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# MONOTONICITY THEOREM, CAUCHY'S INTERLACE THEOREM AND COURANT-FISCHER THEOREM

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## §1 Introduction

In this note the following important theorems on eigenvalues of Hermitian matrices are reworked from a unified viewpoint of exploiting a simple dimensional identity for an easier, quicker and independent proof: Monotonicity Theorem, Caushy's Interlace Theorem and Courant-Fischer Theorem (Minimax Characterization). The usual procedure of invoking the minimax characterization or Sylvester's Law of Inertia to prove either one of the preceeding ones results in a much longer proof (see, for instance, [1, 186-192] or [2, 99-104]). Sylvester's Law of Inertia can be proved from the same viewpoint, as is well-known, and is demonstrated for the reader's convenience (see §5).

Our proofs depend on the following simple dimensional identity:

(1) $\dim(S_1 \cap S_2) = \dim S_1 + \dim S_2 - \dim(S_1 + S_2)$,

where $S_1$ and $S_2$ are subspaces of a finite-dimensional vector space. Thus, this note may also be viewed as a collection of good instances of application of the dimensional identity (1).

Before proceeding, we state the following basic facts used in the subsequent proofs without explicit reference: (a) the eigenvalues of a Hermitian matrix are real and the corresponding eigenvectors may be taken to be orthonormal; (b) letting $\alpha_1 \le \dots \le \alpha_k$ denote a subset of eigenvalues of a Hermitian matrix $A$ and letting $u_1, \dots, u_k$ denote an orthonormal set of corresponding eigenvectors, we have $\alpha_1 \le x^H A x \le \alpha_k$ for any $x$ in the span of $u_1, \dots, u_k$, where $x^H x = 1$. (The symbol "H" denotes conjugate transpose.)

## §2 The Monotonicity Theorem [1, p.191]

Let $A$ and $B$ be Hermitian and let $A+B=C$. Let the eigenvalues of $A$, $B$ and $C$ be $\alpha_1 \le \dots \le \alpha_n$, $\beta_1 \le \dots \le \beta_n$ and $\gamma_1 \le \dots \le \gamma_n$, respectively. Then

(1) $\alpha_j + \beta_{i-j+1} \le \gamma_i$, $\quad (i \ge j)$

(2) $\gamma_i \le \alpha_j + \beta_{i-j+n}$, $\quad (i \le j)$

(3) $\alpha_i + \beta_1 \le \gamma_i \le \alpha_i + \beta_n$, $\quad (i=j)$

Proof. Let

$$A u_i = \alpha_i u_i, \quad B v_i = \beta_i v_i, \quad C w_i = \gamma_i w_i,$$
$$u^H_i u_j = v^H_i v_j = w^H_i w_j = \delta_{ij}, \qquad i,j=1,\dots,n$$

Consider first the case $i \ge j$ and let

$S_1 = \mathrm{span}\{ u_j, \dots, u_n \}$, $\dim S_1 = n-j+1$

$S_2 = \mathrm{span}\{ v_{i-j+1}, \dots, v_n \}$, $\dim S_2 = n-i+j$

$S_3 = \mathrm{span}\{ w_1, \dots, w_i \}$, $\dim S_3 = i$.

Then §1 (1) gives

$$\dim(S_1 \cap S_2 \cap S_3) \ge \dim S_1 + \dim S_2 + \dim S_3 - 2n = 1.$$

This assures the existence of an $x \in S_1 \cap S_2 \cap S_3$ such that $x^H x = 1$. For this $x$,

$$\alpha_j + \beta_{i-j+1} \le x^H A x + x^H B x = x^H C x \le \gamma_i,$$

proving (1). Application of (1) to $(-A)+(-B)=-C$ proves (2). Setting i=j in (1) and (2) gives (3).

Remark: Writing $\gamma_i = \alpha_i + m_i$ and taking trace of $A+B=C$, we find $\Sigma m_i = \mathrm{trace} B$. If, in particular, $B = \tau c c^H$, where $c^H c = 1$, we have $\Sigma m_i = \tau$.

