the sum of $\lambda - \mu_n$ and $\mu_n - \lambda_n$ (it is here that μ_n plays a catalytic role):

$$\lambda - \lambda_n = (\lambda - \mu_n) + (\mu_n - \lambda_n) \quad . \tag{9}$$

We will show that the first term on the right-hand side is the dominant term for large n so that $\lambda - \lambda_n$ may be effectively estimated by

$$\lambda - \lambda_n \cong \lambda - \mu_n \qquad \text{for large } n \quad . \tag{10}$$

Consider computing the first term $\lambda - \mu_n$.

Lemma 5.3.

$$\begin{aligned} \lambda - \mu_n &= \frac{J_n(z)J_{n+1}(z)}{v_n^T v_n} = \frac{J_n(z)J_{n+1}(z)}{\frac{z}{2}iJ_0^2(z) - \frac{z^2}{2}\{J_n^2(z) + J_{n+1}^2(z) - \frac{2n+1}{z}J_n(z)J_{n+1}(z)\}} \\ &\cong \frac{J_n(z)J_{n+1}(z)}{\frac{z}{2}iJ_0^2(z)} \qquad \textit{for large } n \quad . \end{aligned} \tag{11}$$

Proof. It may be directly verified from the relation $Ax = \lambda x$ ($\lambda = 2/z$) in Theorem 1.4 that

$$(A_n - \lambda I)v_n = -[0, 0, \cdots, 0, \frac{1}{\sqrt{n}}J_{n+1}(z), 0, 0, \cdots]^T, \quad \text{(the nonzero component is at the } n^{th} \text{ position)} \tag{12}$$

Multipling v_n^T from left,

$$v_n^T A_n v_n - \lambda v_n^T v_n = -\sqrt{n}J_n(z) \cdot \frac{1}{\sqrt{n}}J_{n+1}(z) = -J_n(z)J_{n+1}(z) \quad . \tag{13}$$

Dividing through by $v_n^T v_n$ and recalling the definition (6) of μ_n, we find

$$\mu_n - \lambda = -\frac{J_n(z)J_{n+1}(z)}{v_n^T v_n} \quad . \tag{14}$$

We next evaluate $v_n^T v_n$ in the denominator in the last expression. We find

$$\begin{aligned} v_n^T v_n &= x^T x - \{(n+1)J_{n+1}^2(z) + (n+2)J_{n+2}^2(z) + \cdots\} \\ &= \frac{z}{2}iJ_0^2(z) - \frac{z^2}{2}\{J_n^2(z) + J_{n+1}^2(z) - \frac{2n+1}{z}J_n(z)J_{n+1}(z)\} \quad , \end{aligned} \tag{15}$$

invoking (3), (4) and the recurrence formula $J_{k-1}(z) - (2k/z)J_k(z) + J_{k+1}(z) = 0$ with $k = n, n+1$. Substitution of (15) into (14) gives Lemma 5.3. ∎

Lemma 5.4. *If* $J_0(z) - iJ_1(z) = 0$ *and* $\mid z \mid >> 1$, *then*

$$J_0^2(z) \cong \begin{cases} -i\dfrac{2}{\pi}, & Re(z) > 0 \\ i\dfrac{2}{\pi}, & Re(z) < 0 \end{cases} \tag{16}$$

Proof. For large j, the j^{th} root z of $J_0(z) - iJ_1(z) = 0$ in the fourth quadrant is given by the following formula quoted in §1 from [8]:

$$z = r_A e^{i[(\pi/2)+\theta_A]}, \quad r_A \cong j\pi, \quad \theta_A \cong -\frac{1}{2}\pi - \frac{\alpha_A}{j\pi}, \quad \alpha_A = \frac{1}{2}ln(4j\pi) \quad . \tag{17}$$

On the other hand, $J_0(z)$ for large $\mid z \mid$ is given from [1, p.364, 9.2.1] by

$$J_0(z) = \sqrt{\frac{2}{\pi z}}\{cos(z - \frac{1}{4}\pi) + e^{Im(z)}O(\mid z \mid^{-1})\}, \qquad \mid arg\ z \mid < \pi \tag{18}$$

