ISE-TR-88-76

A NOTE ON CONSTRUCTION OF THE MINIMAL SUPERMARTINGALE IN CONTINUOUS PARAMETER OPTIMAL STOPPING PROBLEMS

by

Minoru Yoshida

August 23, 1988

INSTITUTE
OF
INFORMATION SCIENCES AND ELECTRONICS

UNIVERSITY OF TSUKUBA

A NOTE ON CONSTRUCTION OF THE MINIMAL SUPERMARTINGALE IN CONTINUOUS PARAMETER OPTIMAL STOPPING PROBLEMS

M. YOSHIDA

Abstract

A construction scheme of the minimal supermartingale lying above continuous parameter reward processes in optimal stopping problems is introduced. It is shown that for some problems the optimal stopping rule can be derived by making use of the construction scheme.

1. Introduction

Let $(\Omega,\mathcal{F},P;\mathcal{F}(t))$ be a complete probability space with right continuous increasing family $(\mathcal{F}(t),t\in T)$ of sub σ-fields of $\mathcal{F}$ each containing P-null sets, where $T=[0,\infty)$. On the probability space suppose that we are given an $\bar{R}=R\cup\{-\infty, \infty\}$ valued stochastic process $X=(X(t),\mathcal{F}(t),t\in T)$, which is adapted to the family $(\mathcal{F}(t),t\in T)$. Let $\mathcal{M}=\{\tau\}$ be a class of stopping times $\tau=\tau(\omega)$ relative to the system $(\mathcal{F}(t),t\in T)$ such that $P(\tau<\infty)=1$.

The optimal stopping problem is described as follows:

(i) Exhibit the cptimal stopping time $\tau^*\in\mathcal{M}$ such that

$$E[X(\tau^*)] = \sup\{E[X(\tau)]: \tau\in\mathcal{M}\}. \qquad (1.1)$$

(ii) Exhibit the maximal expected reward $E[X(\tau^*)]$.

The optimal stopping rule and the maximal expected reward can be

This research was partially supported by Grant-Aid for Research (No. 61740123), Ministry of Education, Japan.

characterized by means of a process called the minimal supermartingale lying above the reward process (MS), of which definition will be given in section 2. The following important theorem is introduced by Irle [2]:

Proposition 1. [2,Corollary in Section 2] *Suppose that for a right continuous process X there exists the MS* $\gamma=(\gamma(t),\mathcal{F}(t),t\in T)$ *lying above X. If the stopping time* τ *defined by* $\tau=\inf\{t\ T:X(t)\geq\gamma(t)\}$ *satisfies*

$$E[X(\sup\sigma_n)]\geq\limsup E[X(\sigma_n)] \tag{1.2}$$

for any sequence of stopping times $(\sigma_n),n=1,2,\ldots,$ *such that* $0\leq\sigma_n\leq\tau$. *And if* $\tau\in\mathcal{M}$. *Then it holds that*

$$E[X(\tau)]\geq\sup\left\{E[X(\sigma)]:\sigma\text{ stopping time},\ \liminf\int_{\{\sigma>t\}}X^-(t)dP=0\right\}\geq\sup\left\{E[X(\sigma)]:\ \sigma\text{ bounded stopping time}\right\}. \tag{1.3}$$

If in addition γ *is regular, then* $E[X(\tau)]=\sup\{E[X(\sigma)]:\ \sigma\in\mathcal{M}\}$.

In Irle [2] it is also shown that for the problems called weakly monotone stopping problems, the optimal stopping rules are expressed without using the explicit form of MS.

In section 2 we shall introduce a construction scheme of MS. We shall also pose two corollaries with which we can derive optimal stopping rules for some special optimal stopping problems, and deal with the relationships between Irle's weakly monotone stopping problems and the problems for which our corollaries can be applied. In section 3 we shall consider two simple examples.

2. Construction scheme of the Minimal Supermartingale

First of all we make the following definitions (c.f. Thompson [5], Mertens [3]):

Definition. (i) A stochastic process $Y=(Y(t),\mathcal{F}(t),t\in T)$ is called simply as a supermartingale when it is an L^1-supermartigale (see Meyer [4]), and especially if Y is a well-measurable process, then it is called as a well-measurable supermartingale.

