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A Simple Pursuit Game with an Obstacle between Players

by

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A Simple Pursuit Game with an Obstacle between Players[1]

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Abstract

The obstacle tag game is studied in a special reduced space. We describe a parametric representation of the domain, where geodesic evasion strategy is non-optimal, and using the tenet of transition solve the game inside the singular domain. We provide equations the roots of which are parameters of the optimal trajectories of players for particular initial states. It is proved that in realistic space the solution contains a focal line, and the optimal paths approach this line tangentially.

Introduction

The simple pursuit game with a circular obstacle (obstacle tag, pursuit around the hole) was posed by R.Isaacs [1]. J.Isbell outlined a solution of the game in [2]. In present paper we give a detailed solution in the case when the line segment joining players intersects the obstacle. The presence of an obstacle changes the topology of the playing space compared with the simplest pursuit game [1]. But the optimal strategy of pursuer is essentially the same. Namely, pursuer must shorten as much as possible taken-along-a-geodesic-line distance apart at any instant of the game.The situation is different with regard to evader. Geodesic evasion strategy, i.e. making the most to lengthen current distance apart, can be non-optimal because it can lead to the positions where current geodesic line is not continuation of the former ones (see [1],p.152), and hence the predected catching time along an initial geodesic line is not guaranteed.

We describe a parametric representation of the outer boundary of the domain **S** where geodesic evasion strategy is non-optimal. Then using "the tenet of transition" [1], we study properties of the value of the game in **S** and provide some numerical results.

The problem

The state $\mathbf{z} = (\mathbf{z}_P, \mathbf{z}_E)$ of the game varies with passing time $t \geq 0$ according to the following differential equation

$$\frac{d}{dt}\mathbf{z} = \mathbf{u},$$

$$\mathbf{z}^0 = (\mathbf{z}_P^0, \mathbf{z}_E^0), \tag{1}$$

where

$$\mathbf{z} \in \mathcal{R}_0 = \{\mathbf{z} = (\mathbf{z}_P, \mathbf{z}_E) \mid \ ||\mathbf{z}_P - \mathbf{z}_c|| \geq R, ||\mathbf{z}_E - \mathbf{z}_c|| \geq R\},$$

$\mathcal{R}_0$ is the playing space, $\mathbf{z}_i = (z_{ix}, z_{iy})$, $i \in \{P, E\}$, are the coordinates of players, $\mathbf{z}_c = (z_{cx}, z_{cy})$ and R are the center and radius of the obstacle, $\mathbf{u} = (\mathbf{u}_P, \mathbf{u}_E)$ is the control vector,

$$\mathbf{u}_i = (u_{ix}, u_{iy}), \quad i \in \{P, E\},$$

$$||\mathbf{u}_P|| \leq 1,$$

$$||\mathbf{u}_E|| \leq \beta < 1.$$

The game terminates when $\mathbf{z}_P^T = \mathbf{z}_E^T$, and the payoff equals T. We shall only study the game in the domain of the playing space where the line segment PE intersects the obstacle. We employ two reduced spaces $\mathcal{R}_1$ and $\mathcal{R}_2$ with the state variables as in Fig.1. In $\mathcal{R}_1$ a state is described by P's abscissa x_P and by E's coordinates (x_E, y_E) in the coordinate system xOy

$$x_P \leq 0,$$

$$x_E \geq 0,$$

$$Rx_E + x_P(y_E - R) \geq 0,$$
$$x_E^2 + (y_E - R)^2 \geq R.$$

In $\mathcal{R}_2$ a state is the vector (r, θ, s) of the parameters of the geodesic line joining players

$$r \geq 0,$$
$$0 \leq \theta \leq tan^{-1}R/r + \pi/2,$$
$$\begin{array}{ll} s \geq 0, & \text{if} \quad 0 \leq \theta \leq tan^{-1}R/r, \\ 0 \leq s \leq Rcot(\theta - tan^{-1}R/r), & \text{otherwise.} \end{array}$$

Geodesic pursuit and evasion

We define strategies of geodesic pursuit for P and geodesic evasion for E as strategies of traversing at the maximal speed a geodesic line between P and E and its tangential straight continuation respectively.

