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DIRECT COMPUTATION OF EXPECTED NUMBERS OF FAILURES AND REPAIRS VIA INTEGRAL EQUATION APPROACH

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Toshiyuki Inagaki

Yasuhiko Ikebe

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INSTITUTE
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Toshiyuki INAGAKI and Yasuhiko IKEBE

Institute of Information Sciences and Electronics
University of Tsukuba
Ibaraki 305 JAPAN

Abstract - This paper develops a method for computing the expected number of failures and the expected number of repairs of a component in a prescribed time interval. Our method computes directly the above mentioned quantities without passing through a conventional step of calculating the unconditional failure and repair intensities over the corresponding time interval. Our method is constructed via integral equation formulation with its operator equation representation. It is shown that the expected number of failures and the expected number of repairs can be computed with the same precision of accuracy as that of the unconditional failure and repair intensities, which cannot be possible by a conventional approach.

1. INTRODUCTION

Probabilistic evaluation of reliability and safety provides essential information for designing and upgrading systems, maintenance policy optimization, and so on. Rigorous methods for reliability quantification have been developed by Vesely [1] for coherent systems and by Inagaki & Henley [2] for non-coherent systems. System reliability characteristics are represented completely in terms of the unconditional failure intensity $w(t)$, the unconditional repair intensity $v(t)$ and the point-unavailability $q(t)$ of relevant components whether the system is coherent or non-coherent.

In addition to reliability parameters $w(t)$, $v(t)$ and $q(t)$ we have two more important parameters for a component: the expected number of failures $W(0,t)$ and the expected number of repairs $V(0,t)$ in a prescribed time interval $[0, t]$. The usefulness of $W(0,t)$ and $V(0,t)$ lies in that they give intuitively understandable measure of reliability and maintainability, and in that $q(t)$ is represented in terms of $W(0,t)$ and $V(0,t)$. It might be believed that $w(u)$ and $v(u)$ for $0 \leq u \leq t$ must be known for computing $W(0,t)$ and $V(0,t)$ since $W(0,t)$ and $V(0,t)$ are given as integrals of $w(u)$ and $v(u)$, respectively, over the time interval $[0, t]$.

This paper shows that $W(0,t)$ and $V(0,t)$ can be computed directly, without passing through a step of calculating $w(u)$ and $v(u)$ for $0 \leq u \leq t$. The direct computation of $W(0,t)$ and $V(0,t)$ is made possible by formulating a system of integral equations that characterizes $W(0,t)$ and $V(0,t)$ as solutions to the integral equations. The integral equations are reformulated as a linear operator equation. A numerical method of solution is given to the linear operator equation. By examining the relation between our

integral equations for W(0,t) and V(0,t) and the well-known integral equations for w(t) and v(t), we show that our method can compute W(0,t) and V(0,t) so that they have the same precision of accuracy as that of w(t) and v(t); the precision of W(0,t) and V(0,t) is generally poorer than that of w(t) and v(t) if we follow a conventional approach that calculates W(0,t) and V(0,t) by numerical integration of w(t) and v(t). An example is given to illustrate our method.

2. PROBLEM STATEMENT

Let w(t) and v(t) denote the unconditional failure intensity and the unconditional repair intensity of a component at time t, respectively:

$$w(t)dt = \Pr[\text{component fails in } [t, t+dt) \mid E_0]$$

$$v(t)dt = \Pr[\text{component repair completes in } [t, t+dt) \mid E_0]$$

where E_0 denotes the event that the component was as good as new at time zero. Let q(t) denote the point-unavailability of a component at time t:

$$q(t) = \Pr[\text{component is not working at time } t]$$

The set of parameters {w(t), v(t), q(t)} of components is necessary for probabilistic evaluation of minimal cut sets (prime implicants) and top-events of systems. In quantifying reliability and safety of systems, we usually compute 1) the existence probability (unavailability) and 2) the unconditional occurrence (failure) intensity for minimal cut sets (prime implicants) and top-events. The reliability parameters 1) and 2) are represented completely in terms of {w(t), v(t), q(t)} of relevant components whether the system is coherent or non-coherent; see [1] or [3, Chapter 7] for coherent cases and [2] for non-coherent cases.

