ISE

ISE-TR-90-84

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September 1, 1990

INSTITUTE
OF
INFORMATION SCIENCES AND ELECTRONICS
UNIVERSITY OF TSUKUBA

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Abstract

We consider computing a prescribed number of least positive zeros of Bessel functions and of their derivatives of a prescribed order within a prescribed relative error. We also consider an inverse problem of computing the order of the Bessel function that has a zero of a prescribed order at a prescribed positive value. The case of Bessel functions of real non-integer order less than -1 is also discussed. Our emphasis in this paper is on algorithm construction and convergence analysis that will be needed for the construction of software for solving the stated problems.

Keywords

Bessel Function, Zeros, Compact Matrix Operator, Eigenvalue Problem, Newton's Method

§1 Introduction.

In this paper we are concerned with the construction of software that will solve the following problems:

Problem I. *Given $m > -1, \epsilon > 0$ and an integer $N > 0$, compute the N least positive zeros of the Bessel function $J_m(x)$ within the relative error ϵ.*

Problem II. *Given $\beta > 0, \epsilon > 0$ and an integer $k > 0$, compute $m > -1$ such that β equals the k^{th} positive zero of $J_m(x)$ within the relative error ϵ.*

Problem III. *Given $m > 0, \epsilon > 0$ and an integer $N > 0$, compute the N least positive zeros of the first derivative of $J_m(x)$ within the relative error ϵ.*

Problem IV. *Given $\beta > 0, \epsilon > 0$ and an integer $k > 0$, compute $m > 0$ such that β equals the k^{th} positive zero of $J'_m(x)$ within the relative error ϵ.*

Problem V. *Given $m < -1$ ($m \neq$integer), $\epsilon > 0$ and an integer $N > 0$, compute all complex zeros and the N least positive zeros of $J_m(x)$ within the relative error ϵ. (There exist precisely $2[|m|]$ complex zeros, where $[|m|]$ denotes the largest integer not exceeding $|m|$.)*

It appears that software for the stated purposes is not known. It is not available from the IMSL library [15]. Our emphasis in this paper is to lay theoretical foundation, including algorithm construction and convergence analysis, that is needed for solving Problems I - V. A paper by Grad and Zakrajšek [5] is our starting point.

In §2 we consider Problem I. The basis of our work, as in [5], is the fact that the problem of computing the zeros of Bessel functions can be reformulated as that of computing the eigenvalues of an infinite real tridiagonal matrix that is obtained from a well-known reccurrence relation among Bessel functions and that may be regarded as a compact operator in the Hilbert space of infinite column vectors with square-summable components. For $m > -1$ the infinite matrix is symmetric as well. See Theorem 2.1. In Theorem 2.2, one of the main theorems of this paper, we give convergence analysis.

In §3 we consider Problem II, an inverse problem to Problem I. We derive an efficient formula for computing $dj_{m,k}/dm$, the derivative of the k^{th} positive zero of $J_m(x)$ with respect to m, in Theorem 3.1. Theorem 3.2 is a theorem on positive definiteness ,which, in particular, is interesting in the sense that it gives a matrix-theoretic proof of the fact that $j_{m,k}$ is an increasing function of m. Theorems 3.1 and 3.2 enable us to solve Problem II with Newton's method.

In §4 Problems III-V are briefly discussed. In particular, we will show that Problems III and IV may be solved in the same manner as for Problems I and II. For Problem V we simply indicate a method of solution that is again similar to the preceeding ones.

All computations were performed in double- or quadruple-precision floating-point arithmetic (14- or 28-digits in hexadecimal) on the FACOM M-780/20 system at University of Tsukuba.

§2 Problem I.

The solution method of choice is the matrix method used in [5]. We will review their technique briefly before we present our results. Take the well-known three-term reccurrence relation among Bessel functions

$$\frac{J_n(x)}{(n+1)(n+2)} + \frac{2J_{n+2}(x)}{(n+1)(n+3)} + \frac{J_{n+4}(x)}{(n+2)(n+3)} = \frac{4}{x^2}J_{n+2}(x) \tag{2.1}$$

which holds good for any x and n, real or complex, excluding $x = 0$ and $n = -1, -2$ or -3.

If we let n take the values $n = m,\ m+2, \ldots$, where $m > -1$, and write these relations in matrix form, we obtain

$$M\mathbf{u} = \frac{4}{x^2}\mathbf{u} + \mathbf{u}_0 \tag{2.2}$$

$$\begin{cases} m_{k,k} = \dfrac{2}{(m+2k-1)(m+2k+1)}, & k = 1, 2, \ldots, \\ m_{k,k-1} = \dfrac{1}{(m+2k-1)(m+2k)}, & k = 2, 3, \ldots, \\ m_{k,k+1} = \dfrac{1}{(m+2k)(m+2k+1)}, & k = 1, 2, \ldots, \\ u = [J_{m+2}(x), J_{m+4}(x), \ldots]^T, & \\ u_0 = [-\dfrac{J_m(x)}{(m+1)(m+2)}, 0, 0, \ldots]^T & \end{cases} \tag{2.3}$$

Here M represents an infinite real tridiagonal matrix whose main diagonal and super- and sub- diagonals converge to 0 for any fixed $m > -1$.

If we multiply $\sqrt{n}$ into (2.1) and regard it as a three-term reccurrence relation in $\sqrt{n}J_n(x)$ for $n = m+2, m+4, \ldots$, we can reformulate these relations as a matrix equation (2.4) below where A is an infinite *real symmetric* tridiagonal matrix whose main diagonal and super- and sub-diagonals again converge to 0:

$$A\mathbf{v} = \frac{4}{x^2}\mathbf{v} + \mathbf{v}_0 \tag{2.4}$$

$$\begin{cases} a_{k,k} = \dfrac{2}{(\alpha_k - 1)(\alpha_k + 1)} \equiv d_k \qquad k = 1, 2, \ldots, \\ a_{k,k-1} = a_{k-1,k} = \dfrac{1}{(\alpha_k - 1)\sqrt{(\alpha_k - 2)\alpha_k}} \equiv f_k, \qquad k = 2, 3, \ldots, \\ \alpha_k = m + 2k, \qquad k = 1, 2, \ldots, \\ v = [\sqrt{m+2}J_{m+2}(x), \sqrt{m+4}J_{m+4}(x), \ldots]^T, \\ v_0 = [-\dfrac{J_m(x)}{(m+1)\sqrt{m+2}}, 0, 0, \ldots]^T \end{cases} \tag{2.5}$$