Let $(r^t, \theta^t, s^t), t > 0$, be the states in $\mathcal{R}_2$ along the geodesic pursuit-evasion trajectory started at (r, θ, s). First of all we cut down the trivial cases where geodesic evasion strategy is optimal.

Proposition 1. If in $\mathcal{R}_2$

$$\theta \leq tan^{-1}R/r, \tag{2}$$

or otherwise if

$$s^t \leq Rcot\left(\theta - tan^{-1}R/r^t\right), \quad t \in [0, r - Rcot\theta], \tag{3}$$

where $r^t = r - t$, $s^t = s + \beta t$, then geodesic pursuit and evasion strategies are optimal, and the value V_0 is described by the formula

$$V_0(r, \theta, s) = \frac{r + R\theta + s}{1 - \beta}. \tag{4}$$

Proof 1. Introduce one more realistic space with the state variables as in Fig.2 where PK_PK_EE is a geodesic line between P and E. If along trajectories corresponding to $\mathbf{u}_P$ and $\mathbf{u}_E$ E does not cross the continuation of the line segment joining P and C then dependences of $l^t, \delta^t, \gamma^t, f^t$ upon $\mathbf{u}_P$ and $\mathbf{u}_E$ are described by differential equations with smooth right-hand parts. For instance, if $l^t, f^t \neq 0$, we have the following state equations

$$\frac{d}{dt}l = \frac{-u_{Px}(l\cos\delta + R\sin\delta) + u_{Py}(l\sin\delta - R\cos\delta)}{l},$$
$$\frac{d}{dt}\delta = \frac{u_{Px}\sin\delta + u_{Py}\cos\delta}{l},$$
$$\frac{d}{dt}\gamma = \frac{-u_{Ex}\sin\gamma + u_{Ey}\cos\delta}{f},$$
$$\frac{d}{dt}f = \frac{u_{Ex}(f\cos\gamma + R\sin\gamma) + u_{Ey}(f\sin\gamma - R\cos\gamma)}{f},$$

and Isaacs' equation

$$-\frac{\sqrt{(V_\delta - RV_l)^2 + l^2 V_l^2}}{(1-\beta)l} + \frac{\beta\sqrt{(V_\gamma - RV_f)^2 + f^2 V_f^2}}{(1-\beta)f} + 1 = 0\,.$$

Using those dependences we can backward verify Proposition 1 at first for initial states $l = 0\,, \delta + \gamma > 0\,, f \geq 0$, and the terminal manifold $l = 0\,, \delta + \gamma = 0\,, f > 0$, where the value equals $f/(1-\beta)$, and then for initial states $l > 0\,, \delta + \gamma > 0\,, f \geq 0$, with the terminal manifold $l = 0\,, \delta + \gamma > 0\,, f > 0$, where the value equals $[\,R(\delta + \gamma) + f\,]/(1-\beta)$ [2]. The fulfilment of (2) or otherwise (3) exactly prevents crossing above-mentioned straight lines.∇

Let **S** denote the manifold of the playing space where $(\theta - \tan^{-1} R/r) \in [\,0\,, \pi/2]$ and (3) is not fulfiled. Let **L** denote the surface of colinear positions of P,C and E. Let **D** denote the part of the outer boundary of **S** where along geodesic pursuit-evasion trajectories equality in (3) definitely takes place one time.

Proposition 2. In $\mathcal{R}_1$ two-dimentional surface **D** has the following parametric representation

$$\begin{aligned} x_P &= -\rho, \\ x_E &= g(\rho,\xi)\cos\xi + R\sin\xi, \\ y_E &= g(\rho,\xi)\sin\xi + R(1-\cos\xi)\,, \end{aligned} \tag{5}$$

