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POLYNOMIAL APPROACH TO H_∞ OPTIMAL CONTROL PROBLEM FOR A SINGLE-INPUT SINGLE-OUTPUT DISCRETE TIME SYSTEM

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ABSTRACT: The H_∞ optimal control problem is solved for a single-input single-output discrete time system by applying an LQG optimal control solution. A polynomial equation is derived in a straightforward way by this method. It is shown that the equation strictly corresponds to that obtained for the continuous time case by Kwakernaak and that the solution exists.

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Keywords: H_∞ optimization, LQG optimal control, polynomial equation

I. Introduction

Sensitivity, stability, and disturbance attenuation are important properties which should be considered in any feedback control design. These are called feedback properties and can be quantified by the H_∞ norm of a sensitivity function S and/or a complemetary sensitivity function T [1], [2], [3]. It is desired to establish a control theory about the minimization of H_∞-norm type cost function of S and T.

Since Zames pointed out the importance of H_∞ norm and considered the minimization of $\max_\omega |V(j\omega)S(j\omega)|$ for a single-input single-output continuous time system [4], there have been many studies reported [3], [5]-[9]. In these studies a few different approaches are reported corresponding to the mathematical tools; interpolation theory, functional analysis, or a polynomial approach. In this paper, the polynomial approach is examined for a single-input single-output discrete time system.

Polynomial approach is first taken by Kwakernaak and he solved the minimization problem of $J_\infty=\max_\omega \{Q\}$ where $Q = |V(j\omega)S(j\omega)|^2 + |W(j\omega)T(j\omega)|^2$ for a single-input single-output continuous time system [3]. The optimum controller is given as the controller which minimizes λ in the class of equalizing controllers which satisfy $Q(s)=\lambda^2$ with the stability condition of the closed loop system. The solution is reduced to the solution of a polynomial equation, and the existence of the solution was shown. Grimble solved the problem by embedding the H_∞ optimization problem in the framework of LQG optimal control problem. The solution also reduced to the solution of another polynomial equation but the existence of the solution was not shown [8]. The relation between both results is not clear. A multi-input multi-output case is reported in [11].

In this paper, the optimization problem is solved for the discrete time case of [3] following the idea of [8], and it is shown that the polynomial equation corresponding to that of [3] can be derived straightforwardly from the equation obtained for the LQG optimal control problem. The existence of the solution is also shown.

In Section II, the H_∞ optimal control problem is derived. In Section III, a ficticious LQG optimal control problem corresponding to the H_∞ optimal control problem is introduced, and then the H_∞ optimal control problem is solved by applying the LQG optimal control solution. In Section IV, the relation between the obtained equation and that of [3] is examined. In Section V, the existence of the solution for the equation is proven.

Notation

$R^{n\times m}$ space of nxm real matrices

$P(z^{-1})$ space of polynomials of z^{-1} with real coefficients

deg(A) degree of $A \in P(z^{-1})$; the largest integer i for which $a_i \neq 0$ where $A(z^{-1})=a_0+a_1z^{-1}+ \ldots + a_nz^{-}$

* adjoint of a rational function of z^{-1};$S^*(z^{-1})=S(z)$

+,0,- superscripts +, 0, - denote that the zeros of the polynomial are inside, on, or outside the unit circle, respectively.

strictly Hurwitz $A \in P(z^{-1})$ is called strictly Hurwitz, if all the zeros of A are inside the unit circle $|z|=1$.

Hurwitz $A \in P(z^{-1})$ is called Hurwitz, if all the zeros of A are inside or on the unit circle.

II. H_∞ optimal control problem

Consider the single-input single-output discrete time system shown in Fig. 1. The transfer functions of the plant and the controller are given by

$$G(z^{-1}) = \frac{B(z^{-1})}{A(z^{-1})} \quad (1)$$

$$H(z^{-1}) = \frac{B_h(z^{-1})}{A_h(z^{-1})} \quad (2)$$

respectively. $A, B, A_h, B_h \in P(z^{-1})$. The plant is assumed to be free of unstable hidden modes. The sensitivity function S and the complementary sensitivity function T are given by

$$S = 1/(1+GH) \quad \text{and} \quad T = GH/(1+GH) \quad (3)$$

respectively. Consider next the H_∞-norm type cost function:

$$J_\infty = \sup_{|z|=1} \{V^* VS^* S + W^* WT^* T\} \quad (4)$$

The weights V and W are

$$V = \frac{B_v(z^{-1})}{A_v(z^{-1})} \quad \text{and} \quad W = \frac{B_w(z^{-1})}{A_w(z^{-1})} \quad (5)$$

where $A_v, B_v, A_w, B_w \in P(z^{-1})$, and A_v and A_w are Hurwitz and have simple poles on the unit circle. This cost function corresponds to that of the continuous time case of [3]. The H_∞ optimal control problem is to obtain a controller which minimizes J_∞ under the constraint that the closed loop system is stable.
Note: A dual performance index of [8] can be represented in the form of

(4) where $V^{*}V = [(A_w A_d B)^{*}(A_w A_d B)]^{-1}L_1$ and $W^{*}W = [(A_w A_d B)^{*}(A_w A_d B)]^{-1}L_2$. Therefore if B has no zeros on the unit circle, i.e. $B = B^{+}B^{-}$, then A_v and A_w of (5) can be chosen strictly Hurwitz, i.e. $A_w = A_v = A_w A_d B^{+}(B^{-})^{*}z^{-n}$ and $B_v^{*}B_v = L_1$ and $B_w^{*}B_w = L_2$. Thus, the results of this paper such as the existence of a solution, which is not proved in [8], can be applied to the problem of [8], if B has no zeros on the unit circle.

III. Solution of H_∞ optimal control problem

In the following, the argument z^{-1} is often omitted for simplicity. The next lemma links the H_∞ optimal control problem and LQG optimal control problem [3],[8].