Example. For $m = 0$ and $m = 1$ the matrix A is given respectively by

$$A = \begin{bmatrix} \frac{2}{1\cdot 3} & \frac{1}{3\sqrt{2\cdot 4}} & & \mathbf{0} \\ \frac{1}{3\sqrt{2\cdot 4}} & \frac{2}{3\cdot 5} & \frac{1}{5\sqrt{4\cdot 6}} & \\ & \frac{1}{5\sqrt{4\cdot 6}} & \frac{2}{5\cdot 7} & \ddots \\ \mathbf{0} & & \ddots & \ddots \end{bmatrix} \qquad (m = 0) \tag{2.6}$$

and

$$A = \begin{bmatrix} \frac{2}{2\cdot 4} & \frac{1}{4\sqrt{3\cdot 5}} & & \mathbf{0} \\ \frac{1}{4\sqrt{3\cdot 5}} & \frac{2}{4\cdot 6} & \frac{1}{6\sqrt{5\cdot 7}} & \\ & \frac{1}{6\sqrt{5\cdot 7}} & \frac{2}{6\cdot 8} & \ddots \\ \mathbf{0} & & \ddots & \ddots \end{bmatrix} \qquad (m = 1) \tag{2.7}$$

Let the symbol H denote the Hilbert space of infinite column vectors $\xi = [\xi_1, \xi_2, \ldots]^T$ such that $\sum |\ \xi_k\ |^2 < \infty$ with the inner product $(\xi, \eta) = \xi^H \eta = \sum \bar{\xi}_k \eta_k$. Then from [2, p.93] M and A both represent compact operators in H. The following theorem is known in [5] and forms the computational basis for solving Problem I.

Theorem 2.1. Let $m > -1$. *Then:*

(a) A positive number α is a zero of $J_m(x)$ if and only if $4/\alpha^2$ is an eigenvalue of A.

(b) A is positive-definite and every eigenvalue of A is simple.

(c) Let $\lambda_1 > \lambda_2 > ... > 0$ be an enumeration of the eigenvalues of A. Then $\lambda_k \to 0 (k \to \infty)$.

(d) Let A_n denote the $n \times n$ principal submatrix of A. The eigenvalues of A_n are simple and positive. Let them be denoted by $\lambda_1^{(n)} > ... > \lambda_n^{(n)} > 0$. These interlace with the eigenvalues of A_{n+1}: $\lambda_1^{(n+1)} > \lambda_1^{(n)} > \lambda_2^{(n+1)} > \lambda_2^{(n)} > ... > \lambda_{n+1}^{(n+1)}$.

(e) Let $\alpha_k^{(n)} = 2/\sqrt{\lambda_k^{(n)}} (k = 1, ..., n)$. We have the monotonic convergence $\alpha_k^{(n)} \downarrow j_{m,k}$ as $n \to \infty$ for each fixed k, where $j_{m,k}$ is the standard notation for the k^{th} positive zero of $J_m(x)$.

The next theorem estimates the rate of convergence and is one of the main theorems of this paper.

Theorem 2.2. *Let α be a positive zero of $J_m(x)$ $(m > -1)$, say $j_{m,k}$, and let α_c be the approximation to α computed from A_n, the $n \times n$ principal submatrix of A. We have then*

$$|\frac{\alpha_c - \alpha}{\alpha}| \leq \frac{|J_{m+2n}(\alpha)||J_{m+2n+2}(\alpha)|}{2J_{m+1}^2(\alpha)(m+2n+1)} + \frac{J_{m+2n+2}^2(\alpha)}{2J_{m+1}^2(\alpha)(m+2n+1)^2(m+2n)}\frac{\alpha^3}{8\pi} \equiv E_1 + E_2 \tag{2.8}$$

where $E_2/E_1 = O((m+2n)^{-4})$ and E_1 is asymptotically dominant as $n \to \infty$ with α fixed.

Proof. Let $\mathbf{v}_n$ denote the column vector consisting of the first n components of $\mathbf{v}$. Then from (2.4)

$$A_n \mathbf{v}_n = \frac{4}{\alpha^2}\mathbf{v}_n + [0, ..., 0, -\frac{J_{m+2n+2}(\alpha)}{(m+2n+1)\sqrt{m+2n}}]^T \tag{2.9}$$

Hence

$$\varphi \equiv \frac{\mathbf{v}_n^T A_n \mathbf{v}_n}{\|\mathbf{v}_n\|^2} = \frac{4}{\alpha^2} - \frac{1}{\|\mathbf{v}_n\|^2}\frac{J_{m+2n}(\alpha)J_{m+2n+2}(\alpha)}{m+2n+1}. \tag{2.10}$$

Let $\lambda = 4/\alpha^2$ and $\lambda_k^{(n)} = 4/\alpha_c^2$. Take the ineqality

$$|\lambda - \lambda_k^{(n)}| \leq |\lambda - \varphi| + |\varphi - \lambda_k^{(n)}|. \tag{2.11}$$

For the first term $|\lambda - \varphi|$ we have the following estimate by (2.10):

$$|\lambda - \varphi| \leq \frac{1}{\|\mathbf{v}_n\|^2}\frac{|J_{m+2n}(\alpha)J_{m+2n+2}(\alpha)|}{m+2n+1}. \tag{2.12}$$

For the second term $|\varphi - \lambda_k^{(n)}|$ we have by [14, §55, p.173]

$$|\varphi - \lambda_k^{(n)}| \leq \frac{\| A_n \mathbf{v}_n - \varphi \mathbf{v}_n \|^2}{a \| \mathbf{v}_n \|^2} \tag{2.13}$$

where a denotes the distance from φ to the nearest eigenvalue of A_n. It is well-known (see, for example, [1, 9.5.2, p.371] or [6, Theorem, p.251]) that the distance between two consecutive zeros of $J_m(x)$ is approximately π. This approximately translates to

$$a = \frac{8\pi}{\alpha^3}. \tag{2.14}$$

Now by [13, p.152]

$$(m+2)J_{m+2}^2(x) + (m+4)J_{m+4}^2(x) + \cdots = \frac{x^2}{4}\{J_{m+1}^2(x) - J_m(x)J_{m+2}(x)\} \tag{2.15}$$

which holds for any complex m. Hence

$$\| \mathbf{v} \|^2 = (m+2)J_{m+2}^2(\alpha) + (m+4)J_{m+4}^2(\alpha) + \cdots = \frac{\alpha^2}{4}J_{m+1}^2(\alpha) \tag{2.16}$$

since $J_m(\alpha) = 0$. For n large, $\| \mathbf{v}_n \|$ may be approximated by $\| \mathbf{v} \|$ since $\| \mathbf{v}_n \| \rightarrow \| \mathbf{v} \|$. Using (2.14), (2.16) and the approximation $\| \mathbf{v}_n \| \approx \| \mathbf{v} \|$ in (2.11), (2.12) and (2.13) we finally obtain the inequality stated in the theorem. ■

Remark. The corresponding bound for $|(\alpha_c - \alpha)/\alpha|$ given in [5] is essentially E_2. In view of Theorem 2.2, this is clearly not rigorous.