(ii) A well-measurable supermartingale Y is called regular, if for any stopping times σ and τ with $P(\sigma\leq\tau<\infty)=1$ the relation $Y(\sigma) \geq E[Y(\tau)|\mathcal{F}(\sigma)]$ a.s. holds.

(iii) For two stochastic processes $S=(S(t),\mathcal{F}(t),t\in T)$ and $R=(R(t),\mathcal{F}(t),t\in T)$ we define $S \left\{\begin{smallmatrix}\leq\\=\end{smallmatrix}\right\} R$ by the requirement that $P((\forall t\in T)\ S(t) \left\{\begin{smallmatrix}\leq\\=\end{smallmatrix}\right\} R(t)) = 1$.

(iv) The minimal supermartingale (MS) $\gamma=(\gamma(t),\mathcal{F}(t),t\in T)$ lying above X is the well-measurable supermartingale such that $\gamma\leq Y$ for any well-measurable supermartingale Y with $Y\geq X$, if especially γ is regular then it is called as the minimal regular supermartingale (MRS) lying above X.

Theorem 1. *Suppose that the process X is a well-measurable process such that $X_t\in L^1$ for any $t\in T$.*

Let

$$\begin{aligned}\xi(t;0) &= \operatorname*{ess\,sup}_{s\geq 0} E[X(t+s)|\mathcal{F}(t)] \qquad \text{and} \\ \xi(t;n) &= \operatorname*{ess\,sup}_{s\geq 0} E[\xi(t+s;n-1)|\mathcal{F}(t)], \quad n=1,2,\ldots, \quad t\in T,\end{aligned} \tag{2.1}$$

and set

$$\xi(t) = \lim_{n\to\infty} \xi(t;n), \qquad \xi = (\xi(t), \mathcal{F}(t), t\in T), \tag{2.2}$$

$$\xi^{+}(t) = \lim_{h\downarrow 0} \xi(t+h), \qquad \xi^{+} = (\xi^{+}(t), \mathcal{F}(t), t\in T) \quad \text{and} \tag{2.3}$$

$$\eta(t) = \max(\xi^{+}(t), X(t)), \qquad \eta = (\eta(t), \mathcal{F}(t), t\in T). \tag{2.4}$$

(i) If $\xi(t)\in L^1$ for any $t\in T$, then η is the MS lying above X.

(ii) If in addition η is regular, then it is the MRS lying above X.

Remark.

If there exists a random variable $U\in L^1$ such that $\eta(t)\geq E[U|\mathcal{F}(t)]$, $\forall t\in T$, then η is regular.

Proof.

From (2.1) we have

$$\xi(t;n) \geq E[\xi(t+s;n-1)|\mathcal{F}(t)] \quad \text{a.s.,} \quad \forall s,t\in T. \tag{2.5}$$

Since $\xi(t;n)\geq\xi(t;n-1)\geq X(t)\in L^1$ a.s. for each $t\in T$, and since (2.5) holds, by the monotone convergence theorem (see for example [1]) we have the followings: $\forall t\in T$, $\exists\, \xi(t)$, $\lim_{n\to\infty} \xi(t;n) = \xi(t)$ a.s. and

$$\xi(t) \geq E[\xi(t+s)|\mathcal{F}(t)] \quad \text{a.s.} \quad \forall s,t\geq 0. \tag{2.6}$$

Thus, if the assumption of (i) is satisfied, then ξ becoms a supermartingale. Since $\mathcal{F}(t)$, $t\in T$, is right continuous it follows, p.95 Meyer [4], that the process ξ^{+} defined by (2.3) is a right continuous, and of course well-measurable, supermartingale such that $\xi^{+}\leq\xi$. Now, it is a straightforward matter to verify that the process η is also a well measurable supermartingale satisfying

$$X \leq \eta \leq \xi, \tag{2.7}$$

in other words it is a well-measurable supermartingale lying above X.

On the other hand, it is known (T4 of Meltens [3]) that there exists the MS $\gamma=(\gamma(t),\mathcal{F}(t),t\in T)$ lying above X, and it must hold that

$$\gamma \leq \eta, \tag{2.8}$$

if the assumptions of this theorem are satisfied.