where

$$\begin{aligned} &\rho \geq \rho_0\,, \\ &\xi_0(\rho) \leq \xi \leq \xi_1(\rho)\,, \\ &\rho_0 = \frac{\sqrt{1-\beta}}{\beta}R, \\ &\xi_0(\rho) = \tan^{-1}\frac{R}{\rho} + \sin^{-1}\frac{R}{\sqrt{\beta(\rho^2+R^2)}}, \\ &\xi_1(\rho) = \tan^{-1}\frac{(1-\beta)R}{\beta\rho} + \sin^{-1}\frac{2\sqrt{\beta}R}{\sqrt{\beta^2\rho^2 + (1+\beta)^2R^2}}, \\ &g(\rho,\xi) = -\beta\rho + (1+\beta)R\cot\xi + \frac{2\sqrt{\beta}}{\sin\xi}, \end{aligned}$$

and the value V_D on **D** is described by the formula

$$V_D(\rho,\xi) = \frac{\rho + R\xi + g(\rho,\xi)}{1-\beta}. \tag{6}$$

Proof 2. Fix state variables r and θ in $\mathcal{R}_2$ as parameters

$$r = \rho, \quad \theta = \xi.$$

The value of the state variable $\tilde{s} = \tilde{s}(\rho, \xi)$ corresponding to the position $(\rho, \xi, \tilde{s})$ in $\mathcal{R}_2$ on **D** is the minimal root of the equation with respect to s

$$\triangle_D(\rho, \xi, s) = 0, \tag{7}$$

where $\triangle_D$ is the discriminant of the quadratic equation

$$\beta t^2 + [s - R\cot\xi - \beta(\rho - R\cot\xi)]t + R(r\cot\xi + R) - s(\rho - R\cot\xi) = 0,$$

describing the difference between the maximal admissible and the actual value of s along geodesic pursuit-evasion trajectories. From (7) and (4) we get the expression for g and (6).$\bigtriangledown$

Proposition 3. Making use of geodesic pursuit and evasion strategies leads a state of the game from an initial position on the surface **D** to the position on the curve **F** with the following parametric representation in $\mathcal{R}_1$

$$\begin{aligned} x_P &= -\rho, \\ x_E &= \sqrt{\beta}\rho, \\ y_E &= (1 + \sqrt{\beta})R, \end{aligned} \tag{8}$$

where $\rho \geq \rho_0$, and the value V_F on **F** is described by the formula

$$V_F(\rho) = \frac{\rho + R(\sin^{-1}\frac{R}{\sqrt{\beta(\rho^2+R^2)}} + \tan^{-1}\frac{R}{\rho}) + \sqrt{\beta\rho^2 - (1-\beta)R^2}}{1-\beta}. \tag{9}$$

Proof 3. At a state on **F** equality in (3) takes place at the instant $t = 0$. Hence on **F** we have

$$\rho - \frac{R}{\sqrt{\beta}\sin\xi} - R\cot\xi = 0,$$

and

$$g(\xi, \rho) = \frac{\sqrt{\beta}R}{\sin\xi} + R\cot\xi.$$

Substituting the expression of g in (5) and (6) we get (8) and (9) respectively. $\bigtriangledown$

Projections $\mathbf{S}_P$,$\mathbf{L}_P$,$\mathbf{D}_P$,$\mathbf{F}_P$ of **S**,**L**,**D**,**F** in $\mathcal{R}_1$ for three positions of P at $x_P = -5, -2.5, -10$ are sketched in Fig.3, $\beta = 0.5$, $R = 1$. Noteworthy that $\mathbf{D}_P$ is tangential to the obstacle as well as to $\mathbf{L}_P$.

Analysis

Our goal now is to fill up **S** with optimal paths in $\mathcal{R}_1$. If an initial state is in **S**, then in order to catch evader, P must approach the obstacle first. We assume that P does it using geodesic pursuit strategy. For sufficiently small $|x_P|$ **S** shrinks to zero. Besides that the state can leave **S** only through the curve **F** because **L** is impenetrable as well as the rim of the obstacle.

Proposition 4. In $\mathcal{R}_1$ along an optimal trajectory starting in **S** the state cannot reach **F** neither directly from inner points of **S** nor along **D**.