Using the fact that

$$J_m(x) = \sqrt{\frac{2}{\pi x}}\{\cos(x - c) + O(x)\} \quad (x \rightarrow \infty), \qquad c = (\frac{m}{2} + \frac{1}{4})\pi, \tag{2.17}$$

we have for large x

$$J_{m+1}(\alpha) = -J_m'(\alpha) \cong \pm\sqrt{\frac{2}{\pi\alpha}}. \tag{2.18}$$

Then the above proof really shows that

$$\frac{\alpha_c - \alpha}{\alpha} \cong \frac{\pi}{4}\frac{\alpha}{m+2n+1}J_{m+2n}(\alpha)J_{m+2n+2}(\alpha) \tag{2.19}$$

for large α and large n such that $\alpha < m+2n$, where E_2 may be ignored against E_1.

The last estimate for the relative error, (2.19), shows that *the smaller zeros converge faster as* $n \to \infty$. See Fig.2.1.

Let n be so large that E_2 may be ignored relative to E_1. Consider counting the number of those computed zeros $\alpha_1^{(n)}, \ldots, \alpha_N^{(n)}$ that have a relative error $< \epsilon$, a given positive number. The bound E_1 would be small while $J_{m+2n}(x)$ (hence $J_{m+2n+2}(x)$) is small in modulus, which is the case until x grows from 0 to about $m+2n$. See Fig.2.4. The first positive zero of $J_m(x)$, $j_{m,1}$, is located near m. Hence N *would equal approximately to* $(m+2n-m)/\pi = 0.64 \cdot n$, *independently of* m *and* ϵ, since the zeros of $J_m(x)$ are approximately π apart as stated in the last proof. Our numerical computations fairly well confirm this prediction, as can be seen from Figs.2.2 and 2.3.

The eigenvalue problem for A_n, a real symmetric tridiagonal matrix of order n,may be solved either by the QR algorithm or by the recent homotopy algorithm of Li and Rhee[8]. For our purpose the implicit QR algorithm routine IMTQL1 in the EISPACK package[12] suits well. The homotopy algorithm, though apparently faster than the QR algorithm, requires careful management of its parameters in order to keep the eigenvalue curves well separated. This is especially critical in our work since the eigenvalues of A_n get increasingly poorly separated as n grows large. For stability's sake we thus prefer the QR algorithm in our work.

§3 **Problem II.**

In Fig 3.1, $j_{m,k}$ and $j'_{m,k}$ are plotted as a function of m for several values of k. Each curve represents an increasing function that is slightly convex upward. For theorical results on the monotonicity and concavity of $j_{m,k}$ and $j'_{m,k}$ see [4],[9],[10] and references given there. Thus either one of the equations

$$j_{m,k} = \beta \qquad (\beta > 0,\ k : a\ positive\ integer) \tag{3.1}$$

or

$$j'_{m,k} = \beta \qquad (\beta > 0,\ k : a\ positive\ integer) \tag{3.2}$$

may be solved for m by Newton's method. For definiteness take (3.1). The algorithm is given by

$$\begin{cases} m^{(i+1)} = m^{(i)} - \dfrac{\alpha(m^{(i)}) - \beta}{\frac{d\alpha}{dm}(m^{(i)})} & ,i = 0,1,2,\ldots \\ m^{(0)} : an\ initial\ guess \end{cases} \tag{3.3}$$

where $\alpha(m)$ denotes $j_{m,k}$.

The computation therefore requires the evaluation of the k^{th} positive zero of $J_m(x)$ and of $d\alpha/dm$ for $m = m^{(0)}, m^{(1)}, \ldots$. The first may be computed by the method of §2. For the evaluation of the derivative $d\alpha/dm$ at a given m, we have the following theorem.

Theorem 3.1. *Let α denote $j_{m,k}$ where k is a fixed positive integer. Then for $m > -1$*

$$\frac{d\alpha}{dm} = -\frac{\alpha}{2}\frac{\mathbf{v}^T\dot{A}\mathbf{v}}{\|\mathbf{v}\|^2} = -\frac{\alpha}{2J_{m+1}^2(\alpha)}\{\dot{d}_1 v_1^2 + \sum_{k=2}^{\infty}(\dot{d}_k v_k^2 + 2\dot{f}_k v_{k-1} v_k)\} \tag{3.4}$$

(the dot '$\cdot$' represents differentiation with respect to m) where

$$\begin{cases} \mathbf{v} = [v_1, v_2, \ldots]^T = [\sqrt{m+2}J_{m+2}(\alpha), \sqrt{m+4}J_{m+4}(\alpha), \ldots]^T \\ \dot{d}_k = -\dfrac{4\alpha_k}{(\alpha_k - 1)^2(\alpha_k + 1)^2} & ,k = 1,2,\ldots \\ \dot{f}_k = -\dfrac{1}{\sqrt{\alpha_k(\alpha_k - 2)}}\{\dfrac{1}{(\alpha_k - 1)^2} + \dfrac{1}{\alpha_k(\alpha_k - 2)}\} = -\dfrac{2}{(\alpha_k - 1)^3}(1 + \sigma_k)(1 + \tau_k) & ,k = 2,3,\ldots \\ \sigma_k = \dfrac{1}{(\alpha_k - 1)\sqrt{\alpha_k(\alpha_k - 2)} + \alpha_k(\alpha_k - 2)} & ,k = 2,3,\ldots \\ \tau_k = \dfrac{1}{2\alpha_k(\alpha_k - 2)} & ,k = 2,3,\ldots \\ \alpha_k = m + 2k & ,k = 1,2,\ldots \end{cases} \tag{3.5}$$

Proof. From (2.4) and (2.5)

$$Av = \lambda v \tag{3.6}$$

where

$$\lambda = \frac{4}{\alpha^2} \tag{3.7}$$

Differentiating (3.6) with respect to m and multiplying $\mathbf{v}^T$ from left we find

$$\mathbf{v}^T \dot{A}\mathbf{v} + \mathbf{v}^T A\dot{\mathbf{v}} = \dot{\lambda}\mathbf{v}^T\mathbf{v} + \lambda\mathbf{v}^T\dot{\mathbf{v}}. \tag{3.8}$$

By symmetry of A, $\mathbf{v}^T A = (A\mathbf{v})^T = (\lambda\mathbf{v})^T = \lambda\mathbf{v}^T$. Substitution into the last equation enables us to cancel the second term from either side, giving

$$\dot{\lambda} = \frac{\mathbf{v}^T\dot{A}\mathbf{v}}{\mathbf{v}^T\mathbf{v}}. \tag{3.9}$$

Substitution of (3.6), (3.7), (3.8) and (2.16) into (3.9) yields (3.4). ■

Remark. In [3, p.108] the following expression is known for $d\alpha/dm$:

$$\frac{d\alpha}{dm} = \frac{2m}{\alpha J^2_{m+1}(\alpha)} \int_0^\alpha \frac{J^2_m(x)}{x} dx. \tag{3.10}$$

On the other hand, using

$$\int_0^\infty J_n(t)J_{n+1}(t)dt = \frac{1}{2} \qquad (Re(n) > -1) \tag{3.11}$$

([11, p.100]) and other well-known reccurrence relations, one may derive

$$2m\int_\alpha^\infty \frac{J^2_m(x)}{x}dx = 1 - J^2_m(\alpha) - 2J^2_{m+1}(\alpha) - 2J^2_{m+2}(\alpha) - \cdots \quad (m > -1,\ \alpha > 0). \tag{3.12}$$

Combining this with (3.10) we find

$$\frac{d\alpha}{dm} = \frac{2}{\alpha J^2_{m+1}(\alpha)}\{J^2_{m+1}(\alpha) + J^2_{m+2}(\alpha) + \cdots\}, \tag{3.13}$$

which is valid only for $m \geq 0$. The series on the right converges twice as slowly compared with the series in (3.4).