From $\gamma \geq X$ and (2.1) by induction we have $\xi(t;n) \leq \gamma(t)$ a.s., for each $t \in T$, $n=0,1,\ldots$, and hence we can conclude that $\xi(t) \leq \gamma(t)$ a.s., for each $t \in T$. Thus from (2.7) it holds that

$$\eta(t) \leq \gamma(t) \quad \text{a.s., for each } t \in T. \tag{2.9}$$

Consequently from (2.8) and (2.9) it must hold that $P((\forall t \in T)\ \eta(t)=\gamma(t))=1$. In addition if η is regular, then it is obvious that η is the MRS lying above X. The proof is complete.

Corollary 1 below follows obviously from the proof of Theorem 1.

Corollary 1. *Let $\xi^n=(\xi(t;n),\mathcal{F}(t),t \in T)$, $n=0,1,\ldots$ Suppose that X satisfies the assumptions of Theorem 1-(i) and that ξ^n, $n=0,1,\ldots$, and ξ are well-measurable processes. Then*

(i) $\xi = \eta$.

(ii) If in addition there exists a non-negative integer $n^* < \infty$ such that $B^{n^*} = B^m$, $m \geq n$, with $B^k = \{(\omega,t): X(t) \geq \xi(t;k)\}$, $k=0,1,\ldots,$ then $\{(\omega,t): X(t) \geq \eta(t)\} = B^{n^*}$.

In the following Corollary 2 we see the relationships between the theorem for weakly monotone stopping problems (Theorem in Section 3 of [2]) and Corollary 1.

Corollary 2. *Suppose that X, ξ^n, $n=0,1,\ldots$, and ξ satisfies the assumptions of Corollary 1-(i). Assume that*

$$C_t \subset C_{t+h}, \quad \forall h \geq 0, \qquad \text{with } C_t = \{\omega : X(t) \geq \xi(t;0)\}, \quad t \in T. \tag{2.10}$$

Then
$$B^m = B^0, \quad m \geq 1. \tag{2.11}$$

Proof. Generally it holds that

$$B^m \subset B^0, \quad m \geq 1. \tag{2.12}$$

If (2.10) is satisfied, then on the set B^0 the followings hold:

$$\xi(t;1) = \operatorname*{ess\,sup}_{h \geq 0} E[\xi(t+h;0) | \mathcal{F}(t)] \leq \operatorname*{ess\,sup}_{h \geq 0} E[X(t+h) | \mathcal{F}(t)] = \xi(t;0)$$
$$\leq X(t).$$

Thus we have $B^1 \supset B^0$, and hence from (2.12), $B^1 = B^0$ holds. By induction we see that (2.11) holds. The proof is complete.

3. Examples

Example 1. Suppose that on a complete probability space $(\Omega, \mathcal{F}, P)$ we are given two independent Poisson processes $Z=(Z(t))$, $t \in T$, $Z(0)=0$, and $Y=(Y(t))$, $t \in T$, $Y(0)=0$, with rate 1 and $\lambda > 0$ respectively. Let $\mathcal{F}(t)$, $t \geq 0$, be the smallest σ-algebra containing $\sigma(Z(s), Y(s); s \leq t)$, $t \geq 0$, and P-null sets of $\mathcal{F}$. Let us define the reward process $X=(X(t), \mathcal{F}(t), t \in T)$ by $X(t) = Z(t) - ((Y(t))^2$.

We have $E[X(t+h) | \mathcal{F}(t)] = -\lambda h^2 + (1-\lambda-2Y(t))h + (Z(t)-((Y(t))^2)$, $h \geq 0$, Especially, if we suppose that $1/3 \leq \lambda < 1$, then

$$\xi(t;0) = \begin{cases} X(t) & \text{if } Y(t) \geq 1, \\ X(t) + (1-\lambda)/4\lambda & \text{if } Y(t) = 0. \end{cases}$$