Let W(0,t) denote the expected number of failures of a component in a prescribed time interval [0, t] and let V(0,t) denote the expected number of repairs of the component in [0, t]. Then W(0,t) and V(0,t) are given by:

$$W(0,t) = \int_0^t w(u)\, du, \quad V(0,t) = \int_0^t v(u)\, du \tag{1}$$

The parameters W(0,t) and V(0,t) are essential for reliability quantification since 1) they provide intuitively understandable measure of reliability and maintainability and 2) the point-unavailability q(t) can be computed by the formula:

$$q(t) = W(0,t) - V(0,t) \tag{2}$$

see, e.g., [3, p. 193]. Thus we may say that the fundamental parameters for probabilistic evaluation of reliability and safety of systems is the set of {w(t), v(t), W(0,t), V(0,t)} of components, rather than the set of {w(t), v(t), q(t)} of components.

Because of the relation (1) it might be believed that W(0,t) and V(0,t) are 'secondary' parameters in the sense that w(u) and v(u) for $0 \leq u \leq t$ must be known for computing W(0,t) and V(0,t). Figure 1 depicts the conventional approach for computing W(0,t) and V(0,t), where w(t) and v(t) are obtained by solving the following system of integral equations [1], [3, p. 193]:

$$\begin{aligned} w(t) - \int_0^t f(t-u)\, v(u)\, du &= f(t) \\ v(t) - \int_0^t g(t-u)\, w(u)\, du &= 0 \end{aligned} \tag{3}$$

where f(t) and g(t) are given probability density functions (pdf) for the time to first failure and the repair time, respectively.

This paper makes the following assertions:

1. $W(0,t)$ and $V(0,t)$ can be computed directly without requiring the knowledge of $w(u)$ and $v(u)$ for $0 \leq u \leq t$. The direct computation of $W(0,t)$ and $V(0,t)$ becomes possible by formulating a system of integral equations that characterizes $W(0,t)$ and $V(0,t)$ as solutions to the integral equations. Figure 2 illustrates the proposed approach for computing $W(0,t)$ and $V(0,t)$.

2. $W(0,t)$ and $V(0,t)$ can be computed so that they have the same precision of accuracy as that of $w(t)$ and $v(t)$. The precision of accuracy of $W(0,t)$ and $V(0,t)$ is generally poorer than that of $w(t)$ and $v(t)$ if we follow the conventional approach in Figure 1: the original precision of $w(u)$ and $v(u)$ for $0 \leq u \leq t$ is lost in the process of numerical integration for producing $W(0,t)$ and $V(0,t)$.

The above mentioned assertions are verified in the subsequent sections.

Remark. In theory, the set of $\{w(t), v(t)\}$ of components provide sufficient information for probabilistic evaluation of systems reliability and safety because of (1) and the relation

$$q(t) = \int_0^t [w(u) - v(u)]\, du \qquad (2')$$

Assertion 2 suggests, however, that the set of $\{w(t), v(t), W(0,t), V(0,t)\}$ of components must be thought as basic information, rather than $\{w(t), v(t)\}$ nor $\{w(t), v(t), q(t)\}$ of components for performing probabilistic evaluation of reliability and safety with high precision of accuracy.

3. INTEGRAL EQUATIONS

The expected number of failures $W(0,t)$ and the expected number of repairs $V(0,t)$ of a component in the time interval $[0, t]$ can be described

as solutions to a system of integral equations:

$$W(0,t) - \int_0^t f(t-u)\, V(0,u)\, du = F(t)$$
$$V(0,t) - \int_0^t g(t-u)\, W(0,u)\, du = 0, \quad 0 \leq t \leq T \tag{4}$$

where T denotes an arbitrary time point and F(t) denotes a cumulative distribution function (cdf) for the time to first failure:

$$F(t) = \int_0^t f(u)\, du \tag{5}$$

The integral equations (4) are derived by integrating (3) through use of Dirichlet formula:

$$\int_a^b \left[\int_a^x h(x,y)\, dy \right] dx = \int_a^b \left[\int_y^b h(x,y)\, dx \right] dy$$

where h(x,y) is continuous on a triangular region $a \leq y < x \leq b$.

