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NUMERICAL SOLUTION OF FREE BOUNDARY PROBLEM FOR UNSTEADY SLAG FLOW IN THE HEARTH

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ABSTRACT

A numerical method for solving a problem in unsteady slag flow in the hearth of a blast furnace is presented. This problem is reduced to some free boundary problem for an elliptic system. The potential problem for a given free boundary is approximated by the penalty method. The derivatives of the potential function on the free boundary is approximated by the integration of the penalty term, and then the subsequent shape of the free boundary is obtained by solving the differential equation for the motion of the free boundary. The finite difference method is used to solve the penalized problem. A numerical example is given.

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Makoto NATORI and Hideo KAWARADA

1. Introduction

We present a numerical method for solving a problem in unsteady flow of molton slag in the hearth region of iron producing blast furnaces during the tapping operation[1]. This problem is reduced to some free boundary problems for an elliptic system. This type of problem is similar to the porous flow of underground water in which the water surface is a free boundary. The numerical calculations of this type of problem were done by various researchers[2-4]. The three-dimensional problem of the slag flow in the hearth was solved by using the finite element method by Ichihara and Fukutake[5]. They concluded that their computation scheme is not so efficient as it is applied for the practical use.

The object of this paper is to settle this computational instability by using the penalty method developed by Kawarada and Natori [6-9].

2. Formulation

We consider two-dimensional slag flow in the hearth which is bounded by impermeable boundaries $y=0$, $x=0$ and $x=a$. One of vertical boundaries, $x=0$, has a tapping hole near the bottom. While $y=g(x,t)$ denotes the free surface of the slag region(See Fig.1).

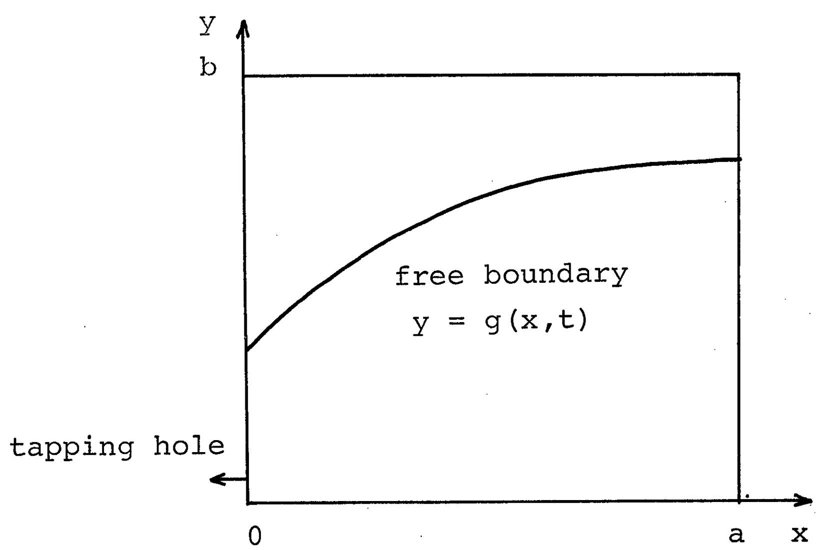

Fig. 1

A velocity potential can be defined by

$$\phi = \frac{p}{\gamma} + y$$

where p is the fluid pressure and γ is the specific weight of the fluid. If it is assumed that Darcy's law holds, the potential is given by

(1) $\Delta \phi = 0$

(2) $\phi = y$ on $y = g(x,t)$

(3) $\phi_y = 0$ on $y = 0$

(4) $\phi_x = 0$ on $x = 0$ and $x = a$, except on the tapping hole

(5) $\phi_x = k\ (>0)$ on the tapping hole.

where k is some constant.

The motion of the free surface is given by

$$(6)\quad g_t = - (\phi_y - \phi_x g_x) \Big|_{y=g(x,t)} = - \sqrt{1 + g_x^2}\ \frac{\partial \phi}{\partial n} \Big|_{y=g(x,t)}$$

The initial shape of the free surface, $y = g(x,0)$, is given and forms an initial condition for equation (6).