Theorem 3.2. *For $m > -1$, $-\dot{A}$ is positive-definite.*

Proof. We will prove the theorem by showing the eigenvalues of $-\dot{A}$ are all positive. To do this, we will show the similarity transform $D^{-1}(-\dot{A})D$ is strongly diagonally dominant (namely, the property that the sum of the moduli of the off-diagonals is strictly less than the modulus of the diagonal element holds for each row) where

$$D = diag\{(m+1)^{1.5}, (m+3)^{1.5}, (m+5)^{1.5}, \ldots\}. \tag{3.14}$$

We note that $-\dot{A}$ itself is not strongly diagonally dominant, since it fails to be so for the second row when $m = 0$. $D^{-1}(-\dot{A})D$ has the form

$$D^{-1}(-\dot{A})D = \begin{bmatrix} -\dot{d}_1 & p_2 & & \mathbf{0} \\ q_2 & -\dot{d}_2 & p_3 & \\ & q_3 & -\dot{d}_3 & \ddots \\ \mathbf{0} & & \ddots & \ddots \end{bmatrix} \tag{3.15}$$

where

$$\begin{cases} p_k = \dfrac{(m+2k-1)^{1.5}}{(m+2k-3)^{1.5}}(-\dot{f}_k) & ,k = 2, 3, \ldots \\ q_k = \dfrac{(m+2k-3)^{1.5}}{(m+2k-1)^{1.5}}(-\dot{f}_k) & ,k = 2, 3, \ldots. \end{cases} \tag{3.16}$$

Note that the diagonal elements and the super- and sub-diagonal elements of $D^{-1}(-\dot{A})D$ are positive. Therefore we must verify for $m > -1$

$$-\dot{d}_1 > p_2 \tag{3.17}$$

and

$$-\dot{d}_k > q_k + p_{k+1} \qquad ,k = 2, 3, \ldots. \tag{3.18}$$

The verification of (3.17) is straightforward by noting $\sigma_2 < (2\sqrt{3}+3)^{-1} = 0.154\cdots$ and $\tau_2 < 1/6 = 0.166\cdots$ for any $m > -1$.

To prove (3.18) we let $u = 2/(m+2k-1)$. Then $0 < u < 1$ since $k \geq 2$ and $m > -1$. We note that $\sigma_k < u^2/6$, $\tau_k < u^2/6$, $\sigma_{k+1} < u^2/8$ and $\tau_{k+1} < u^2/8$. After some computation one can show that to prove (3.18) it is enough to show

$$\Psi(u) \equiv \{(1-u^2)^{1.5} + 1\}(1+u)^{0.5}(1+\frac{u^2}{6})^2/(2+u) \leq 1 \qquad for\ 0 \leq u \leq 1. \tag{3.19}$$

One can show that as u increases from 0 to 1 $\Psi(u)$ decreases steadily from 1 to a unique positive minimum value and increases until it reaches $0.64\cdots$ at $u = 1$. See Fig.3.2. ■

Corollary. *$j_{m,k}$, $k = 1, 2, \ldots$, is an increasing function of m for $m > -1$.*

Example. Let $\beta = 10.1734681350627 = j_{1,3}$, i.e., the third positive zero of $J_1(x)$. We consider solving $j_{m,3} - \beta = 0$ by Newton's method (3.3), where we denote $j_{m,3}$ by $\alpha(m)$. The answer is clearly $m = 1$. We take $m^{(0)} = 0$. The result is summarized in Table 3.1 below.

i	$m^{(i)}$	$j_{m^{(i)},3}$	*Relative Error*
0	0.000000000000000	8.6537279129110	0.100×10^{1}
1	0.969080032750020	10.1277295540507	0.309×10^{-1}
2	0.999976638332601	10.1734336031533	0.234×10^{-4}
3	0.999999999986751	10.1734681350431	0.132×10^{-10}

Table 3.1. Newton's method (3.3) as applied to the last example.

The last table shows that the 3^{rd} iterate $m^{(3)} = 0.99\cdots$ is accurate to 10 digits. The next iterate will have an accuracy of 20 digits or more.

§4 Problems III, IV and V.

We will be brief in this section, giving only the main results with occasional explanatory comments.

We consider Problem III first. By combining the three-term reccurrence relations (2.1) for $n = m-1, m+1, \ldots$, where $m > -1$ and $m \neq 0$, with the well-known relation

$$2J'_m(x) = J_{m-1}(x) - J_{m+1}(x) \tag{4.1}$$

we obtain the matrix equation

$$B\mathbf{w} = \frac{4}{x^2}\mathbf{w} + \mathbf{w}_0 \tag{4.2}$$

where

$$\begin{cases} B = \begin{bmatrix} g_1 & h_2 & & \mathbf{0} \\ h_2 & g_2 & h_3 & \\ & h_3 & g_3 & \ddots \\ \mathbf{0} & & \ddots & \ddots \end{bmatrix} \\ g_1 = \dfrac{4+3m}{m(m+1)(m+2)} \\ g_k = \dfrac{2}{(m+2k-2)(m+2k)} \qquad , k = 2, 3, \ldots \\ h_k = \dfrac{1}{(m+2k-2)\sqrt{(m+2k-3)(m+2k-1)}} \qquad , k = 2, 3, \ldots \\ \mathbf{w} = [\sqrt{m+1}J_{m+1}(x), \sqrt{m+3}J_{m+3}(x), \ldots]^T \\ \mathbf{w}_0 = [-\dfrac{2}{m\sqrt{m+1}}J'_m(x), 0, \ldots]^T \end{cases} \tag{4.3}$$

Writing $B(m)$ for B and $A(m)$ for A (for the definition of A see §2), to emphasize their dependency on m, we note that

$$B(m) = A(r) + \begin{bmatrix} \frac{1}{(r+1)(r+2)} & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad r \equiv m-1 \tag{4.4}$$

Theorem 4.1. *A positive number α is a zero of $J'_m(x)$ if and only if $4/\alpha^2$ is an eigenvalue of B. Approximate zeros of $J'_m(x)$ may thus be computed from an $n \times n$ principal submatrix of B, B_n, as in §2.*

Theorem 4.2. *Let $m > 0$ and let α denote $j'_{m,k}$, the k^{th} positive zero of $J'_m(x)$, and let α_c be the approximation to α computed from B_n. We have then*

$$\left| \frac{\alpha_c - \alpha}{\alpha} \right| \leq \frac{|J_{m+2n}(\alpha)||J_{m+2n-1}(\alpha)|}{2\{1-(m/\alpha)^2\}J_m^2(\alpha)(m+2n)} + \frac{|J_{m+2n}(\alpha)||J_{m+2n+1}(\alpha)|}{2\{1-(m/\alpha)^2\}J_m^2(\alpha)(m+2n)^2(m+2n-1)}\frac{\alpha^3}{8\pi} \tag{4.5}$$

Proof is similar to that of Theorem 2.2.