Following Irle [2], if we set $C_t = \{\omega : X(t) \geq \xi(t;0)\}$, then $C_t \subset C_{t+h}$ for any $t, h \geq 0$ and $\bigcup_{t \in T} C_t = \Omega$, and hence we see that the process X forms a

weakly monotone process. According to Theorem in Section 3 of Irle [2], the stopping time $\tau=\inf\{t:X(t)\geq\xi(t;0)\}=\inf\{t:Y(t)\geq 1\}$ has the optimality property as is described in Proposition 1. Indeed since the stopping time τ is totally inaccesible with respect to the family $(\mathcal{F}(t)),t\in T$, [4, p.139], it must hold that $\lim_{n\to\infty} X(\sigma_n,\omega)=X(\sup\sigma_n,\omega)$ for a.a. $\omega\in A$, where (σ_n),n=1,2,..., is any sequence of stopping times such that $0\leq\sigma_n\leq\tau$ and $\sigma_n(\omega)=\tau(\omega)$ holds for a.a. $\omega\in A$ and large enough n with $A=\{\omega:\sup\sigma_n=\tau\}$. On the other hand for a.a. $\omega\in A^c$, since $X(\sigma_n,\omega)$ increasis as n increases, it holds that
$\sup X(\sigma_n,\omega)=X(\sup\sigma_n,\omega)$, a.s. $\omega\in A^c$. Consequently we have
$E[X(\sup\sigma_n)] = E[\lim\sup X(\sigma_n)] \geq \lim\sup E[X(\sigma_n)]$. Thus the stopping time τ satisfies (1.2), and it is obvious that $\tau\in\mathcal{M}$, hence the relation (1.3) holds.

Now, let us apply Theorem 1 to this example and derive the MS. Let $K_1=(1-\lambda)^2/4\lambda$ and the number $h_n\geq 0$ be the unique solution, of which existence is obvious, of the following equation: $U(K_n,h_n)=0$, n=1,2,..., where $K_{n+1}=R(K_n,h_n)$, n=1,2,..., with
$R(k,h)=Ke^{-\lambda h}+h(1-\lambda(h+1))$ and $U(k,h)=\frac{\partial}{\partial h}R(K,h)=-K\lambda e^{-\lambda h}+1-\lambda(1+2h)$.
By induction it is easy to see that $E[\xi(t+h;n-1)|X(t)\geq 0,\ Y(t)=0]$ $=X(t)+R(K_n,h)$, $\xi(t;n)=\begin{cases} X(t) & \text{if } Y(t)\geq 1, \\ X(t)+K_n & \text{if } Y(t)=0,\ n=1,2,..., \end{cases}$ and K_n increases as n increases and $\lim_{n\to\infty} K_n= (1-\lambda)/\lambda =K^*$. Consequently, from Theorem 1 we have that the MS $\eta=\xi$ and $\xi(t)=\begin{cases} X(t) & \text{if } Y(t)\geq 1, \\ X(t)+K^* & \text{if } Y(t)=0. \end{cases}$
Note that the assumption of Corollary 1-(ii) is satisfied with $n^*=0$.

Example 2. Suppose that on a complete probability space

$(\Omega, \mathcal{F}, P)$ we are given two independent Poisson processes $Z=(Z(t)), t\in T$, $Z(0)=0$, and $Y=(Y(t)), t\in T$, $Y(0)=0$, with common rate 1, and an exponentially distributed random variable θ (with rate $\mu>0$) which is independent of Z and Y. We define a process $I=(I(t)), t\in T$, by

$I(t)=\begin{cases} 1 & \text{if } t<\theta, \\ 0 & \text{if } t\geq\theta. \end{cases}$ Let $\mathcal{F}(t)$ be the smallest σ-algebra containing $\sigma(Z(s),Y(s),I(s);s\leq t)$ and all P-null sets of $\mathcal{F}$, $t\in T$. Let us define the reward process $X=(X(t),\mathcal{F}(t),t\in T)$ by $X(t)=I(t)\{(Z(t))^2-(Y(t))^2\}$. We denote $N=\{0,1,\ldots\}$ and define $K(z,y)=z^2-y^2$, $z,y\in N$. Let

$R_0(z,y)=\begin{cases} K(z,y) & \text{if } z>y,\ z,y\in N, \\ 0 & \text{if } z\leq y,\ z,y\in N, \end{cases}$ and $R_n(z,y)=\sup_{h\geq 0} \Phi_n(z,y;h)$, with

$$\Phi_n(z,y;h)=e^{-(\mu+2)h}\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} R_{n-1}(z+j,y+i)\frac{h^j}{j!}\frac{h^i}{i!}\right\},\quad n=1,2,\ldots,\quad z,y\in N.$$

Then it holds that $\xi(t;n)=I(t)\cdot R_n(Z(t),Y(t))$, $t\in T$.