Lemma 1 [8]

Consider the auxiliary problem of minimizing

$$J = (1/2\pi j) \oint_{|z|=1} Q(z^{-1})\Sigma(z^{-1})z^{-1}\, dz \qquad (6)$$

Suppose that for some real rational $\Sigma(z^{-1}) = \Sigma^{*}(z^{-1}) > 0$ on $|z|=1$ is minimized by a function $Q(z^{-1})=Q^{*}(z^{-1})$ for which $Q(z^{-1})=\lambda^2$ where λ is a real constant. Then this function also minimizes $\sup_{|z|=1} Q(z^{-1})$. ■

Then the next ficticious problem can be introduced. The system is given in Fig. 2 where ξ is a white noise with zero mean and unity spectrum. $A_\sigma \in P(z^{-1})$ and $B_\sigma \in P(z^{-1})$ are assumed to be strictly Hurwitz. The cost function is given by

$$J = (1/2\pi j) \oint_{|z|=1} \{ V^* V \Phi_{ee} + W^* W \Phi_{yy} \} z^{-1} \, dz \tag{7}$$

where V and W are defined by (5) and Φ_{ee} and Φ_{yy} are the spectra of e and y, respectively. Φ_{ee} and Φ_{yy} are represented as

$$\Phi_{ee} = S^* S \Sigma \Phi_{ss} \text{ and } \Phi_{yy} = T^* T \Sigma \Phi_{ss} \tag{8}$$

where $\Sigma \triangleq B_{\sigma}{}^* B_{\sigma} (A_{\sigma}{}^* A_{\sigma})^{-1}$ and $\Phi_{ss} = 1$ by assumption. By substituting (8) into (7) obtain

$$J = (1/2\pi j) \oint_{|z|=1} \{ V^* V S^* S + W^* W T^* T \} \Sigma z^{-1} \, dz \tag{9}$$

From (4), identify $Q = V^* V S^* S + W^* W T^* T$ so that (9) strictly corresponds to (6).

The controller which minimizes J of (9) subject to the constraint that the closed loop system is stable is found by solving the LQG optimal control problem. The solution is given by the next theorem.

Theorem 1

Introduce factorizations:

$$A = A^+ A^0 A^- \quad \text{and} \quad B = B^+ B^0 B^- \tag{10}$$

The spectral factor $D \in P(z^{-1})$ is defined by

$$D^* D = A_v{}^* A_v B_w{}^* B_w + A_w{}^* A_w B_v{}^* B_v \tag{11}$$

where D is assumed to be strictly Hurwitz. Introduce the polynomial equations with respect to $X_1, Y_1, Z_1 \in P(z^{-1})$:

$$D^* B^- z^{-n_2} X_1 + Z_1 A_v A_\sigma = B_v{}^* B_v A^{-*} B^{-*} A_w{}^* B_\sigma z^{-n_3} \qquad (12)$$

$$D^* A^- z^{-n_2} Y_1 - Z_1 A_w A_\sigma = B_w{}^* B_w A^{-*} B^{-*} A_v{}^* B_\sigma z^{-n_3} \qquad (13)$$

If $n_2 = \deg(D)$ and $n_3 = \deg(DA^- B^-)$, equations (12) and (13) can be solved for the unique minimal degree solution with respect to Z_1 (a solution Z_1 which satisfies $\deg(Z_1) < n_3$ always exists and is the unique minimal degree solution). Hence, the optimal controller which minimizes (9) is given by

$$H = (B^\varnothing B^+ A_v Y_1)^{-1} (A^\varnothing A^+ A_w X_1) \qquad (14)$$

and the minimum value of the cost function is given by

$$J_{min} = (1/2\pi j) \oint_{|z|=1} \left\{ \frac{Z_1{}^* Z_1}{D^* DA^{-*} A^- B^{-*} B^-} + \frac{B_w{}^* B_w \; B_v{}^* B_v}{D^* D} \Sigma \right\} z^{-1} \, dz \qquad (15)$$

■

Proof) Appendix A.

The proof of Theorem 1 is similar to that of Theorem 3.1 of [8]. The different points are that the factorizations of A and B in the form of (10) are introduced for the exact proof, and that attention is given to reducing the degrees of the coefficient polynomials and the unknown polynomials in the derivation of (12) and (13). This simplification is necessary to show the relation between this theorem and the Kwakernaak's condition in Section IV.

Application of Lemma 1 and Theorem 1 gives the next H_∞ optimal control solution.

Theorem 2

Let D, n_2, and n_3 be defined as in Theorem 1. Assume that $D^*D\lambda^2 - B_w^*B_wB_v^*B_v$ is positive on $|z|=1$, and that there exists a solution (X_4, Y_4, Z_1^-, $A_{\sigma 1}$, λ) which satisfies

$$D^*DB^-z^{-n_2}X_4 + Z_1^-A_vA_{\sigma 1} = B_v^*B_vA_w^*Z_{1s}^-z^{-n_2} \tag{16}$$

$$D^*DA^-z^{-n_2}Y_4 - Z_1^-A_wA_{\sigma 1} = B_w^*B_wA_v^*Z_{1s}^-z^{-n_2} \tag{17}$$

$$A_{\sigma 1}^*A_{\sigma 1} = D^*D\lambda^2 - B_w^*B_wB_v^*B_v \tag{18}$$

where $\deg(Z_1^-) < n_3$, Z_1^- has all its zeros outside the unit circle or constant, $A_{\sigma 1}$ is strictly Hurwitz, $Z_{1s}^- = (Z_1^-)^*z^{-n_4}$ and $n_4=\deg(Z_1^-)$. Then the optimal controller which minimizes (4) is given by

$$H = (B^@B^+A_vY_4)^{-1}(A^@A^+A_wX_4) \tag{19}$$

and the minimum value of J_∞ is λ^2. ■

Proof) Appendix B.

IV. Relation to Kwakernaak's equation

Kwakernaak considered the class of controllers which satisfy $Q = \lambda^2$ and obtained the controller which minimizes λ with condition that the closed loop system is stable. Optimality was proved by applying the continuous time case of Lemma 1 [3]. Grimble considered the class of controllers which are the solution of LQG optimal control problem and obtained the controller which satisfies $Q = \lambda^2$ depending upon Lemma1. Theorem 2 was derived following Grimble's idea for the discrete time case of Kwakernaak's problem. The relation between Theorem 2 and the result of [3] is clarified by the next Lemma.