Remark. The zeros of $J_0'(x)$ are the same as those of $J_1(x)$, since $J_0'(x) = -J_1(x)$. For $-1 < m < 0$, B has a simple negative eigenvalue with all other eigenvalues positive. The positive zeros of $J_m'(x)$ for this case correspond to the positive eigenvalues.

Theorem 4.3. *Let α denote a positive zero of $J_m'(x)$. Then for $m > -1 (m \neq 0)$*

$$\frac{d\alpha}{dm} = -\frac{\alpha \mathbf{w}^T \dot{B} \mathbf{w}}{2\{1-(m/\alpha)^2\} J_m^2(\alpha)} \tag{4.6}$$

where the dot '$\cdot$' represents differentiation with respect to m and where $\mathbf{w}$ is evaluated at $x = \alpha$.

Theorem 4.4. *For $m > 0$, $-\dot{B}$ is positive-definite.*

Proof follows from (4.4) and the fact that $-\dot{A}(r)$ is positive-definite for $r > -1$ (see Theorem 3.2). Problem IV may thus be solved using Newton's method as in §3.

Consider the last problem, *i.e.*, Problem V. Let $m < -1$. Since $J_{-m}(x) = (-1)^m J_m(x)$ if m is an integer, we may assume that m is not an integer.

Theorem 4.5. *A complex number α is a zero of $J_m(x)$ where $m < -1$ and $m \neq$ integer, if and only if $4/\alpha^2$ is an eigenvalue of $\tilde{A}$, where if $m < -2$*

$$\left\{\begin{aligned} \tilde{A} &= \begin{bmatrix} d_1 & f_2 & & & & \mathbf{0} \\ f_2 & \ddots & \ddots & & & \\ & \ddots & d_{p-1} & \tilde{f}_p & & \\ & & -\tilde{f}_p & d_p & \ddots & \\ \mathbf{0} & & & \ddots & \ddots & \end{bmatrix}, \quad \frac{|m|}{2} < p < \frac{|m|}{2} + 1 \\ \tilde{f}_p &= \frac{1}{(m+2p-1)\sqrt{-(m+2p-2)(m+2p)}} = |f_p|, \end{aligned}\right. \tag{4.7}$$

which differs from A in §2 only at the p^{th} off-diagonal pair $(-\tilde{f}_p, \tilde{f}_p)$, and if $-2 < m < -1$, $\tilde{A} = A$ but with $d_1 < 0$.

By [13, §15.27] $J_m(x)$ has precisely $2[|m|]$ complex zeros, of which two are purely imaginary if $[|m|]$ is odd, and the rest of the zeros being real. Our numerical experiments indicate that all complex zeros and a given number of positive zeros of $J_m(x)$ may be approximately computed from an $n \times n$ principal submatrix of $\tilde{A}$ within a given relative error by taking n large enough. Further investigation is needed here.

§5 **Conclusion.**

We have outlined in the last three sections computational methods for solving Problems I-V with convergence analysis where appropriate. We are experimenting with a pilot version of software that is intended to solve these problems. Results will be reported elsewhere.