From (17) we find, for large j, $z \cong j\pi - i\alpha_A$ so that (18) gives

$$\begin{aligned} J_0^2(z) &\cong \frac{2}{\pi(j\pi - i\alpha_A)}\cos^2(j\pi - i\alpha_A - \frac{1}{4}\pi) \\ &= \frac{2}{\pi(j\pi - i\alpha_A)}\{(-1)^j\frac{1}{2}(e^{\alpha_A - i\frac{\pi}{4}} + e^{-\alpha_A + i\frac{\pi}{4}})\}^2 \\ &\cong \frac{2}{\pi(j\pi - i\alpha_A)}(-ij\pi) \cong -i\frac{2}{\pi} \end{aligned} \tag{19}$$

An arbitraly third quadrant root z', being the reflection of some fourth quadrant root z about the imaginary axis, has the form $z' = -\bar{z}$. Hence,

$$J_0^2(z') = J_0^2(-\bar{z}) = \overline{J_0^2(z)} \cong i\frac{2}{\pi} \quad \blacksquare \tag{20}$$

Using the last lemma in (11), we obtain, for a fixed root z of $J_0(z) - iJ_1(z) = 0$ with $\mid z \mid >> 1$,

$$\lambda - \mu_n \cong \frac{J_n(z)J_{n+1}(z)}{\frac{z}{2}iJ_0^2(z)} \cong \pm\frac{\pi}{z}J_n(z)J_{n+1}(z) \ , \qquad \text{for large } n \quad , \tag{21}$$

where the plus sign is for z with $Re(z) > 0$ and the minus sign for z with $Re(z) < 0$.

We return to the estimate (8) for $\mid \mu_n - \lambda_n \mid$.

Lemma 5.5.

$$\| (A_n - \mu_n I)v_n \| \cong | J_{n+1}(z) | / \sqrt{n} \quad \textit{for large } n \ . \tag{22}$$

Proof. We take the decomposition

$$(A_n - \mu_n I)v_n = (A_n - \lambda I)v_n + (\lambda - \mu_n)v_n \ . \tag{23}$$

The first term on the right-hand side is given by (12); and $\lambda - \mu_n$ in the second term may be estimated by Lemma 5.3. Hence,

$$\begin{aligned}
\frac{\|\text{ the second term }\|}{\|\text{ the first term }\|} &= \frac{| \lambda - \mu_n | \, \| v_n \|}{\| (A_n - \lambda I)v_n \|} \\
&= \frac{| J_n(z) | \, | J_{n+1}(z) | \, \| v_n \| / | v_n^T v_n |}{| J_{n+1}(z) | / \sqrt{n}} \\
&= \frac{\sqrt{n} \, | J_n(z) |}{| v_n^T v_n |} \| v_n \| \cong \frac{\sqrt{n} \, | J_n(z) |}{| x^T x |} \| x \| \qquad \text{for large } n \\
&\equiv O(\sqrt{n} J_n(z)) \to 0 \qquad (n \to \infty)
\end{aligned} \tag{24}$$

Hence, (24) implies

$$\| (A_n - \mu_n I)v_n \| \cong \| (A_n - \lambda I)v_n \| = | J_{n+1}(z) | / \sqrt{n} \qquad \text{for large } n \ ,$$

proving the lemma. ∎

Using the last lemma in (8), we find

$$| \mu_n - \lambda_n | \leq \frac{1}{| x^T x |} \frac{| J_{n+1}(z) |^2}{n} \beta \cong \frac{\pi \, | J_{n+1}(z) |^2}{| z | \, n} \beta \qquad (\beta = \| (A - \lambda I)^{-1} \|_S) \ , \tag{25}$$

where, in the last approximate equality, (4) and Lemma 5.4 are used under the assumption that $| z | >> 1$ with n taken correspondingly large.

We now have the estimate (21) for $\lambda - \mu_n$ and the estimate (25) for $\mu_n - \lambda_n$. Taking the ratio,

$$\frac{| \mu_n - \lambda_n |}{| \lambda - \mu_n |} \leq \frac{\dfrac{\pi \, | J_{n+1}(z) |^2}{| z | \, n} \beta}{\dfrac{\pi}{| z |} | J_n(z) | \, | J_{n+1}(z) |} = \frac{| J_{n+1}(z) |}{n \, | J_n(z) |} \beta \cong \frac{1}{n} \frac{| z |}{2n} \beta = \frac{| z | \, \beta}{2n^2} = O(n^{-2}) \tag{26}$$

It follows that $|\mu_n - \lambda_n| << |\lambda - \mu_n|$ for large n. Hence, the claim $\lambda - \lambda_n \cong \lambda - \mu_n$ is justified for large n (see (10)), and

$$\lambda - \lambda_n \cong \lambda - \mu_n \cong \pm\frac{\pi}{z} J_n(z) J_{n+1}(z) \tag{27}$$

by (21). Substitution $\lambda = 2/z$ and $\lambda_n = 2/z_n$ and the approximation $z_n \cong z$ for large n finally proves Theorem 1.5.