Lemma 2

Equation (16) and (17) is equivalent to

$$A_w B^- X_4 + A_v A^- Y_4 - Z_{1s}^- = 0 \tag{20}$$

$$B_w^* B_w A_v^* B^- z^{-n_2} X_4 - B_v^* B_v A_w^* A^- z^{-n_2} Y_4 + Z_1^- A_{\sigma 1} = 0 \tag{21}$$

■

Proof) Appendix C.

From $Z_{1s}^- = (Z_1^-)^* z^{-n_4}$, (21) is represented as

$$B_w^* B_w A_v^* B^- z^{-n_2} X_4 - B_v^* B_v A_w^* A^- z^{-n_2} Y_4 = -(Z_{1s}^-)^* z^{-n_4} A_{\sigma 1} \tag{22}$$

By setting

$$B_w = \alpha_2,\ A_v = \beta_1,\ B^- = \psi_+,\ X_4 = \xi,\ B_v = \alpha_1,\ A_w = \beta_2$$

$$A^- = \phi_+,\ Y_4 = \theta,\ Z_{1s}^- = \chi,\ A_{\sigma 1} = \lambda \pi_\lambda,\ Z_1^- = \chi^* \tag{23}$$

the equations (3.8), (3.15), and (3.16) of [3] are obtained from (18), (20), and (22), and vice versa. Thus the exact correspondence between these polynomial equations are shown.

The above results can be summarized as the next theorem, which is the alternative representation of Theorem 2.

Theorem 3

Let D, n_2, and n_3 be defined as in Theorem 1. Assume that $D^* D \lambda^2 - B_w^* B_w B_v^* B_v$ is positive on $|z|=1$, and that there exists a solution $(X_4, Y_4, Z_1^-, A_{\sigma}, \lambda)$ which satisfies

$$A_w B^- X_4 + A_v A^- Y_4 - Z_{1s}^- = 0 \tag{24}$$

$$B_w^* B_w A_v^* B^- z^{-n_2} X_4 - B_v^* B_v A_w^* A^- z^{-n_2} Y_4 + Z_1^- A_{\sigma 1} = 0 \tag{25}$$

$$A_{\sigma 1}^* A_{\sigma 1} = D^* D \lambda^2 - B_w^* B_w B_v^* B_v \tag{26}$$

where $\deg(Z_1^-) < n_3$, Z_1^- has all its zeros outside the unit circle or constant, A_{σ} is strictly Hurwitz, $Z_{1s}^- = (Z_1^-)^* z^{-n_4}$ and $n_4 \triangleq \deg(Z_1^-)$. Then the optimal controller which minimizes (4) is given by

$$H = (B^{0} B^+ A_v Y_4)^{-1} (A^{0} A^+ A_w X_4) \tag{27}$$

and the minimum value of J_∞ is λ^2. ■

V. Existence of a solution of the polynomial equation in Theorem 3

Consider first, the permitted range of λ. Since D is assumed to be strictly Hurwitz, there exists a value of λ which satisfies $D^* D \lambda^2 - B_w^* B_w B_v^* B_v \geq 0$ for all z on $|z|=1$. Denote the minimum value of $|\lambda|$ as λ_ℓ, i.e.

$$\lambda_\ell = \sqrt{\max_{\substack{0 \leqq \theta \leqq 2\pi \\ z = e^{j\theta}}} B_w^* B_w B_v^* B_v / (D^* D)}$$

$$= \sqrt{\max_{\substack{0 \leqq \theta \leqq \pi \\ z = e^{j\theta}}} V^* V W^* W / (V^* V + W^* W)} \qquad (28)$$

This shows that λ_ℓ is determined by the weights V and W and is not affected by the system parameters A and B.

For $|\lambda| > \lambda_\ell$ $A_{\sigma 1}$ can always be well-defined as the strictly Hurwitz factor and Theorem 3 can be applied. When the optimal value of λ coincides with λ_ℓ , $A_{\sigma 1}$ cannot be strictly Hurwitz and Theorem 3 cannot be applied in this special case.

The existence of a solution for the polynomial equations in Theorem 3 is examined next. Solving (24) and (25) for $B^- x_4$ and $A^- Y_4$ yields:

$$\begin{pmatrix} B^- X_4 \\ \\ A^- Y_4 \end{pmatrix} = \frac{1}{D^* D z^{-n_2}} \begin{pmatrix} B_v^* B_v A_w^* z^{-n_2} & A_v \\ \\ B_w^* B_w A_v^* z^{-n_2} & -A_w \end{pmatrix} \begin{pmatrix} Z_{1s}^- \\ \\ -Z_1^- A_{\sigma 1} \end{pmatrix} \qquad (29)$$

From this equality, the degrees of X_4 and Y_4 need to satisfy

$$\deg(X_4) \leqq \deg(Z_1^-) - \deg(B^-) - \min(\deg(A_w), \deg(B_v)) \qquad (30)$$

$$\deg(Y_4) \leqq \deg(Z_1^-) - \deg(A^-) - \min(\deg(B_w), \deg(A_v)) \qquad (31)$$

Since $\deg(Z_1^-) < n_3$ from Theorem 3, the degree of Z_1^- is bounded by $\deg(Z_1^-) \leqq \deg(DA^-B^-)-1$. Thus the degrees of X_4, Y_4, and Z_1^- satisfy

$$\deg(X_4) \leqq n_x \triangleq \deg(A^-) + \max(\deg(A_v), \deg(B_v)) - 1 \qquad (32)$$

$$\deg(Y_4) \leqq n_y \triangleq \deg(B^-) + \max\ (\deg(A_w),\ \deg(B_w)) - 1 \tag{33}$$

$$\deg(Z_1^-) \leqq n_z \triangleq \deg(A^-)+\deg(B^-)+ \max\ (\deg(A_v B_w),\ \deg(A_w B_v))-1 \tag{34}$$

The next theorem corresponds to Theorem 1 of [3].