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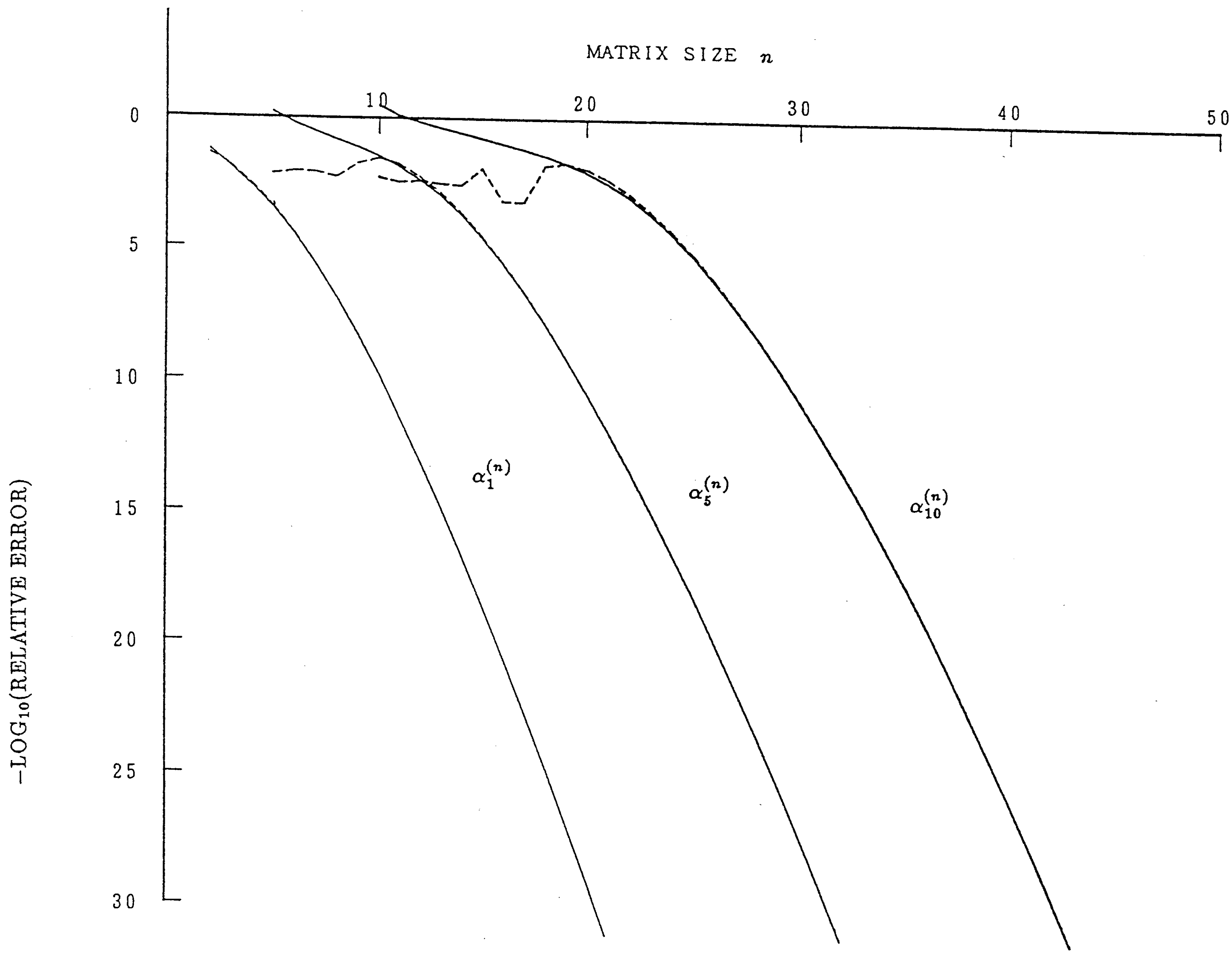
List of Figures .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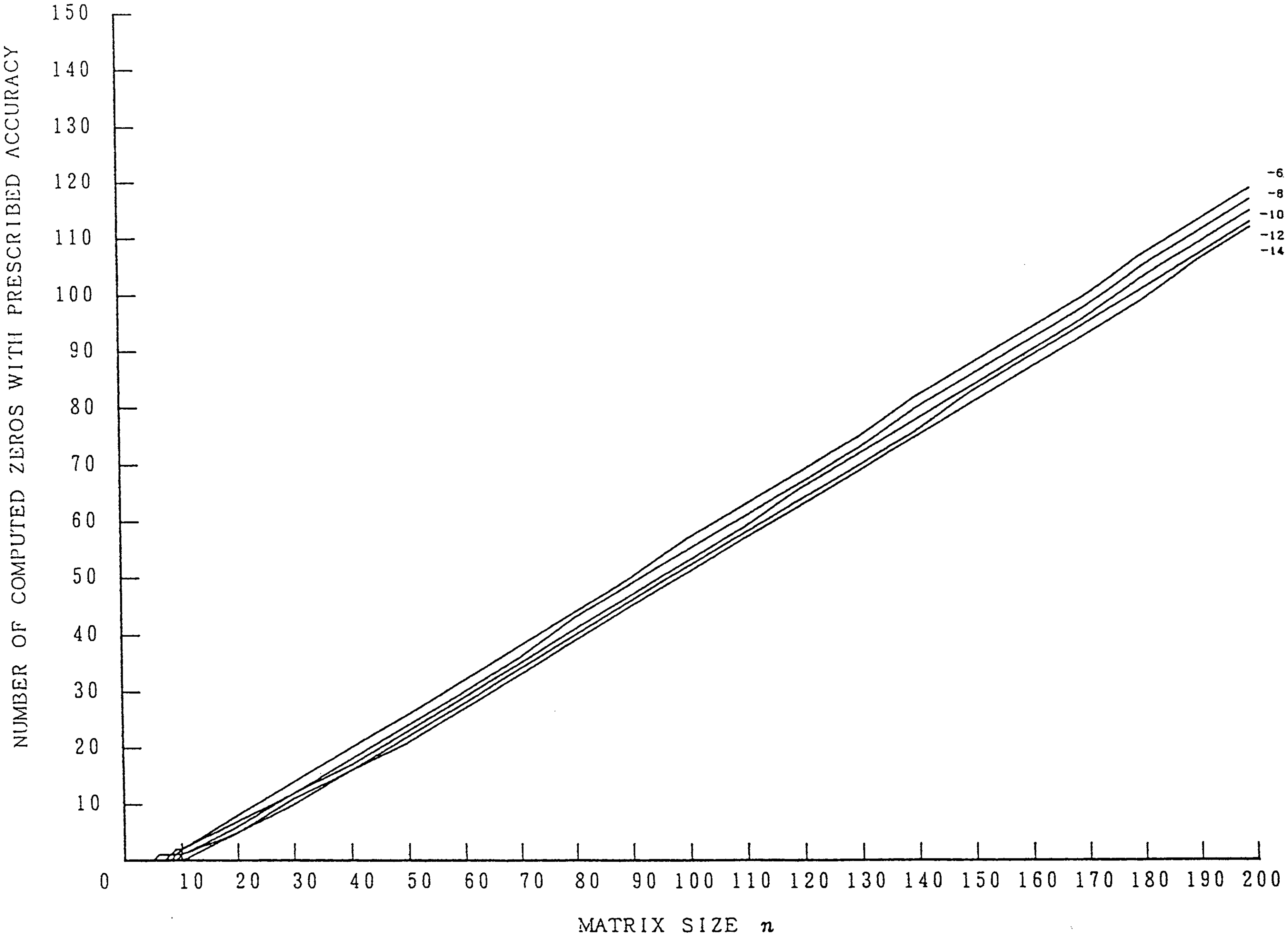
0
5
10
15
20
25
30
$-\mathrm{LOG}_{10}$(RELATIVE ERROR)
$\alpha_1^{(n)}$
$\alpha_5^{(n)}$
$\alpha_{10}^{(n)}$

Fig. 2.1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80
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20
10
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10 20 30 40 50 60 70 80 90 100 110 120 130 140 150 160 170 180 190 200
MATRIX SIZE n
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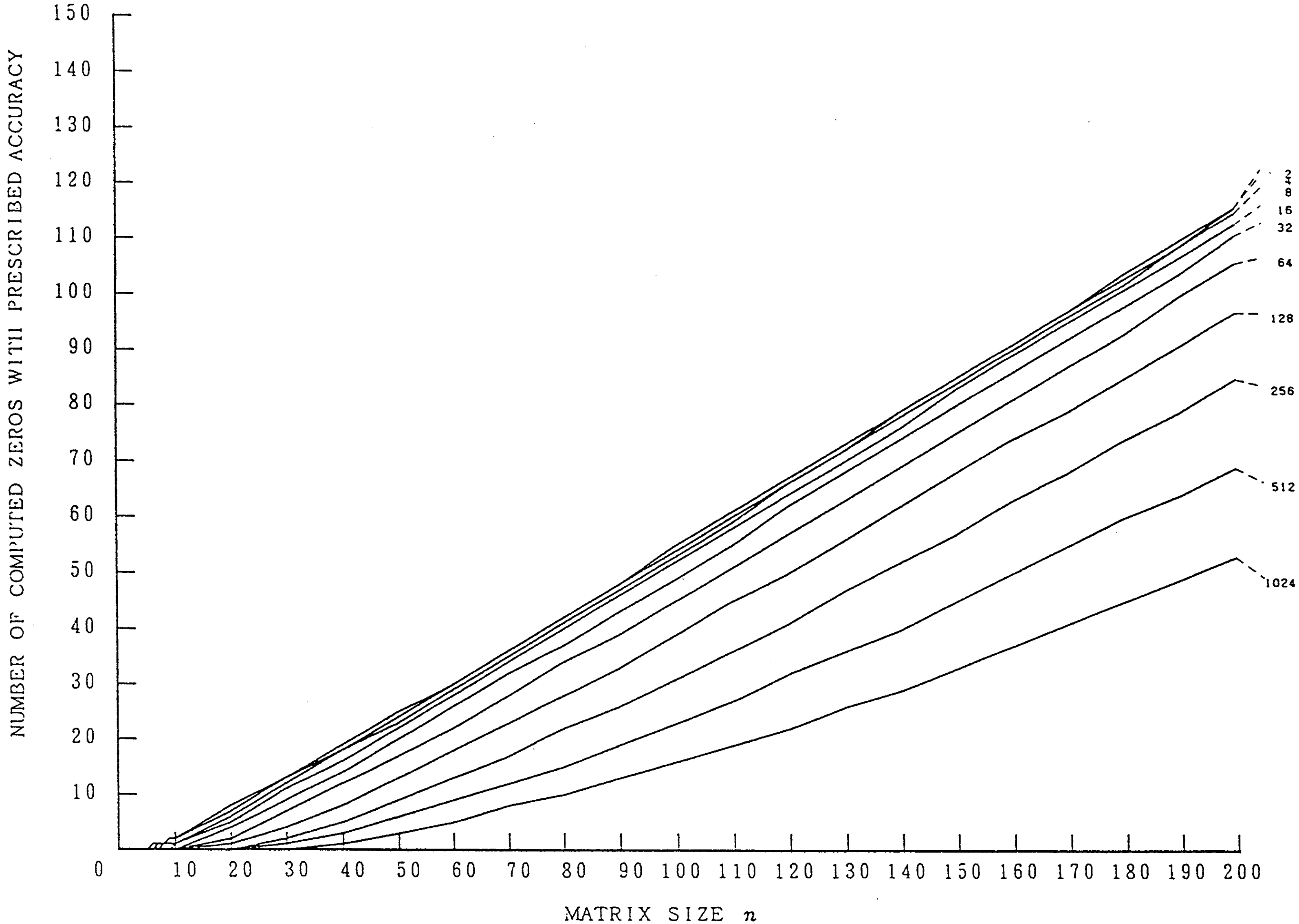
-8
-10
-12
-14