Suppose especially that $\mu\geq 4$, then we have the followings:

(i) $0 \leq R_n(z,y+1) \leq R_n(z,y) \leq R_n(z+1,y)$, $\forall z,y\in N$, $n=0,1,\ldots$.

(ii) $R_n(z,y) = K(z,y)$, $\forall z,y\in N$: $z>y$, $n=0,1,\ldots$.

(iii) $\xi(t) = \begin{cases} I(t)K(Z(t),Y(t)) & \text{if } Z(t)>Y(t), \\ I(t) \lim_{n\to\infty} R_n(Z(t),Y(t)) & \text{if } Z(t)\leq Y(t). \end{cases}$

(iv) $\eta = \xi = (\xi(t),\mathcal{F}(t),t\in T)$ is the MRS lying above X.

(v) $$\tau^*=\inf\{t:\xi(t)\leq X(t)\}=\inf\{t:Z(t)>Y(t)\} \tag{3.1}$$

is optimal:

$$E[X(\tau^*)] = \sup\{E[X(\tau)]:\tau\in\mathcal{M}\}. \tag{3.2}$$

Indeed, by induction it is easy to see that (i) holds. In order to see that (ii) holds for any n, firstly we note that (ii) holds for n=0. Suppose that (ii) holds for $n=m\geq 0$, that is

$$R_m(z,y) = K(z,y), \quad \forall z,y\in N: z>y. \tag{3.3}$$

Under this assumption from the relation (i) it also holds that

$$0 \leq R_m(z,y) \leq K(y+1,y), \quad \forall z,y\in N: z\leq Y. \tag{3.4}$$

For some k=1,2,..., let $z\geq k$ and $y=z-k$, since (i), (3.3) and (3.4) hold, we have the followings:

$$\frac{\partial}{\partial h}\Phi_m(z,z-k;h) = -(\mu+2)\Phi_m(z,z-k;h)+\Phi_m(z+1,z-k;h)+\Phi_m(z,z-k;h)$$

$$\leq -(\mu+1)\Phi_m(z,z-k;h)+\Phi_m(z+1,z-k;h) \leq e^{-(\mu+2)h}[\sum_{i=0}^{\infty}\sum_{j=(i+1-k)\,0}^{\infty} \Psi(i,j,k;\mu,z)]$$

where (3.5)

$$\Psi(i,j,k;\mu,z)=-(\mu+1)K(z+j,z-k+i)+K(z+j+1,z-k+i)+K(z-k+j+1,z-k+j).$$

But $\Psi(i,j,k;\mu,z)\leq 0$ for $\mu\geq 4$, k=1,2,..., $j\geq i+1-k$. Hence, the left hand side of (3.5) is non-positive for any $h\geq 0$, $z\in N$, k=1,2,..., and we have $R_{m+1}(z,z-k)=\Phi_m(z,z-k;0)=K(z,z-k)$, $z\in N$, k=1,2,... . Thus, by induction we see that (ii) holds.

(iii) is a direct consequence of (i) and (ii). Now, we see the validities of (iv) and (v). Since $\xi\geq 0$ and (3.3) and (3.4) hold and ξ is a right continuous process (see (iii)), from Corollary 1-(i) it follows that ξ is the MRS lying above X. If we define τ^* by (3.1), then after similar discussion as was done for the stopping time τ in Example 1 (see also Example (1) of Irle [2]) we see that τ^* satisfies (1.2). And hence, from Proposition 1, we see that (3.2) holds.

We note that in this example the assumption of Corollary 1-(ii) is satisfied with $n^*=0$ if $\mu\geq 4$, but the reward process X does not form a weakly monotone process (see Corollary 2), and hence Theorem in Section 3 of Irle [2] can not be applied to this example.

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INSTITUTE OF INFORMATION SCIENCES AND ELECTRONICS
UNIVERSITY OF TSUKUBA
TSUKUBA-SHI, IBARAKI 305 JAPAN

REPORT DOCUMENTATION PAGE	REPORT NUMBER ISE-TR-88-76

TITLE

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AUTHOR(S)

Minoru Yoshida

REPORT DATE	NUMBER OF PAGES
August 23, 1988	10
MAIN CATEGORY	**CR CATEGORIES**
Mathematics of Computing	G.3

KEY WORDS

Optimal stopping problems , Minimal supermartingale

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SUPPLEMENTARY NOTES