Theorem 4

a) For $|\lambda| \geqq \lambda_\ell$ the polynomial equations (24), (25), and (26) have a family of solutions X_4, Y_4, Z_1^-, $A_{\sigma 1}$, with $\deg(X_4) \leqq n_x+1$, $\deg(Y_4) \leqq n_y+1$, and $\deg(Z_1^-) = n_z+1$, unique within multiplication by a continuous function of λ, whose coefficients are continuous functions of λ, such that $|\lambda|$ sufficiently large the polynomial Z_{1s}^- has degree n_z+1 and has all its zeros inside the unit circle.

b) Let $Z_{1s+\lambda\ell}$ and $Z_{1s-\lambda\ell}$ be the solutions Z_{1s}^- of (24), (25), and (26) with $\deg(Z_{1s}^-) = n_z+1$ for $\lambda = \lambda_\ell$ and $\lambda = -\lambda_\ell$, respectively. Suppose that $Z_{1s+\lambda\ell}$ and $Z_{1s-\lambda\ell}$ have at least one zero outside the unit circle. Then there exists a λ with $|\lambda| > \lambda_\ell$ such that $\deg(X_4) \leqq n_x$, $\deg(Y_4) \leqq n_y$, $\deg(Z_1^-) \leqq n_z$ and Z_{1s}^- has all its zeros inside the unit circle.

■

Proof) Appendix D.

Note that the coprimeness of A_v and A_w is assumed in [3] and this assumption is not required. Theorem 4 can be proved without this assumption by slightly changing the proof of Lemma A.1 of [3]. Also note that there is no restrictions on the degrees of A_v, A_w, B_v, B_w, A, and B whereas there are some restrictions on these degrees in [3]. As a result the degrees of the unknown polynomials X_4, Y_4, and Z_1^- are given in a more general form as (32), (33), and (34) than those of [3].

The proof of Theorem 4a is exactly the same as the last half of the proof of Theorem 1a of [3]. The first half of the proof of Theorem 1a of [3] is not necessary, because the factorized form (25) has already been obtained.

Theorem 4b assures that if $Z_{1s+\lambda\ell}$ or $Z_{1s-\lambda\ell}$ have at least one zero outside the unit circle, there exists a solution which satisfies the condition of Theorem 3. This means that a controller can be found which gives $Q = \lambda^2$ in the class of LQG optimal controller where A_σ and B_σ are parameters, so the H_∞ optimal control solution is also the LQG optimal control solution for appropriately chosen A_σ and B_σ.

Consider the case that $Z_{1s+\lambda\ell}$ and $Z_{1s-\lambda\ell}$ have all their zeros inside the closed unit circle. Since the zeros of $Z_{1s+\lambda\ell}$ on the unit circle are cancelled in the equations (24) and (25) as shown in the proof of Theorem 4b, it can be assumed that the solution Z_{1s}^- of (24) (25), and (26) is strictly Hurwitz and $\deg(Z_{1s}^-) \leqq n_z+1$. The minimum value λ_ℓ can be attained by this solution, because the solution satisfies $Q = \lambda^2$ and the poles of the closed loop system are the zeros of Z_{1s}^-. This result corresponds to Theorem 2 of [3].

The above results can be summarized as follows.

Theorem 5

a) If $Z_{1s+\lambda\ell}$ or $Z_{1s-\lambda\ell}$ have at least one zero outside the unit circle, the H_∞ optimal controller exists and is given by Theorem 3.

b) If $Z_{1s+\lambda\ell}$ and $Z_{1s-\lambda\ell}$ have all their zeros inside the closed unit circle, the minimum value of $|\lambda|$ is λ_ℓ and the H_∞ optimal controller is given by $H = (B^\theta B^+ A_v Y_4)^{-1}(A^\theta A^+ A_w X_4)$ where X_4 and Y_4 are the solutions of (24), (25), (26) with $\lambda = \lambda_\ell$ and $\deg(Z_1^-) \leqq n_z+1$.

VIII. Conclusion

Firstly, the LQG optimal control problem is solved where much attention is paid to decreasing the degrees of the coefficient polynomials contained in the polynomial equations and to the factorizations of A and B (Theorem 1). Then, the H_∞ optimal control problem is solved by using the LQG optimal control solution with the same attention to degree (Theorem 2). Secondly, it is shown that the solution of the obtained polynomial equation agrees with that of Kwakernaak's (Theorem 3). Whereas Kwakernaak determined the form of the controller by several insights to transfer functions, in this paper the form can be determined straightforwardly by the LQG optimal control solution. Lastly, the existence of a solution of the equation is proved under a less restrictive condition than that of Kwakernaak about the coprimeness of the polynomial denominators of the weights in the cost function and the degrees of the coefficient polynomials (Theorem 4 and Theorem 5).

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Appendix A (Proof of Theorem 1)

Define F as $F = H(1+HG)^{-1}$, then $S = 1-GF$ and $T = GF$ from (3). J defined by (7) is expanded with respect to F:

$$J = (1/2\pi j) \oint_{|z|=1} \{(V^*V+W^*W)\Sigma G^*GF^*F - V^*V\Sigma G^*F^* - V^*V\Sigma GF + V^*V\Sigma\} z^{-1}\, dz$$

By setting

$$M^*M \triangleq (V^*V+W^*W)G^*G \tag{A.1}$$

$$N^*N \triangleq \Sigma \tag{A.2}$$

$$U \triangleq V^*V\Sigma G^* \tag{A.3}$$

$$J = (1/2\pi j) \oint_{|z|=1} \{(MFN - \frac{U}{M^*N^*})(MFN - \frac{U}{M^*N^*})^* + V^*V\Sigma - \frac{U^*U}{M^*MN^*N}\} z^{-1}\, dz$$

Define D_c and D_f as

$$D_c = DD_bB^{\emptyset} \quad \text{and} \quad D_f = D_aA^{\emptyset}B_{\sigma} \tag{A.4}$$

where strictly Hurwitz spectral factors D_a and D_b are defined by $D_a{}^*D_a = (A^+A^-)^*(A^+A^-)$ and $D_b{}^*D_b = (B^+B^-)^*(B^+B^-)$. Define

$$M = D_c/(A_vA_wA) \quad \text{and} \quad N = D_f/(AA_{\sigma}) \tag{A.5}$$

then

$$\frac{U}{M^*N^*} = \frac{B_v{}^*B_vD_f{}^*D_fB^*A_w{}^*}{D_c{}^*D_f{}^*A_vA_{\sigma}A} \tag{A.6}$$