§6 Discussions on Numerical Computation.

For this section, we have two things in mind.

We will first give the numerical values for a selected set of roots of $J_0(z) - iJ_1(z) = 0$ in the fourth quadrant, computed from the $n \times n$ matrix $\hat{A}_n$ according to the procedure stated in §1, where n is to be taken large enough in accordance with the prescribed relative accuracy ϵ. Table 1 below gives the numerical values of the first 10 roots in the fourth quadrant, correct to 15 digits.

Secondly, and in the remainder of this section, our discussion centers on the relative error $(z_n - z)/z$ and its estimate $\pm(\pi/2)J_n(z)J_{n+1}(z)$. (We keep the same notation as in Theorem 1.5 in §1.)

To begin with, we will give a sample numerical comparison to verify the degree to which the two quantities under consideration agree. See Table 2 below, where the values of the relative error $(z_n - z)/z$ associated with the approximate root z_n are tabulated against its estimate $(\pi/2)J_n(z)J_{n+1}(z)$ for the first and second roots z in the fourth quadrant and $n = 4, 8, 12, 16, 20$ and for the fifth root and $n = 12, 16, 20, 24, 28$. It may be seen from the table that the two quantities agree to about one digit for large n, even for the roots near the origin, the fact that is quite satisfactory for all practical purposes for estimating the correct number of digits of a given approximate root.

We next consider the related problem of finding the minimum value of n, the size of the truncated matrix $\tilde{A}_n$, such that the k^{th} root for a given k is guaranteed to be computed correctly to a given number of digits. To this end, we first note, with the aid of numerical computation, that, for a given value of n, the estimate $(\pi/2)J_n(z)J_{n+1}(z)$ has smaller modulus for the root z with samller modulus. This implies that the smaller $| z |$ is, the smaller n may be sufficient to give the desired accuracy.

In fact, if we let N_ϵ denote the number of approximate roots z_n computed from the $n \times n$ matrix $\tilde{A}_n$, having the prescribed accuracy ϵ (counting all those in the third and fourth quadrant), it may be observed that N_ϵ is roughly proportional to n and is rather insensitive to ϵ. See Fig. 1 below. The observation we just made may be roughly explained as follows. Using the known asymptotic expansion [1, p.365, 9.3.1]

$$J_n(z) \sim \frac{1}{\sqrt{2\pi n}}(\frac{ez}{2n})^n \qquad (z \text{ fixed}, \ n \to \infty),$$

we find

$$\frac{\pi}{2}J_n(z)J_{n+1}(z) \sim \frac{1}{4n}(\frac{ez}{2n})^{2n+1} \quad .$$

This indicates that the quantitiy on the left-hand side rapidly decreases when $| \ ez/2n \ |$ decreases past 1. This would in turn imply that those z, the roots of $J_0(z) - iJ_1(z) = 0$, which satisfy $| \ ez/2n \ |< 1$, would give those whose approximate values computed from $\tilde{A}_n$ have the prescribed accuracy ϵ, unless ϵ is not too small. Since the roots of $J_0(z) - iJ_1(z) = 0$ distribute approximately π apart as stated in §1, a rough estimate for N_ϵ would be

$$2(\frac{2n}{e})/\pi = \frac{4n}{e\pi} \cong 0.47n \quad .$$

This roughly confirms the actual situation that Fig. 1 shows, where the actual slope of the curves shown varies from 0.5 to 0.6, approximatery.

All computations were performed in double- or quadruple-precision floating-point arithmetic (14- or 28-digits in hexadecimal) on the FACOM M-780/20 system at University of Tsukuba.