Eq. (4) states that W(0,t) and V(0,t) can be computed directly without passing through a step of calculating w(u) and v(u) for $0 \leq u \leq t$, which verifies Assertion 1 given in the previous section.

4. OPERATOR EQUATIONS

The system of integral equations (4) is reformulated as a linear operator equation:

$$Lx = b \tag{6}$$

where

$$L = \begin{bmatrix} I & -H_f \\ -H_g & I \end{bmatrix}, \quad x = \begin{bmatrix} W(0,t) \\ V(0,t) \end{bmatrix}, \quad b = \begin{bmatrix} F(t) \\ 0 \end{bmatrix}$$

I denotes the identity operator, H_f and H_g are Volterra integral operators defined by:

$$H_f[V(0,t)] = \int_0^t f(t-u)\, V(0,u)\, du$$
$$H_g[W(0,t)] = \int_0^t g(t-u)\, W(0,u)\, du$$

The integral equations (4) or operator equation (6) is usually difficult to solve analytically if f(t) and/or g(t) is a non-exponential pdf; some numerical method is needed for the cases.

Let P_n denote a bounded linear projection ($P_n^2 = P_n$) from a Banach space C[0, T] of real-valued functions on the interval [0, T] onto an n-dimentional subspace S_n of C[0, T]. We have the approximate operator equation:

$$L_n x_n = b_n \tag{7}$$

where

$$L_n = \begin{bmatrix} I & -P_n H_f \\ -P_n H_g & I \end{bmatrix}, \quad x_n = \begin{bmatrix} W_n \\ V_n \end{bmatrix}, \quad b_n = \begin{bmatrix} P_n F(t) \\ 0 \end{bmatrix}$$

and P_n (n = 1, 2,...) are assumed to satisfy the conditions:

$$\text{(a)} \quad \| P_n z - z \| = \sup_{0 \le t \le T} | P_n z(t) - z(t) | \to 0 \quad \text{as } n \to \infty$$

for every $z \in C[0, T]$, and

$$\text{(b)} \quad \| P_n \| = \sup \{ \| P_n z \| : \| z \| = 1 \} = 1, \quad n = 1, 2,... \tag{8}$$

Eq. (7) is uniquely solvable for x_n since the inverse L_n^{-1} exists for any n. Actually, L_n^{-1} is given by:

$$L_n^{-1} = \begin{bmatrix} (I - P_n H_f P_n H_g)^{-1} & P_n H_f (I - P_n H_g P_n H_f)^{-1} \\ P_n H_g (I - P_n H_f P_n H_g)^{-1} & (I - P_n H_g P_n H_f)^{-1} \end{bmatrix}$$

Moreover, a sequence $\{x_n\}$ of solutions to (7) converges to the solution x to the original operator equation (6) as $n \to \infty$. The unique solvability and convergence are proven under assumptions (a) and (b) for P_n and the following fact [4]:

$$\| H_f \| = F(T) < 1, \quad \| H_g \| = G(T) < 1 \tag{9}$$

where F(t) is given in (5) and G(t) is the pdf for repair time:

$$G(t) = \int_0^t g(u)\, du$$

It is important to note that the integral equations (3) for w(t) and v(t) are expressed as follows [4]:

$$Lx' = b' \tag{10}$$

where $b' = \mathrm{col}(f(t), 0)$ and the solution x' gives col(w(t), v(t)). The approximate operator equation that associates with (10) is given by:

$$L_n x_n' = b_n' \tag{11}$$

where $b_n' = \mathrm{col}(P_n f(t), 0)$. It is proven [5] that a sequence $\{x_n'\}$ of solutions to (11) converges to the true solution x' to (10). The difference between (6) and (10), or between (7) and (11) lies solely in the right sides of the equations. Thus we have the following observations:

1. The numerical method [5] developed for solving (11) directly applies to our operator equation (7) for W(0,t) and V(0,t).