Fig. 2.2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20
10
0
10 20 30 40 50 60 70 80 90 100 110 120 130 140 150 160 170 180 190 200
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128
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512
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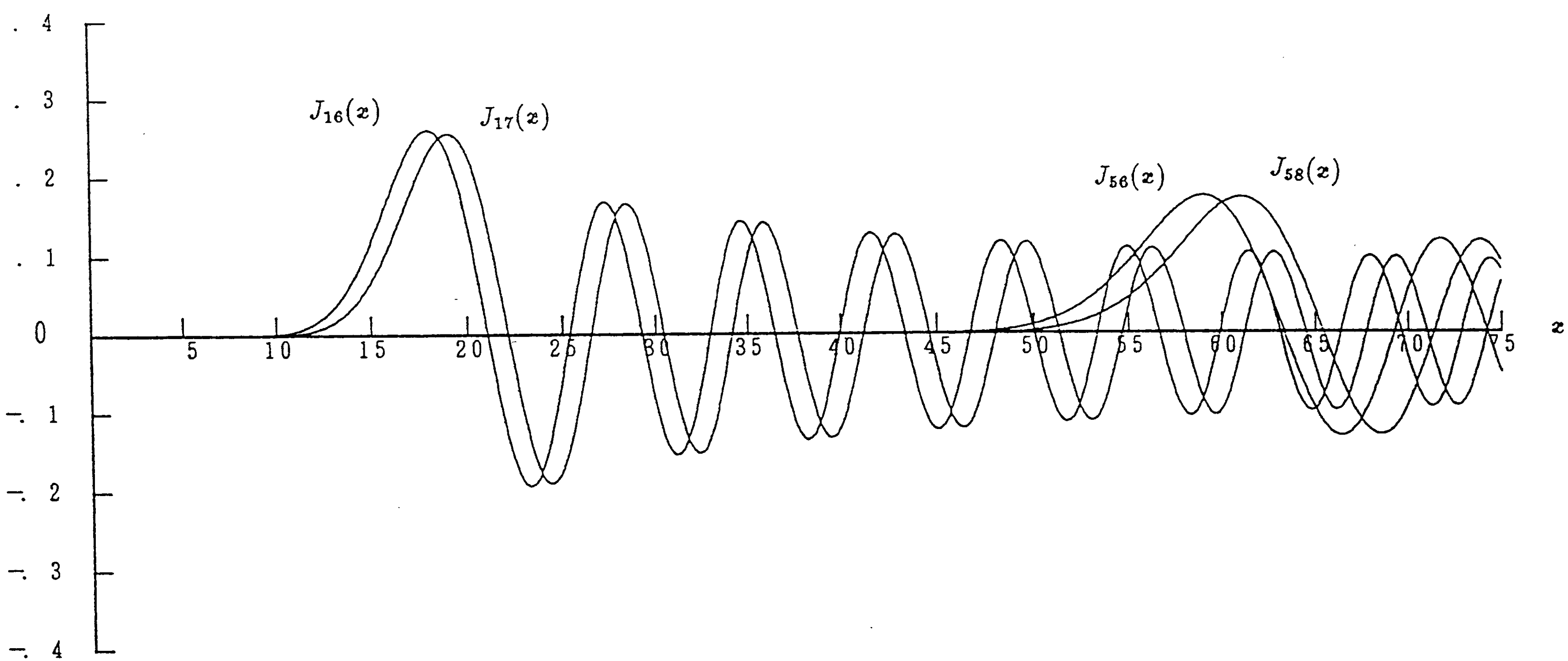
Fig. 2.3