	Real	Imaginary
1	$.29803\ 82414\ 79049 \times 10^1$	$-.12796\ 02540\ 29915 \times 10^1$
2	$.61751\ 53070\ 95484 \times 10^1$	$-.16187\ 14384\ 47149 \times 10^1$
3	$.93419\ 60983\ 46134 \times 10^1$	$-.18188\ 72787\ 77295 \times 10^1$
4	$.12498\ 50706\ 39585 \times 10^2$	$-.19614\ 59538\ 01999 \times 10^1$
5	$.15650\ 10438\ 53098 \times 10^2$	$-.20723\ 09817\ 83076 \times 10^1$
6	$.18798\ 91168\ 36963 \times 10^2$	$-.21630\ 10983\ 27459 \times 10^1$
7	$.21945\ 97998\ 43811 \times 10^2$	$-.22397\ 72492\ 27609 \times 10^1$
8	$.25091\ 88576\ 39076 \times 10^2$	$-.23063\ 12806\ 67550 \times 10^1$
9	$.28236\ 97314\ 53980 \times 10^2$	$-.23650\ 36120\ 66197 \times 10^1$
10	$.31381\ 46098\ 96480 \times 10^2$	$-.24175\ 86986\ 36241 \times 10^1$

Table 1. The first 10 roots of $J_0(z) - iJ_1(z) = 0$ in the fourth quadrant.

For the first root:

	$(z_n - z)/z$		$(\pi/2)J_n(z)J_{n+1}(z)$	
n	Real	Imaginary	Real	Imaginary
4	-0.181×10^{-01}	-0.385×10^{-02}	-0.213×10^{-01}	-0.498×10^{-02}
8	$+0.262 \times 10^{-06}$	-0.867×10^{-07}	$+0.267 \times 10^{-06}$	-0.543×10^{-07}
12	-0.620×10^{-13}	$+0.393 \times 10^{-13}$	-0.651×10^{-13}	$+0.297 \times 10^{-13}$
16	$+0.111 \times 10^{-20}$	-0.101×10^{-20}	$+0.121 \times 10^{-20}$	-0.820×10^{-21}
20	-0.245×10^{-29}	$+0.366 \times 10^{-29}$	-0.320×10^{-29}	$+0.293 \times 10^{-29}$

For the second root:

4	$-0.216 \times 10^{+00}$	$-0.561 \times 10^{+00}$	$+0.584 \times 10^{+00}$	$+0.163 \times 10^{+00}$
8	-0.482×10^{-02}	-0.147×10^{-03}	-0.576×10^{-02}	$+0.288 \times 10^{-03}$
12	$+0.428 \times 10^{-06}$	$+0.305 \times 10^{-06}$	$+0.439 \times 10^{-06}$	$+0.335 \times 10^{-06}$
16	-0.234×10^{-12}	-0.318×10^{-11}	-0.584×10^{-13}	-0.326×10^{-11}
20	-0.197×10^{-17}	$+0.158 \times 10^{-17}$	-0.209×10^{-17}	$+0.147 \times 10^{-17}$

For the fifth root:

12	$+0.116 \times 10^{-00}$	-0.360×10^{-00}	$+0.177 \times 10^{-00}$	$+0.306 \times 10^{-00}$
16	-0.741×10^{-02}	-0.219×10^{-01}	-0.151×10^{-01}	-0.392×10^{-01}
20	-0.146×10^{-03}	$+0.648 \times 10^{-04}$	-0.173×10^{-03}	$+0.941 \times 10^{-04}$
24	$+0.290 \times 10^{-07}$	$+0.908 \times 10^{-07}$	$+0.350 \times 10^{-07}$	$+0.103 \times 10^{-06}$
28	$+0.110 \times 10^{-10}$	-0.409×10^{-12}	$+0.120 \times 10^{-10}$	-0.436×10^{-12}

Table 2. The relative error $(z_n - z)/z$ v.s. its estimate $(\pi/2)J_n(z)J_{n+1}(z)$ for the first, second and fifth roots z in the fourth quadrant: $n = 4, 8, 12, 16, 20$ for the first and second roots and $n = 12, 16, 20, 24, 28$ for the fifth root.