2. W(0,t) and V(0,t) can be computed with the same precision of accuracy as that of w(t) and v(t). This is the Assertion 2 that we stated in Section 2. We verify the assertion by the discussions given below.

Operator equations (7) and (11) have the same condition number [6, p.20], $\mathrm{cond}(L_n)$, which is defined by:

$$\mathrm{cond}(L_n) = \| L_n \| \, \| L_n^{-1} \|$$

The condition number is determined independently of a numerical method of solution and provides information for estimating the precision of numerical solutions. The upper bound for $\mathrm{cond}(L_n)$ is obtained, in a similar manner to [4], as follows:

$$\mathrm{cond}(L_n) \leq \max \left\{ \frac{(1 + \| P_n H_f \|)^2}{1 - \| P_n H_f \| \| P_n H_g \|}, \quad \frac{(1 + \| P_n H_g \|)^2}{1 - \| P_n H_f \| \| P_n H_g \|} \right\}$$

or, by applying (8) and (9),

$$\mathrm{cond}(L_n) \le \max\left\{\frac{(1 + F(T))^2}{1 - F(T)G(T)}, \frac{(1 + G(T))^2}{1 - F(T)G(T)}\right\}$$

Let $\tilde{x}_n$ and $\tilde{x}_n'$ denote approximate solutions to (7) and (11), respectively, that are obtained in any manner whatsoever. According to the 'backward error analysis' [6, 7], $\tilde{x}_n$ and $\tilde{x}_n'$ are considered the exact solutions to slightly perturbed equations

$$L_n\tilde{x}_n = b_n + \Delta b_n, \quad L_n\tilde{x}_n' = b_n' + \Delta b_n'$$

respectively, where $\tilde{x}_n = x_n + \Delta x_n$ and $\tilde{x}_n' = x_n' + \Delta x_n'$. The magnitude of relative errors are evaluated as:

$$\frac{\|\Delta x_n\|}{\|x_n\|} \le \mathrm{cond}(L_n)\,\frac{\|L_n\tilde{x}_n - b_n\|}{\|b_n\|} \tag{12}$$

and

$$\frac{\|\Delta x_n'\|}{\|x_n'\|} \le \mathrm{cond}(L_n)\,\frac{\|L_n\tilde{x}_n' - b_n'\|}{\|b_n'\|} \tag{13}$$

Suppose we work on a base β, floating-point computing system. Let $\mathrm{cond}(L_n) \le \beta^p$. If $\|\Delta b_n\| / \|b_n\|$ and $\|\Delta b_n'\| / \|b_n'\|$ are assumed to be of the same order, say β^{-q}, then (12) and (13) imply that approximately at least the first q-p digits of $\tilde{x}_n$ and $\tilde{x}_n'$ are correct.

5. METHOD OF SOLUTION

The numerical method of solution given in [5] is summarized for the case of solving (7). Let $\{e_i(t) : i = 1, \ldots, n\}$ be a basis for an n-dimensional subspace S_n of $C[0, T]$. Since (7) implies $W_n \in S_n$ and $V_n \in S_n$, W_n and V_n are expressed uniquely as:

$$W_n = \sum_{i=1}^{n} c_i\, e_i(t), \quad V_n = \sum_{i=1}^{n} d_i\, e_i(t)$$

where c_i and d_i are real numbers; in other words, $c = \mathrm{col}(c_1, \ldots, c_n)$ and $d = \mathrm{col}(d_1, \ldots, d_n)$ determines W_n and V_n. Projection P_n that maps $C[0, T]$ into S_n can be written as follows [8]:

$$P_n = e_1 e_1^* + \ldots + e_n e_n^*$$

where e_i^* are linear functionals defined by:

$$e_1^*(z)\, e_1 + \ldots + e_n^*(z)\, e_n = P_n z$$

for any $z \in C[0, T]$. Thus the problem of solving (7) is reduced to that of solving a linear system of equations