The righthand side of (A.6) can be expanded by use of the polynomial equation:

$$D_{fc}X + ZAA_vA_{\sigma} = B_v^* B_v D_f^* D_f B^* A_w^* z^{-n_0} \qquad (A.7)$$

where $n_0 = \deg(D_cD_f)$ and $D_{fc} = D_f^* D_c^* z^{-n_0}$. Then,

$$\frac{U}{M^* N^*} = \frac{X}{AA_vA_{\sigma}} + \frac{Z}{D_{fc}} \qquad (A.8)$$

The term MFN is expanded as

$$MFN = \frac{D_cD_fB_h}{AA_vA_{\sigma}A_w(A_hA+B_hB)} \qquad (A.9)$$

From (A.8) and (A.9),

$$MFN - \frac{U}{M^* N^*} = \frac{(D_cD_f-XBA_w)B_h-XA_wAA_h}{AA_vA_{\sigma}A_w(A_hA+B_hB)} - \frac{Z}{D_{fc}} \qquad (A.10)$$

Further introduce the polynomial equation:

$$D_{fc}Y - ZBA_wA_{\sigma} = B^* BA_v^* B_w^* B_wA^* B_{\sigma}^* B_{\sigma} z^{-n_0} \qquad (A.11)$$

Then (A.10) becomes

$$MFN - \frac{U}{M^* N^*} = \frac{YA_vB_h-XA_wA_h}{A_vA_{\sigma}A_w(A_hA+B_hB)} - \frac{Z}{D_{fc}} \qquad (A.12)$$

From (A.7) or (A.11), Z has the form : $Z = Z_1(A^0B^0A^+B^+B_{\sigma})^* z^{-n_1}$ and $n_1 \triangleq \deg(A^0B^0A^+B^+B_{\sigma})$. Equations (A.7) and (A.11) become

$$D^* B^- z^{-n_2} X + Z_1A^0A^+A_vA_{\sigma} = B_v^* B_vA^+A^{-*} A^0B^{-*} A_w^* B_{\sigma} z^{-n_3} \qquad (A.13)$$

$$D^* A^- z^{-n_2} Y - Z_1B^0B^+A_wA_{\sigma} = B_w^* B_wB^+B^0A^{-*} B^{-*} A_v^* B_{\sigma} z^{-n_3} \qquad (A.14)$$

where $n_2 \triangleq \deg(D)$ and $n_3 \triangleq \deg(DA^- B^-)$. From (A.13) and (A.14), X and Y have the form:

$$X = A^{\emptyset} A^{+} X_1 \qquad \text{and} \qquad Y = B^{\emptyset} B^{+} Y_1 \tag{A.15}$$

Then (A.13) and (A.14) can be simplified as

$$D^{*} B^{-} z^{-n_2} X_1 + Z_1 A_v A_{\sigma} = B_v{}^{*} B_v A^{-*} B^{-*} A_w{}^{*} B_{\sigma} z^{-n_3} \tag{A.16}$$

$$D^{*} A^{-} z^{-n_2} Y_1 - Z_1 A_w A_{\sigma} = B_w{}^{*} B_w A^{-*} B^{-*} A_v{}^{*} B_{\sigma} z^{-n_3} \tag{A.17}$$

By using (A.15), each term of (A.12) becomes

$$T_1(z) \triangleq \frac{Y A_v B_h - X A_w A_h}{A_v A_{\sigma} A_w (A_h A + B_h B)}$$

$$= \frac{Y_1}{A_w A_{\sigma}} \left[\frac{B_h B^{\emptyset} B^{+}}{A_h A + B_h B} \right] - \frac{X_1}{A_v A_{\sigma}} \left[\frac{A_h A^{\emptyset} A^{+}}{A_h A + B_h B} \right]$$

$$T_2(z) \triangleq \frac{Z}{D_{fc}} = \frac{-Z_1 (A^{+} B^{+})^{*} z^{-(n_1 - n_0)}}{D^{*} D_b{}^{*} D_a{}^{*}} \tag{A.18}$$

The rational function $T_1(z)$ has all its poles inside or on the unit circle and the poles on the unit circle are simple, because A_{σ} is strictly Hurwitz, A_v and A_w are Hurwitz, their poles on the unit circle are simple, and the assumption of the asymptotic stability of the closed loop system guarantees the asymptotic stability of the rational functions inside the parenthesis [ ]. The rational function $T_2(z)$ has clearly all their poles outside the unit circle. J can be written as

$$J = (1/2\pi j) \oint_{|z|=1} \{ T_1{}^{*} T_1 + T_2{}^{*} T_2 - T_1 T_2{}^{*} - T_1{}^{*} T_2$$

$$+ V^{*} V \Sigma - \frac{U^{*} U}{M^{*} M N^{*} N} \} z^{-1}\, dz \tag{A.19}$$

From the above consideration $T_1^* T_2$ is regular inside the unit circle, so

$$(1/2\pi j) \oint_{|z|=1} \{T_1^* T_2\} z^{-1}\, dz = T_1^*(0) T_2(0) + \alpha \quad \text{and}$$

$$(1/2\pi j) \oint_{|z|=1} \{T_1 T_2^*\} z^{-1}\, dz = T_1^*(0) T_2(0) - \alpha$$ where α is the sum of the residues at the poles on the unit circle. Let F be the minimal degree solution of (A.7) and (A.11), then $\deg(Z) < \deg(D_{fc})$ and $T_1(0)=0$. Since $T_2(z)$ is regular inside the unit circle, $T_1^*(0)T_2(0)=0$. Thus J has the minimum value for $T_1=0$ from (A.19), which means that the controller is given by

$$H = A_h^{-1} B_h = (YA_v)^{-1} XA_w$$

$$= (B^0 B^+ A_v Y_1)^{-1} (A^0 A^+ A_w X_1) \qquad \text{(A.20)}$$

and J becomes

$$J = (1/2\pi j) \oint_{|z|=1} \left\{ \frac{Z_1^* Z_1}{D^* DA^{-*} A^- B^{-*} B^-} + \frac{B_w^* B_w B_v^* B_v}{D^* D} \Sigma \right\} z^{-1}\, dz \qquad \text{(A.21)}$$