Proof 4. Suppose that an optimal trajectory reaches **F** or another point of **D** from within of **S** directly avoiding **L**. In any that case we can rewrite the payoff as

$$\tau + \frac{l^\tau + R(\delta^\tau + \gamma^\tau) + f^\tau}{(1-\beta)}, \tag{10}$$

where τ is the first instant of reaching **D**. In the subsidiary game with payoff (10) there are no state constraints, and geodesic pursuit and evasion strategies are optimal. But according to the definition of **S** making use of these strategies leads to reaching a colinear to C position of the players before reaching **D**. This is a contradiction.$\bigtriangledown$

Thus from any inner position in **S** the state must definitely reach **F** along **L**. Hence in $\mathcal{R}_1$ the value V_S for an inner position of **S** can be rewritten as

$$V_S(x_P, x_E, y_E) = -x_P + x_P{}^{\tau_F} + V_F(-x_P{}^{\tau_F}), \tag{11}$$

where $\tau_F = \tau_F(x_P, x_E, y_E)$ is the first instant of reaching **F** along **L**. Left-hand side of (11) is a monotonic increasing function of $-x_P{}^{\tau_F}$. It means E must try to maximize the abscissa $x_E{}^{\tau_F}$ of position on **F** (see (8)). From within of **S** E reaches projection of **L** along a straight line and then continue along **L**, because a temporary leaving **L** leads to the resulting decrease in $x_E{}^{\tau_F}$ and hence in the payoff. Let $\tilde{\alpha} = \tilde{\alpha}(x_P, x_E, y_E)$ denote the optimal angle of E to the $x-$axis at the state (x_P, x_E, y_E).

Proposition 5. In **S** $\tilde{\alpha}$ is the maximal root of the equation with respect to α

$$\triangle_S(x_P, x_E, y_E, \alpha) = 0, \tag{12}$$

where $\triangle_S$ is the discriminant of the quadratic equation

$$(\beta \sin\alpha)\tau^2 - [R - y_E - \beta(x_P \sin\alpha + R\cos\alpha)]\tau + [R(+x_E - x_P) + x_P y_E] = 0. \tag{13}$$

Proof 5. The minimal positive root of (13) determines the instant of reaching **L** along the straight line at an angle α from initial position $(x_P, x_E, y_E) \in$ **S**. The less value of α brings the less value of the payoff. Indeed, the greater α leads to the sooner reaching of **L** and to the greater y_E at the instant of reaching **L**. Along **L** at any instant t E must play at an angle of

$$\sin^{-1}\frac{y_E{}^t - R}{\beta\sqrt{(x_P+t)^2 + R^2}} + \sin^{-1}\frac{R}{\sqrt{(x_P+t)^2+R^2}}. \tag{14}$$

It means that from the begining of the game until reaching **F** the angle of the motion remains greater for the greater value of α. As a result, y_E on **F** is the less for the greater α. The minimal value of α with which E still reaches **L** corresponds to the maximal solution of (12). Note that coefficients of (13) are the same for all positions along corresponding straight line.$\bigtriangledown$

Fig.4 shows the initial part of the optimal trajectory of E started at the position $x_P = -10, x_E = 1.5, y_E = 1$, in **S**, $\beta = 0.5, R = 1$. In the reduced space $\mathcal{R}_1$ the optimal trajectories of E started in **S** both approach $\mathbf{L_P}$ along straight lines (see Proposition 5) and leave $\mathbf{L_P}$ at positions on $\mathbf{F_P}$ (see Proposition 3) tangentially. In realistic space positions of $\mathbf{S} \cap \mathbf{L} \setminus \mathbf{F}$ constitute a focal line (FL) [3]. The other part of **L** is a dispersal line (DL) (see Proposition 1) [1]. **L** is a locus of discontinuity of the value gradient. At each point of **L** there are two different ways of the optimal play. Along the FL

E faces a perpetuated dilemma [1]. Making use of geodesic pursuit strategy at positions on $\mathbf{L} \setminus \mathbf{F}$ P may switch at will between "southern" and "northern" paths. To respond optimally, E must know P's decision instaneously. If let E play that way the obtained pair of strategies is an upper saddle point [3]. To approximate the discriminating strategy by ordinary one, E may never reach projection of the FL from within of $\mathbf{S}$, but only its one-side ϵ-neighbourhood, or respond with time-delay ϵ. In both the cases the resulting decrease in the x_E on $\mathbf{F}$ tends to zero as $\epsilon \to 0$ [1,4].