§3 The Cauchy's Interlace Theorem [1, p.186]

Let

$$A=\begin{bmatrix} B & C \\ & \\ C^H & D \end{bmatrix}$$

be an n×n Hermitian matrix, where $B$ is m×m (m<n). Let the eigenvalues of $A$ and $B$ be $\alpha_1\leq\ldots\leq\alpha_n$ and $\beta_1\leq\ldots\leq\beta_m$, respectively. Then

$$\alpha_k\leq\beta_k\leq\alpha_{k+n-m}, \quad k=1,\ldots,m.$$

Proof. Let

$$Au_i=\alpha_i u_i, \quad u^H_i u_j=\delta_{ij}, \quad i,j=1,\ldots,n,$$

$$Bv_i=\beta_i v_i, \quad v^H_i v_j=\delta_{ij}, \quad i,j=1,\ldots,m,$$

$$w_i=\begin{pmatrix} v_i \\ \\ 0 \end{pmatrix}, \quad i=1,\ldots,m.$$

Let $1\leq k\leq m$ and let

$S_1$=span{ $u_k,\ldots,u_n$ }, dim $S_1$=n-k+1

$S_2$=span{ $w_1,\ldots,w_k$ }, dim $S_2$=k

Again by §1 (1), the existence of an $x\in S_1\cap S_2$ such that $x^H x=1$ is assured and we have

$$\alpha_k\leq x^H A x\leq\beta_k$$

Application of this to $-A$ gives $\beta_k\leq\alpha_{k+n-m}$.

§4 The Courant-Fischer Theorem (Minimax Characterization) [1, p. 188]

Let $A$ be Hermitian and let $\alpha_1\leq\ldots\leq\alpha_n$ be the eigenvalues of $A$. Then for $k=1,\ldots,n$,

$$\alpha_k = \min_{S^k} \max\{v^H A v : v \in S^k, v^H v = 1\}$$
$$= \max_{S^{k-1}} \min\{v^H A v : v \perp S^{k-1}, v^H v = 1\}$$

where $S^k$ denotes an arbitrary k-dimensional subspace of complex n-vectors.

Proof. Let

$$Au_i = \alpha_i u_i, \quad u^H{}_i u_j = \delta_{ij}, \quad j=1,\ldots,n.$$

Let

$$S_1 = \text{span}\{ u_k, \ldots, u_n \} \text{ and } S_2 = S^k, \text{ (any k-dimensional subspace)}$$

Then §1 (1) guarantees the existence of an $x \in S_1 \cap S^k$, $x^H x = 1$, giving $x^H A x \geq \alpha_k$.

On the other hand, for any $u \in \text{span}\{ u_1, \ldots, u_k \}$, a k-dimensional subspace, we have $u^H A u \leq \alpha_k$ and $u^H{}_k A u_k = \alpha_k$, proving the first equality of the theorem.

To prove the second, choose

$$S_1 = \text{span}\{ u_1, \ldots, u_k \}, \; S_2 = (S^{k-1})^{\perp}$$

and proceed in a similar line of argument as above.

§5 The Sylvester's Law of Inertia

Let $A$ be an $n \times n$ Hermitian matrix and let

$$V^H{}_i A V_i = \text{diag}\{ d_1{}^{(i)}, \ldots, d_n{}^{(i)} \}, \; i=1,2,$$

where the $V_i$ are nonsingular. Then

$$\pi_1 = \pi_2 \text{ and } \mu_1 = \mu_2,$$

where $\pi_i$ (resp. $\mu_i$) denotes the number of the positive (resp.

negative) $d_j^{(i)}$'s, (i, fixed), i=1,2. (The symbol diag{ $d_1^{(i)}$,... } denotes the diagonal matrix with $d_j^{(i)}$ as the jth diagonal element.)

Proof. Suppose $\pi_1 > \pi_2$ and let $S_1 = \mathrm{span}\{ v_j^{(1)} : d_j^{(1)} > 0 \}$ and $S_2 = \mathrm{span}\{ v_j^{(2)} : d_j^{(2)} \leq 0 \}$ (dim $S_1 = \pi_1$, dim $S_2 = n - \pi_2$), where $v_j^{(i)}$ denote the jth column of $V^{(i)}$. By §1 (1), there is an $x \in S_1 \cap S_2$ such that $x^H x = 1$ and we have $x^H A x > 0$ (since $x \in S_1$) and $x^H A x \leq 0$ (since $x \in S_2$), a contradiction. Hence $\pi_1 = \pi_2$. Similarly $\mu_1 = \mu_2$.

References

1 Parlett,B.N., The Symmetric Eigenvalue Problem, Prentice-Hall, 1980

2 Wilkinson,J.H., The Algebraic Eigenvalue Problem, Oxford, 1965

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ABSTRACT

Monotonicity Theorem, Cauchy's Interlace Theorem, Courant-Fischer Theorem and Sylvester's Law of Inertia are reworked from a unified viewpoint of exploiting a simple dimensional identity for an easier, quicker and independent proof.

SUPPLEMENTARY NOTES