A solution (X_0, Y_0, Z_0) of (A.13) and (A.14) which satisfies $\deg(z_0) < \deg(DA^- B^-)$ always exists and is unique, and Z_0 is the least degree representative of Z_1 modulo $DA^- B^-$, i.e. Z_1 can be expressed as $Z_1 = Z_0 + DA^- B^- p(z^{-1})$ where $p(z^{-1})$ is an arbitrary polynomial. Since clearly $Z_1^* Z_1 \geqq Z_0^* Z_0$ on the unit circle, J of (A.21) takes the minimum value if Z_1 is the minimum degree solution Z_0. ■

Appendix B (Proof of Theorem 2)

By Lemma 1 the LQG optimal controller which satisfies $V^*VS^*S+W^*WT^*T=\lambda^2$ is the H_∞ optimal controller. This equality can be represented as

$$\frac{Z_1^* Z_1}{B^{-*}B^- A^{-*}A^-(D^* D\lambda^2 - B_w^* B_w B_v^* B_v)} = \frac{B_\sigma^* B_\sigma}{A_\sigma^* A_\sigma} \tag{B.1}$$

Note that this equation can be also obtained by setting the integrand of (15) equal to $\Sigma\lambda^2$ as in [8]. Since A_σ and B_σ need to be strictly Hurwitz and the denominator of the righthand side of (B.1) has no zeros on the unit circle by the positivity of $D^*D\lambda^2 - B_w^*B_wB_v^*B_v$ on the unit circle, the numerator Z_1 cannot contain any zeros on the unit circle and so Z_1 can be factorized as $Z_1=Z_1^+Z_1^-$. Then choose B_σ as

$$B_\sigma = Z_1^+ Z_{1s}^- \tag{B.2}$$

where $Z_{1s}^- \triangleq (Z_1^-)^* z^{-n_4}$ and $n_4 \triangleq \deg(Z_1^-)$. A_σ is the strictly Hurwitz factor defined by

$$A_\sigma^* A_\sigma = B^{-*}B^- A^{-*}A^-(D^* D\lambda^2 - B_w^* B_w B_v^* B_v) \tag{B.3}$$

Define $X_1=X_2Z_1^+$ and $Y_1=Y_2Z_1^+$, then (18) and (19) are simplified to

$$\begin{aligned} D^* B^- z^{-n_2} X_2 + Z_1^- A_v A_\sigma &= B_v^* B_v A^{-*} B^{-*} A_w^* Z_{1s}^- z^{-n_3} \\ D^* A^- z^{-n_2} Y_2 - Z_1^- A_w A_\sigma &= B_w^* B_w A^{-*} B^{-*} A_v^* Z_{1s}^- z^{-n_3} \end{aligned} \tag{B.4}$$

This equation has the same form as (12) and (13) by identifying X_2, Y_2, Z_1^-, Z_{1s}^- as X_1, Y_1, Z_1, B_σ. Therefore if (B.4) has a solution with $\deg(Z_1^-)<n_3$ and Z_{1s}^- is strictly Hurwitz, the controller

$H = (B^{@}B^{+}A_vY_2)^{-1}(A^{@}A^{+}A_wX_2)$ is optimal from Theorem 1.

By defining $X_2 = X_3A^{-*}B^{-*}z^{-n_5}$ and $Y_2 = Y_3A^{-*}B^{-*}z^{-n_5}$ where $n_5 \triangleq \deg(A^-B^-)$, (B.4) is further simplified to

$$D^*B^-z^{-n_2}X_3 + Z_1^-A_vA_{\sigma 1} = B_v^*B_vA_w^*Z_{1s}^-z^{-n_2} \tag{B.5}$$

$$D^*A^-z^{-n_2}Y_3 - Z_1^-A_wA_{\sigma 1} = B_w^*B_wA_v^*Z_{1s}^-z^{-n_2} \tag{B.6}$$

where $A_{\sigma 1}$ is the strictly Hurwitz spectral factor defined by

$$A_{\sigma 1}^*A_{\sigma 1} = D^*D\lambda^2 - B_w^*B_wB_v^*B_v \tag{B.7}$$

X_3 and Y_3 are shown to have a common factor D as follows. Evaluating (B.5) and (B.6) at a zero of D^*, which is denoted $z=z_c$, gives

$$\begin{aligned} Z_1^-A_vA_{\sigma 1} &= B_v^*B_vA_w^*Z_{1s}^-z^{-n_2} \quad \text{for } z=z_c \\ -Z_1^-A_wA_{\sigma 1} &= B_w^*B_wA_v^*Z_{1s}^-z^{-n_2} \quad \text{for } z=z_c \end{aligned} \tag{B.8}$$

It follows from (B.7) that $A_{\sigma 1}^*A_{\sigma 1} = -B_w^*B_wB_v^*B_v$ for $z=z_c^{-1}$, and

$$Z_1^{-*}A_v^*A_{\sigma 1}^* = Z_{1s}^-z^{n_6}A_v^*A_{\sigma 1}^* = B_v^*B_vA_wZ_1^-z^{n_2+n_6} \quad \text{for } z=z_c^{-1}$$

$$-Z_1^{-*}A_w^*A_{\sigma 1}^* = -Z_{1s}^-z^{n_6}A_w^*A_{\sigma 1}^* = B_w^*B_wA_vZ_1^-z^{n_2+n_6} \quad \text{for } z=z_c^{-1}$$

where $n_6 = \deg(Z_1^-)$. Substituting in (B.5) and (B.6) gives

$$D^*B^-z^{-n_2}X_3A_{\sigma 1}^* = B_v^*B_v(Z_1^-A_vB_w^*B_w + A_w^*Z_{1s}^-z^{-n_2}A_{\sigma 1}^*) = 0 \quad \text{for } z=z_c^{-1}$$

$$D^* A^- z^{-n_2} Y_3 A_{\sigma 1}^* = B_w^* B_w (-Z_1^- A_w B_v^* B_v + A_v^* Z_{1s}^- z^{-n_2} A_{\sigma 1}^*) = 0 \quad \text{for } z = z_c^{-1}$$