Fig.5 and 6 show the pieces of the projections of the value and the upper part of $\tilde{\alpha}$ (a) and cross-sections of these surfaces (b) in the neighbourhood of $\mathbf{L}$ (y'-axis in Fig.1), $x_p = -5$, $\beta = 0.5$, $R = 1$. The difference between two values of the optimal angle along y'-axis equals

$$2 \sin^{-1} \frac{y_E - R}{\beta\sqrt{x_P{}^2 + R^2}}$$

for the FL, and

$$2 \sin^{-1} \frac{R}{\sqrt{x_E^2 + (y_E - R)^2}}$$

for the DL, where (x_P, x_E, y_E) is the state in $\mathcal{R}_1$.

Conclusion

We have described some properties and an algorithm of calculation of the solution of the simple pursuit game with an obstacle between players. We have focused here only on the investigation of the "local" features of the value in the domain of the playing space where the segment joining players intersects the obstacle. On the whole, our solution agrees with the Isbell's scheme [2]. We have not dealt with such questions as (in the fashion of [4,5])

- proof of optimality of the solution using the "viscosity condition";
- inclusion of rundom velocity errors in (1);
- perturbation of the shape of obstacle.

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Bibliography

[1] Isaacs, R., *Differential Games* , John Wiley, New York, N.Y., 1967.

[2] Isbell, J.R.,*Pursuit Around a Hole*, Naval Research Quarterly, Vol. 14, 1967.

[3] Bernhard, P., *Singular Surfaces in Differential Games*, in P.Hagedorn et al. eds., Differential Games and Applications, Springer lecture notes in Control and Information Sciences,**3**, 1977.

[4] Breakwell, J.V., *Time-Optimal Pursuit Inside a Circle*, in T. Basar and P. Bernhard eds., Differential Games and Applications, Springer lecture notes in Control and Information Sciences, **119**, 1989.

[5] Breakwell, J.V., Bernhard, P., *A Simple Game With a Singular Focal Line*, INRIA-Sophia Antipolis, **944**, 1988.

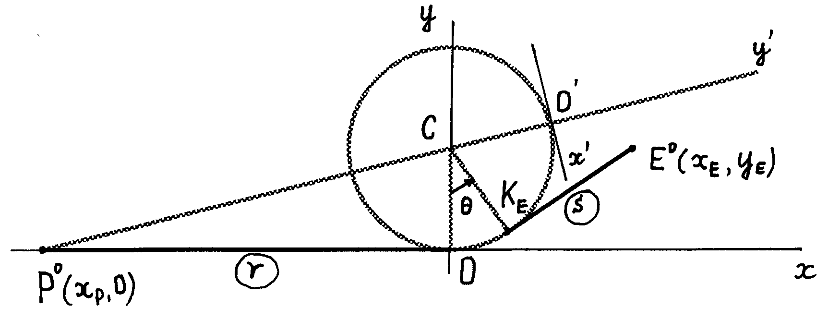

Fig.1. State variables in the reduced spaces.

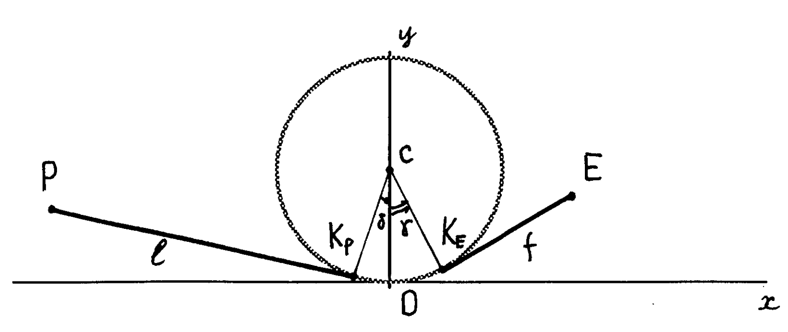

Fig.2. State variables in the realistic space.