$$\begin{bmatrix} I & -A_n \\ -B_n & I \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} h \\ 0 \end{bmatrix} \tag{14}$$

where I denotes the identity matrix of order n, A_n is an n by n matrix with $e_i^* H_f e_j(t)$ as its (i,j)-element, B_n is an n by n matrix with $e_i^* H_g e_j(t)$ as its (i,j)-element, and h is a column vector of order n defined by $\mathrm{col}(e_1^* F(t), \ldots, e_n^* F(t))$. The linear system of equations (14) can be solved by direct methods such as Gaussian elimination, or by iterative methods such as Gauss-Seidel method.

6. EXAMPLE

Let f(t) and g(t) be exponential pdf's:

$$f(t) = \lambda \exp(-\lambda t), \quad g(t) = \mu \exp(-\mu t)$$

where λ and μ are constant failure rate and repair rate, respectively. For the exponential cases we know the analytical expressions for W(0,t) and V(0,t):

$$\begin{aligned} W(0,t) &= \frac{\lambda\mu}{\lambda+\mu} t + \frac{\lambda^2}{(\lambda+\mu)^2} \{1 - \exp[-(\lambda+\mu)t]\} \\ V(0,t) &= \frac{\lambda\mu}{\lambda+\mu} t + \frac{\lambda\mu}{(\lambda+\mu)^2} \{1 - \exp[-(\lambda+\mu)t]\} \end{aligned} \tag{15}$$

see, e.g., Henley & Kumamoto [3, p. 203]. We illustrate the effectiveness of our method for computing W(0,t) and V(0,t) by comparing the result obtained via our method with true values calculated by (15).

Let P_n be a piecewise linear interpolatory projection:

$$P_n z = \sum_{i=1}^{n} z(t_i)\, e_i(t), \quad z \in C[0, T]$$

where t_i are knot points for which $0 = t_1 < t_2 < \ldots < t_n = T$. We assume knot points are equally spaced: $t_i - t_{i-1} = T/(n-1) = s$ for any i. Then $e_i(t)$ are given by:

$$e_i(t) = \begin{cases} (t - t_{i-1})/s, & t_{i-1} \le t \le t_i \\ (t_{i+1} - t)/s, & t_i \le t \le t_{i+1} \end{cases}$$

The i-th element of vector h and the (i,j)-element of matrix A_n in (12) are given as:

$$e_i * F(t) = 1 - \exp[-(i-1)\lambda s], \quad i = 1, \ldots, n$$

$$e_i * H_f e_j(t) = \begin{cases} 0, & i = 0 \text{ or } i \le j - 1 \\ 1 - [1 - \exp(-\lambda s)]/(\lambda s), & i = j \ne 1 \\ \{[\exp(\lambda s)-1]/(\lambda s) - 1\}\exp[-(i-1)\lambda s], & j = 2,\ i = 2, \ldots, n \\ [\exp(-2\lambda s) - 2\exp(-\lambda s) + 1]/\{\lambda s \exp[-(i-j-1)\lambda s]\} & \\ \qquad i \le j + 1,\ j = 2, \ldots, (n-1) & \end{cases} \tag{16}$$

The (i,j)-element $e_i * H_g e_j(t)$ of matrix B_n is obtained by replacing λ in (16) by μ.

Let $\lambda = 0.001\ \text{hr}^{-1}$ and $\mu = 0.1\ \text{hr}^{-1}$. Eq. (14) is solved by Gauss-Seidel method under the following convergence criterion:

$$|c^{(k+1)} - c^{(k)}| < \varepsilon \text{ and } |d^{(k+1)} - d^{(k)}| < \varepsilon$$

where $c^{(k)}$ and $d^{(k)}$ denote the k-th iterate of c and d, respectively, and we set $c^{(0)} = d^{(0)} = \text{col}(0, \ldots, 0)$ and $\varepsilon = 1.0 \times 10^{-10}$.