It follows that X_3 and Y_3 have a common factor D. Namely $X_3 = DX_4$ and $Y_3 = DY_4$ and equations (16) and (17) are obtained. ■

Appendix C (Proof of Lemma 2)

Equation (16) multiplied by A_w plus equation (17) multiplied by A_v gives

$$D^* D(A_w B^- X_4 + A_v A^- Y_4) = D^* D Z_{1s}^-$$

This equation can be simplified to (20). Equation (16) multiplied by $B_w^* B_w A_v^*$ minus equation (17) multiplied by $B_v^* B_v A_w^*$ gives

$$D^* D(B_w^* B_w A_v^* B^- z^{-n_2} X_4 - B_v^* B_v A_w^* A^- z^{-n_2} Y_4) = -D^* D\, Z_1^- A_{\sigma 1}$$

This equation can be simplified to (21). Equations (23) and (24) can also be obtained from (27) and (28). ■

Appendix D (Proof of Theorem 4)

First prove the next lemma which corresponds to Lemma A.1 of [3], but which does not assume the coprimeness of A_v and A_w.

Define new variables $X \triangleq X_4/\lambda$, $Y \triangleq Y_4/\lambda$, $Z \triangleq Z_{1s}^-$, and $A_\mu \triangleq A_{\sigma 1}/\lambda$, and rewrite (24), (25), and (26) as

$$A_w B^- X + A_v A^- Y - \mu Z = 0 \tag{D.1}$$

$$B_w{}^* B_w A_v{}^* B^- z^{-n_z} X - B_v{}^* B_v A_w{}^* A^- z^{-n_z} Y + Z^* z^{-n_z} A_\mu = 0 \tag{D.2}$$

$$A_\mu{}^* A_\mu = D^* D - \mu^2 B_w{}^* B_w B_v{}^* B_v \tag{D.3}$$

From the above equations,

$$B_v{}^* B_v A^{-*} A^- Y^* Y + B_w{}^* B_w B^{-*} B^- X^* X = Z^* Z \tag{D.4}$$

is obtained. Consider the existence of a solution of (D.1) and (D.4).

Lemma D.1

For $|\mu|$ sufficiently small, the polynomial equations (D.1) and (D.4) have a unique family of solutions X, Y, and Z with $\deg(X) = n_x+1$, $\deg(Y) = n_y+1$, and $\deg(Z) = n_z+1$ where n_x, n_y, n_z are defined by (32), (33), and (34), whose coefficients depend continuously on μ, such that Z is monic and has all its zeros inside the unit circle. ■

Proof Lemma D.1)

First show that (D.4) has a solution at $\mu = 0$ and then use the implicit function theorem to show its existence for $|\mu|$ sufficiently small. For $\mu = 0$, (D.1) reduces to $A_w B^- X + A_v A^- Y = 0$. Let the greatest common devisor of A_w and A_v be A_g, namely $A_w = A_{w1} A_g$ and $A_v = A_{v1} A_g$. Since $A_{w1} B^-$ and $A_{v1} A^-$ is coprime, the equation $A_{w1} B^- X + A_{v1} A^- Y = 0$ has the general solution $X = A_{v1} A^- a$ and $Y = -A_{w1} B^- a$, with a an arbitrary polynomial. Substituting (D.4) yields

$$Z_* Z = (B_{v*} B_v A_{w1*} A_{w1} + B_{w*} B_w A_{v1*} A_{v1}) A^-_* A^- B^-_* B^- a_* a \tag{D.5}$$

From (D.1) Z always contains a factor A_g, so $a_* a$ needs to include $A_{g*} A_g$ as a factor from (D.5) by continuity, i.e., $a_* a = A_{g*} A_g a_{1*} a_1$. Thus $Z_0 = a_1 D A^-_* B^-_* z^{-n_7}$ where $n_7 \triangleq \deg(A^- B^-)$, $X_0 = A_v A^- a_1$, $Y_0 = -A_w B^- a_1$ are obtained. It is shown that (D.1) and (D.4) have a solution such that Z has degree n_z+1, has all its zeros inside the unit circle and is monic where a_1 is a constant such that Z is monic. Now apply the implicit function theorem to the set of equations for the coefficients of the polynomials X, Y, and Z that results when (D.1) and (D.4) are expanded in powers of z^{-1} and the coefficients of like powers are equated. It is verified that (D.1) and (D.4) yield the same number of equations as unknown coefficients if we look for a solution X, Y, Z of degrees n_x+1, n_y+1, n_z+1, respectively with Z monic. The number of equations is given by $\max\{\deg(A_w B^-)+n_x+1, \deg(A_v A^-)+n_y+1, n_z+1\}$ + $\max\{\deg(B_v A^-)+n_y+1, \deg(B_w B^-)+n_x+1, n_z+1\}$, and this agrees with the number of unknowns, i.e. $n_x+n_y+n_z+2$. It follows from the implicit function theorem that these equations have a solution for $|\mu|$ sufficiently small, if the Jacobian of these equations is nonsingular at the solution that is obtained at $\mu = 0$. This Jacobian is nonsingular if and only if the homogeneous set of equations:

$$A_w B^- \overline{X} + A_v A^- \overline{Y} = 0 \tag{D.6}$$

$$B_{v*} B_v A^-_* A^- (Y_0 \overline{Y}_* + Y_{0*} \overline{Y}) + B_{w*} B_w B^-_* B^- (X_0 \overline{X}_* + X_{0*} \overline{X}) - (Z_0 \overline{Z}_* + Z_{0*} \overline{Z}) = 0 \tag{D.7}$$

with $\deg(\overline{Z})=n_z$, $\deg(\overline{X}) = n_x+1$, $\deg(\overline{Y}) = n_y+1$, has the unique solution $\overline{X} = 0$, $\overline{Y} = 0$, $\overline{Z} = 0$. (D.6) has a general solution $\overline{X} = A_{v1} A^- c$ and $\overline{Y} = -A_{w1} B^- c$ with c an arbitrary polynomial. Substituting these into