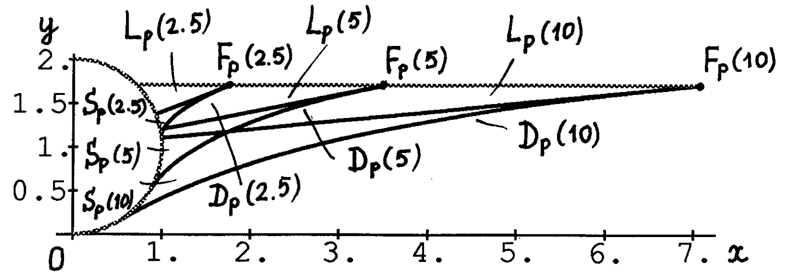

Fig.3. Projections of **S** , **L** , **D** , **F** in $\mathcal{R}_1$.

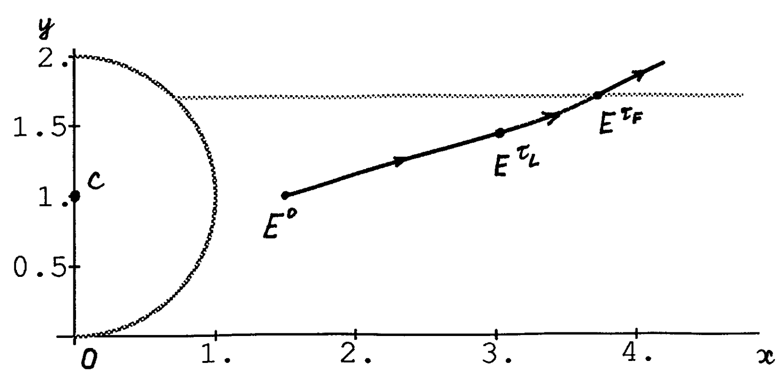

Fig.4. An example of the optimal trajectory of E in $\mathcal{R}_1$ for the initial state from **S** .

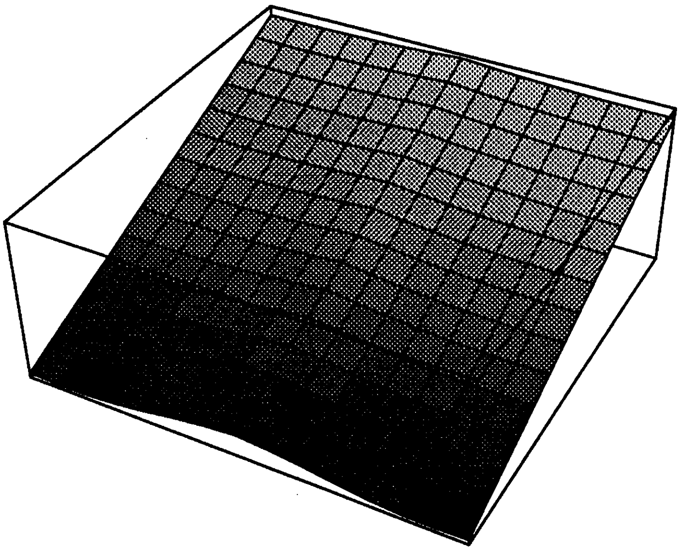

Fig.5a. Surface of the value projection.

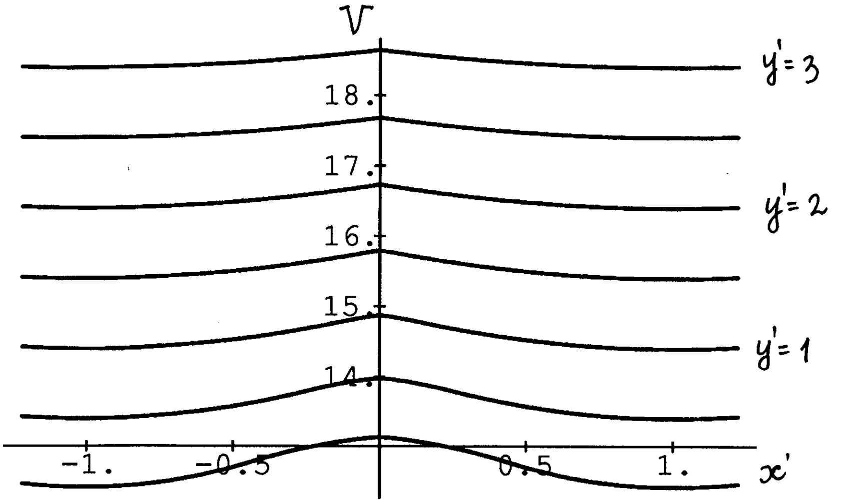

Fig.5b. Cross-sections of the value projection.