Suppose we want to evaluate $W(0,t)$ and $V(0,t)$ at, say, $t = 10$ hr. We set $T = 10$. Figure 3 lists approximate solutions $W_n(0,10)$ and $V_n(0,10)$ for various values of n that are obtained by solving (14). Figure 3 also indicates the true values of $W(0,10)$ and $V(0,10)$ that are computed by (15). The relative errors of solutions for a case of $n = 128$, for instance, are evaluated as:

$$|W(0,10) - W_{128}(0,10)| \,/\, |W(0,10)| = 1.0 \times 10^{-7}$$

$$|V(0,10) - V_{128}(0,10)| \,/\, |V(0,10)| = 1.0 \times 10^{-7}$$

7. CONCLUSION

This paper has given a system of integral equations that describe $W(0,t)$ and $V(0,t)$ as solutions to the integral equations. It is shown that the integral equation approach enables us to compute $W(0,t)$ and $V(0,t)$ directly, without assuming a knowledge of $w(u)$ and $v(u)$ for $0 \leq u \leq t$. It is also shown that $W(0,t)$ and $V(0,t)$ can be computed so that they have the same precision of accuracy as that of $w(t)$ and $v(t)$, in spite of the relation that $W(0,t)$ and $V(0,t)$ are defined as integrals of $w(u)$ and $v(u)$ over the time interval $[0, t]$.

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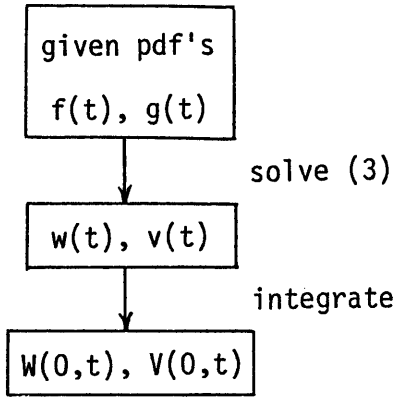

Figure 1 Conventional approach for computing W(0,t) and V(0,t)

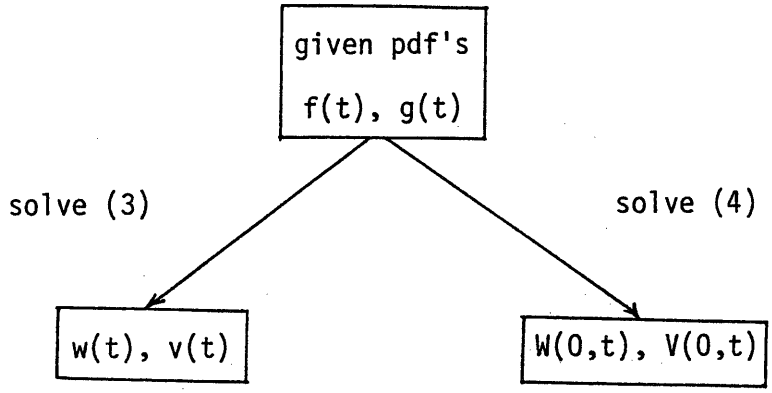

Figure 2 Newly developed approach for computing W(0,t) and V(0,t)

# of knot points	$W_n(0,10) \times 10^2$	$V_n(0,10) \times 10^2$	# of iterations for convergence
16	0.99633386	0.36684500	4
32	0.99633209	0.36684530	4
64	0.99633168	0.36684537	4
128	0.99633158	0.36684538	4

true values: $W(0,10) = 0.99633148 \times 10^{-2}$

$V(0,10) = 0.36684542 \times 10^{-2}$

Figure 3 Numerical results for a case of t = 10

INSTITUTE OF INFORMATION SCIENCES AND ELECTRONICS
UNIVERSITY OF TSUKUBA
SAKURA-MURA, NIIHARI-GUN, IBARAKI 305 JAPAN

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AUTHOR(S)

Toshiyuki Inagaki; Institute of Information Sciences and Electronics, University of Tsukuba

Yasuhiko Ikebe; Institute of Information Sciences and Electronics, Universiyt of Tsukuba

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SUPPLEMENTARY NOTES