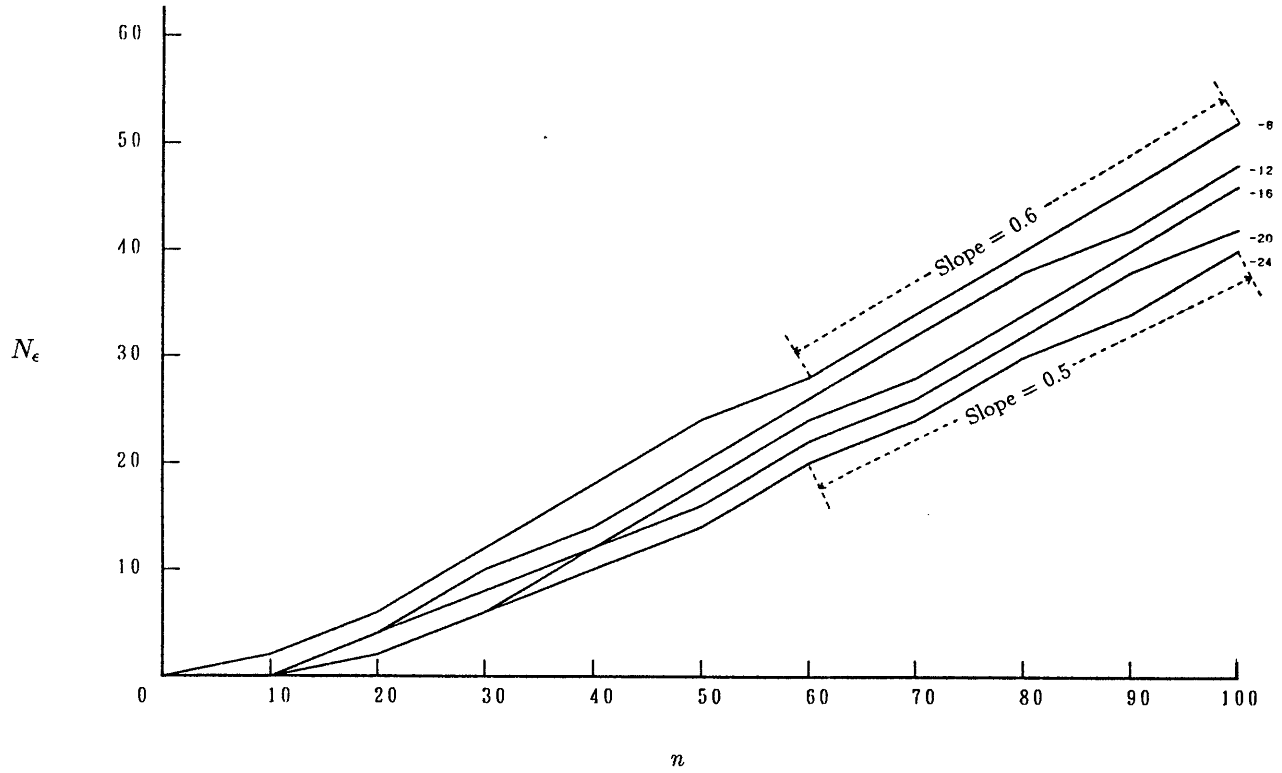

Fig. 1. The number N_ϵ of approximate roots with relative error ϵ or less, computed from the $n \times n$ matrix $\hat{A}_n$ v.s. n, for $\epsilon = 10^{-8}, 10^{-12}, 10^{-16}, 10^{-20}, 10^{-24}$. The number attached to each curve indicates $log_{10}\epsilon$.

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INSTITUTE OF INFORMATION SCIENCES AND ELECTRONICS
UNIVERSITY OF TSUKUBA
TSUKUBA-SHI, IBARAKI 305 JAPAN

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ABSTRACT

Consider computing the eigenvalues of a given compact infinite matrix regarded as operating in the complex Hilbert space l^2 by computing the eigenvalues of the truncated finite matrices and taking an obvious limiting process. In this paper, we deal with a special case where the given matrix is complex and symmetric (but not necessarily Hermitian) and where each of its eigenvalues is simple. We give a complete error analysis where the resolvent functions defined only on a proper closed invariant subspace (the *singular resolvent*) play a critical role. In fact, it is proved that the sequence of norms of singular resolvents of the truncated matrices converges to the norm of the corresponding singular resolvent for the given exact matrix. As an application, the numerical solution of $J_0(z) - iJ_1(z) = 0$, which appears in the analysis of the solitary wave runup on a sloping beach, reformulated as an eigenvalue problem for a compact complex symmetric tridiagonal matrix is given together with a full error analysis. The corresponding results for the case where the given matrix is Hermitian is concisely presented.

SUPPLEMENTARY NOTES