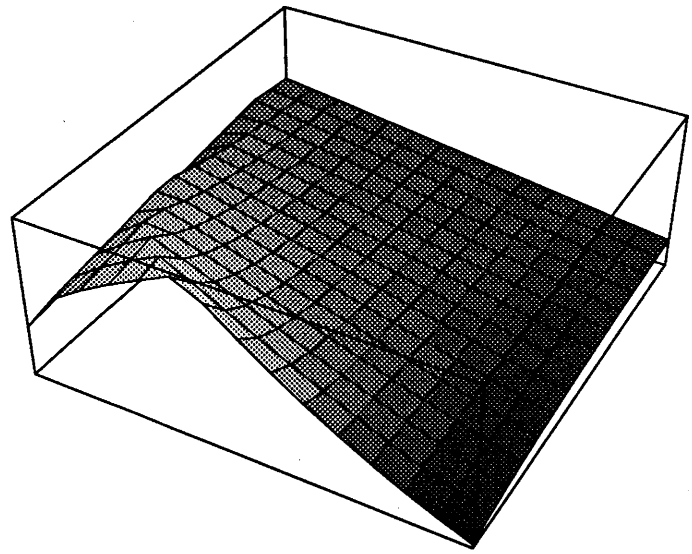

Fig.6a. Surface of the upper part of $\tilde{\alpha}$ projection.

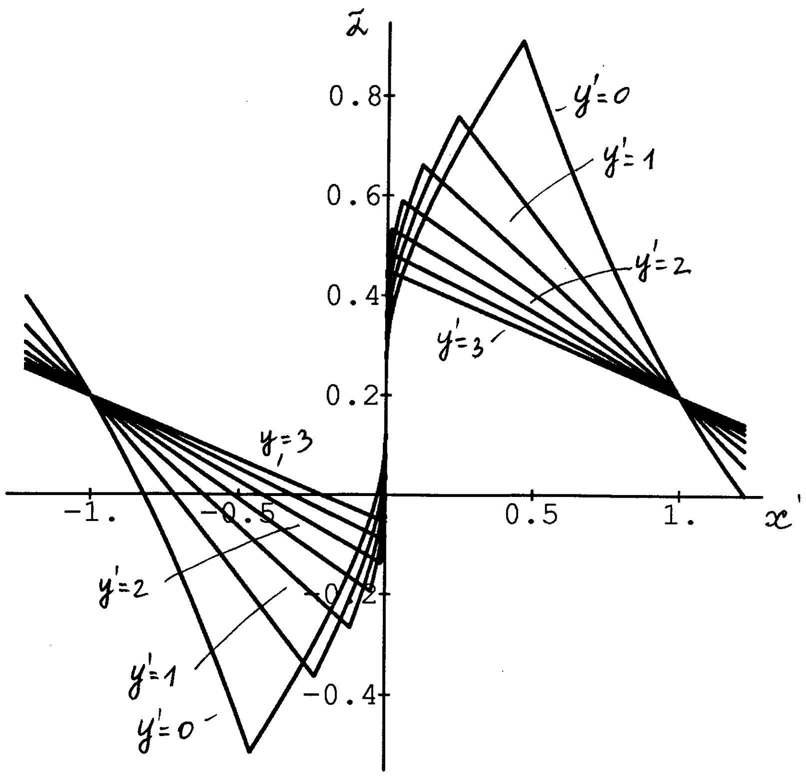

Fig.6b. Cross-sections of $\tilde{\alpha}$ projection.

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SUPPLEMENTARY NOTES