(D.7), we obtain after rearrangement

$$A_\sigma A^{-*}B^{-*}D_1(c^* A^- B^- D_1^* - \overline{Z}^*) + (c^* A^- B^- D_1^* - \overline{Z}^*)^* A_\sigma^* A^- B^- D_1^* = 0 \quad \text{(D.8)}$$

where $D_1^* D_1 = B_v^* B_v A_{w1}^* A_{w1} + B_w^* B_w A_{v1}^* A_{v1}$ and D_1 is strictly Hurwitz. Now if a is a polynomial that has no common roots with a^*, the polynomial equation $ax+a^*x^* = 0$ has the general solution $x = a^*w$ where w is any polynomial such that $w^* = -w$. As a result it follows from (D.8) that $cA^{-*}B^{-*}D_1 - \overline{Z} = A_\sigma A^{-*}B^{-*}D_1 w$ where w is any polynomial such that $w^* = -w$. From (D.1) $\overline{Z}$ always has a factor A_σ, so $\overline{Z}$ also has this factor. Therefore c can be expressed as $c = A_\sigma c_1$ and $\overline{X} = A_v A^- c_1$ and $\overline{Y} = A_w B^- c_1$. Since $\overline{Y}$, A_w, $B^- \in P(z^{-1})$ and A_w and B^- do not contain z^{-1} as a factor, $c_1 \in P(z^{-1})$. $\overline{Z}$ is represented as $\overline{Z} = (c_1-w)A^{-*}B^{-*}D$. Since $\deg(\overline{Z}) = n_z$, necessarily $c_1 - w = 0$. From $w^* = -w$ and $c_1 = w \in P(z^{-1})$, $c_1 = 0$ and $w = 0$, which proves that (D.6) and (D.7) have the unique solution $\overline{X} = 0$, $\overline{Y} = 0$, $\overline{Z} = 0$. This completes the proof.

■

Proof of Theorem 4a)

For a given μ, A_σ is given, and (D.1) and (D.2) form a set of homogeneous linear equations in the coefficients of the unknown polynomials X, Y, and Z. This linear set of equations is over-determined. Let us arrange the unkown coefficients of the polynomials X, Y, and Z in a vector x, then (D.1) and (D.2) can be put into the form $\Gamma x = 0$, where Γ is a rectangular matrix. This equation has a nontrivial solution if and only if $\Gamma^T \Gamma$ is singular. Since a nontrivial solution exists for μ small enough, it follows that $\det(\Gamma^T \Gamma)$ is an analytic function of μ for $|\mu| \leq 1/\lambda_0$, and hence (D.1) and (D.2) have a nontrivial solution with the properties stated in Theorem 4a. ■

Proof of Theorem 4b)

Suppose that $Z_{1s+\lambda\ell}$ has a zero outside the unit circle. Track the solution of (D.2) with deg(Z) = n_z+1 as μ is increased from 0 to $1/\lambda_0$. Since the coefficients if Z vary continuously with μ, for some $\mu < 1/\lambda_0$ at least one zero of Z crosses over to the outside of the unit circle. Consider what happens to the polynomials X and Y. Solving (D.1) and (D.2) for B^-X and A^-Y yields

$$\begin{bmatrix} B^-X \\ A^-Y \end{bmatrix} = \frac{1}{D^*Dz^{-n_2}} \begin{bmatrix} B_v \cdot B_v A_w \cdot z^{-n_2} & A_v \\ B_w \cdot B_w A_v \cdot z^{-n_2} & -A_w \end{bmatrix} \begin{bmatrix} \mu Z \\ -z^{-n_z} Z^* A_\mu \end{bmatrix} \qquad (D.9)$$

Suppose that a real zero of Z crosses over the unit circle. Then the zero must be 1 or -1 and Z has the factor $z^{-1}+1$ or $z^{-1}-1$. From the above equation, X and Y also include this factor and this factor can be cancelled in (D.1) and (D.2), leading to a solution of reduced degree. Suppose that a complex zero of Z, whose imaginary part is not zero, crosses over the unit circle. Then the zero has the form $e^{j\theta}$ and Z must have the complex conjugate zero $e^{-j\theta}$ at the same time. Therefore X and Y have a factor $(z^{-1}-e^{j\theta})(z^{-1}-e^{-j\theta})$ and this factor can be cancelled in (D.1) and (D.2), leading to a solution of reduced degree. ■

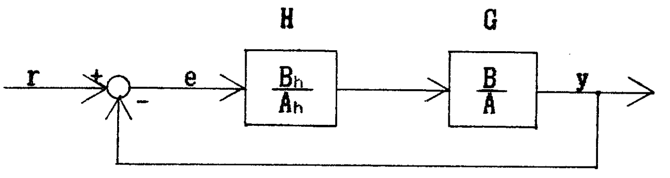

Fig. 1 Feedback system

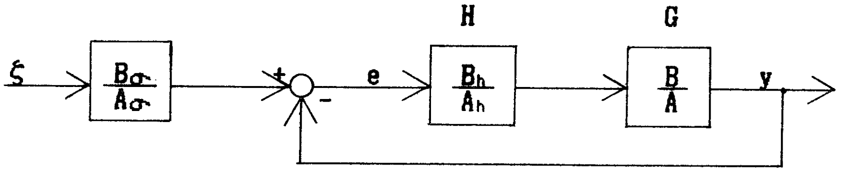

Fig. 2 Fictitious feedback system
for LQG optimal control design

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TITLE

POLYNOMIAL APPROACH TO H_∞ OPTIMAL CONTROL PROBLEM FOR A SINGLE-INPUT SINGLE-OUTPUT DISCRETE TIME SYSTEM

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ABSTRACT

The H_∞ optimal control problem is solved for a single-input single-output discrete time system by applying an LQG optimal control solution. A polynomial equation is derived in a straightforward way by this method. It is shown that the equation strictly corresponds to that obtained for the continuous time case by Kwakernaak and that the solution exists.

SUPPLEMENTARY NOTES