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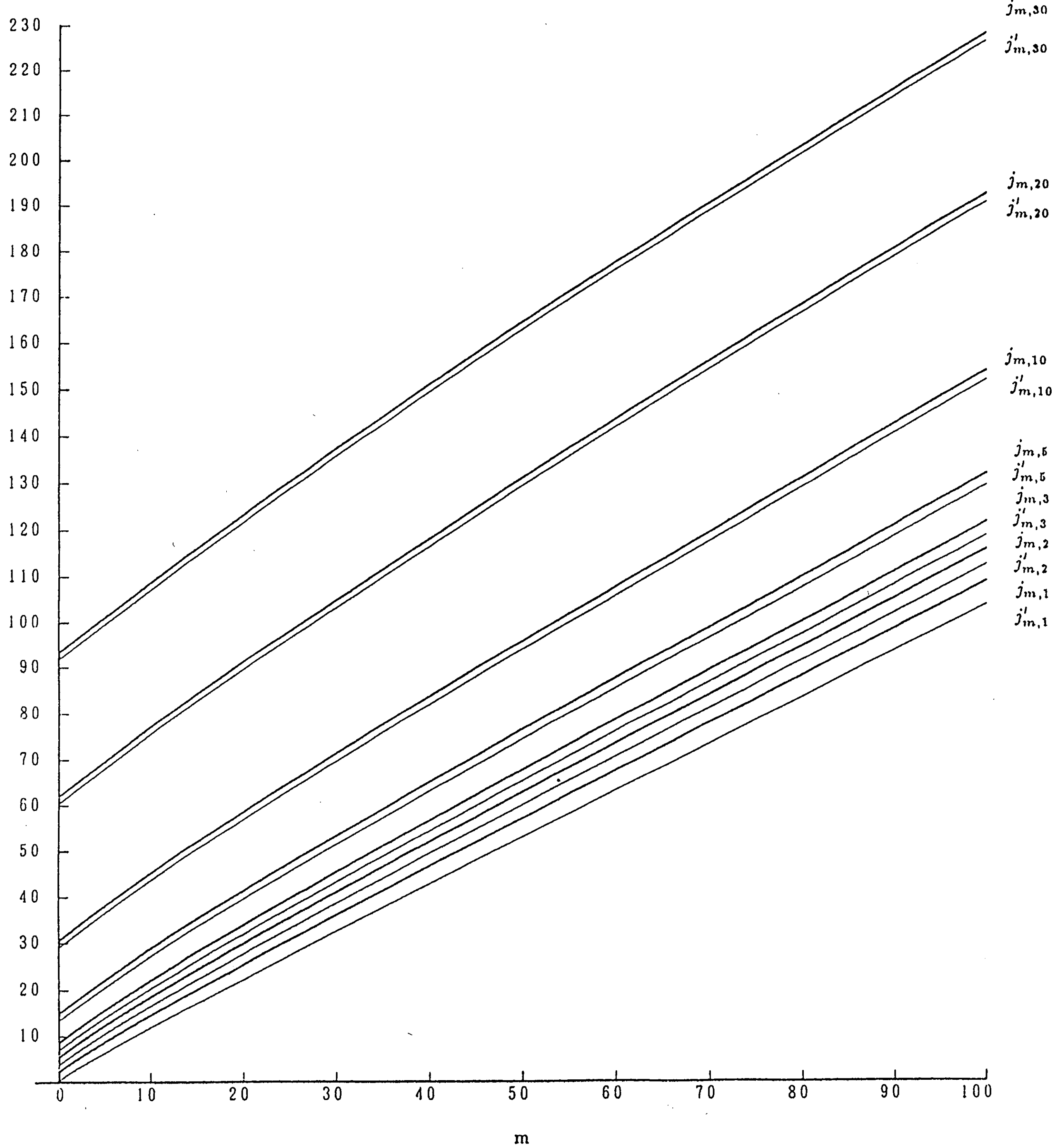
J17(x)
J56(x)
J58(x)

Fig. 2.4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$j'_{m,1}$

Fig. 3.1

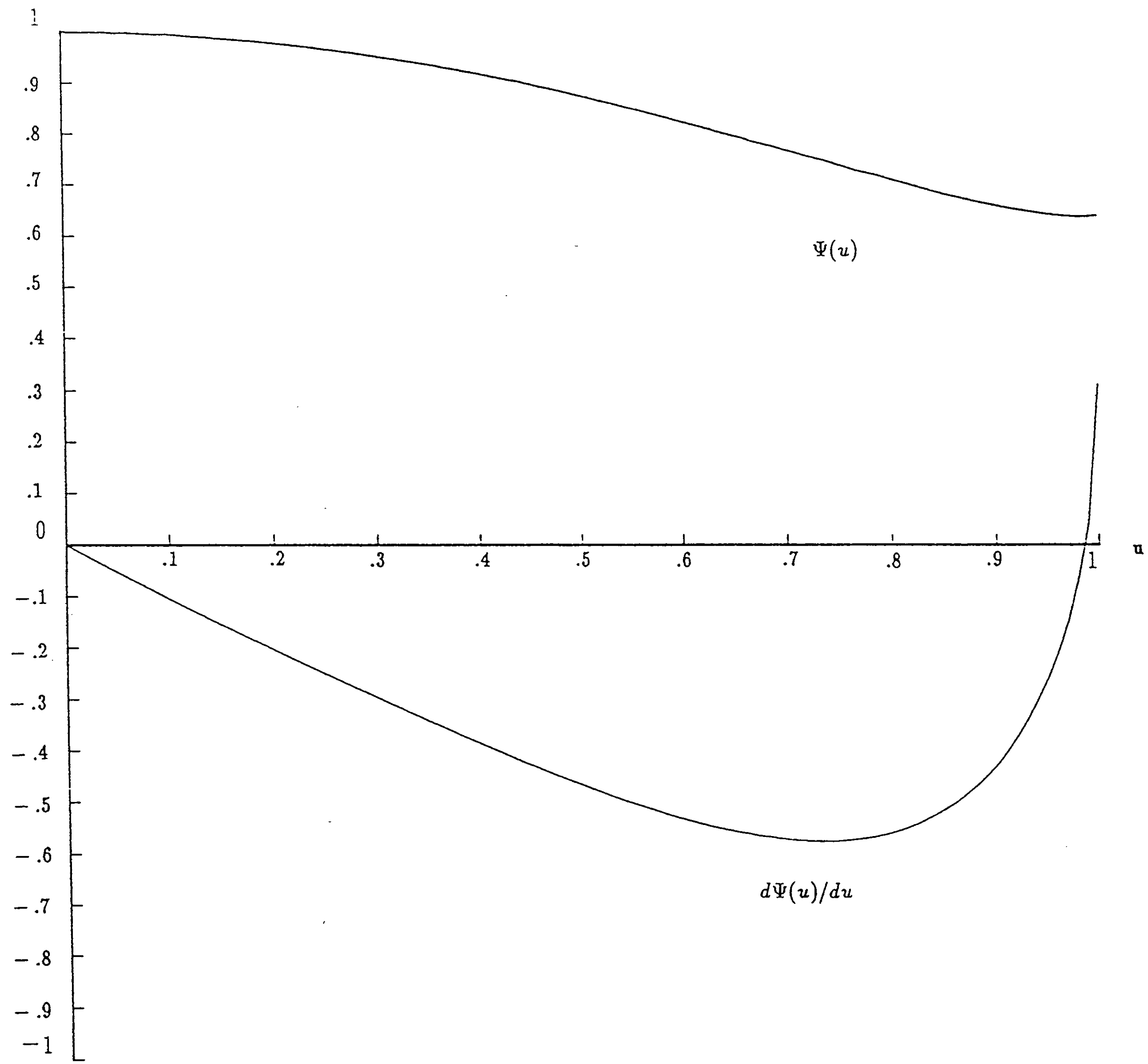

Fig. 3.2

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REPORT DOCUMENTATION PAGE	REPORT NUMBER ISE-TR-90-84

TITLE

Computing Zeros and Orders of Bessel Functions

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REPORT DATE	NUMBER OF PAGES
September 1, 1990	14
MAIN CATEGORY	CR CATEGORIES
Numerical Analysis	

KEY WORDS

Bessel Function, Zeros, Compact Matrix Operator, Eigenvalue Problem, Newton's Method

ABSTRACT

Abstract

We consider computing a prescribed number of least positive zeros of Bessel functions and of their derivatives of a prescribed order within a prescribed relative error. We also consider an inverse problem of computing the order of the Bessel function that has a zero of a prescribed order at a prescribed positive value. The case of Bessel functions of real non-integer order less than -1 is also discussed. Our emphasis in this paper is on algorithm construction and convergence analysis that will be needed for the construction of software for solving the stated problems.

SUPPLEMENTARY NOTES