When we try to solve the problem formulated above, a numerical procedure must contain a routine for solving the potential problem (1)-(5) for a given free boundary $y = g(x,t)$. When this is done, the derivatives of the potential function can be calculated on the free boundary, and then the equation (6) can be solved for the subsequent shape of the free boundary.

If we use the method of the integrated penalty to solve the potential problem, then the derivatives of the potential function on the free boundary are easily obtained. This is the reason of our application of the penalty method to the free boundary problems.

3. Penalty Method

3.1 Penalized problem

We define the characteristic function $\chi^{\varepsilon}(x,y,t)$ such as

$$(7)\quad \chi^{\varepsilon}(x,y,t) = \begin{cases} 1 & \text{in } \Omega - \Omega_g^{\varepsilon} \\ 0 & \text{in } \Omega_g^{\varepsilon} \end{cases}$$

where the domains $\Omega_g\varepsilon$ and Ω, which includes $\Omega_g\varepsilon$, are defined by

$$\Omega_g\varepsilon = \{(x,y)\,|\,0 < x < a, \quad 0 < y < g^{\varepsilon}(x,t)\}$$

and

$$\Omega = \{(x,y)\,|\,0 < x < a, \quad 0 < y < b\}$$

as shown in Fig.2.

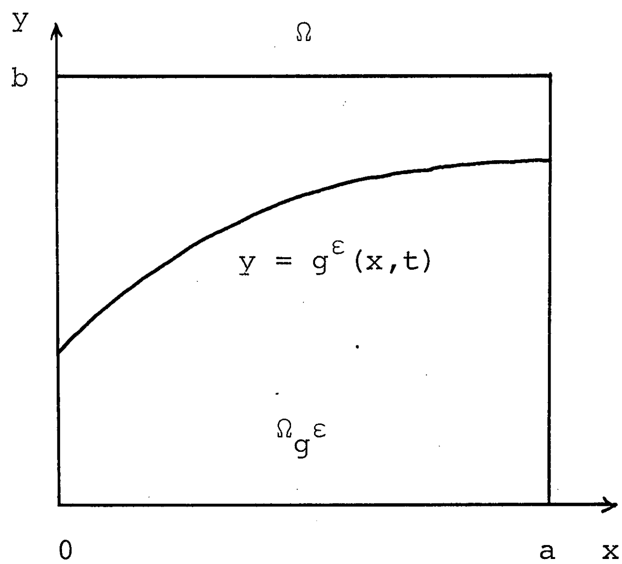

Fig. 2

By the use of χ^{ε}, equation (1) is approximated by

$$(8) \qquad \Delta\phi^{\varepsilon} - \frac{1}{\varepsilon}\chi^{\varepsilon}(\phi^{\varepsilon} - y) = 0 \quad \text{in } \Omega,$$

where ε is a positive constant. We add a new boundary condition:

$$(9) \qquad \phi^{\varepsilon} = y \quad \text{on} \quad y = b,$$

to the boundary conditions (3)-(5).

In fact, if we let ε be sufficiently small then we know that ϕ^{ε} approximates ϕ in $\Omega_g\varepsilon$ and ϕ^{ε} is nearly equal to y in $\Omega - \Omega_g\varepsilon$. Therefore the boundary condition (2) is approximately satisfied.

If we use the method of integrated penalty, equation (6) is approximated by

$$(10) \qquad g_t^{\varepsilon} = -(1 - \frac{1}{\varepsilon} \int_0^b \chi^{\varepsilon}(\phi^{\varepsilon} - y)\, dy).$$

Now, the second term in the right hand side of (10) is called the integrated penalty.

3.2 Discretization of the penalized problem

The penalized problem (8) with the boundary conditions (3)-(5) and (9) is discretized by the finite difference. Also the free boundary equation (10) is solved by Euler's method. The intervals $0 \leqq x \leqq a$ and $0 \leqq y \leqq b$ are divided into N and M equal subintervals of width h. The mesh size of time is denoted by Δt. We use the following notations in the discretized system:

$$x_i = ih \quad , \quad 0 \leqq i \leqq N$$

$$y_j = jh \quad , \quad 0 \leqq j \leqq M$$

$$t_k = k\Delta t \quad , \quad 0 \leqq k$$

$$\phi_{ijk} = \phi^{\varepsilon}(x_i, y_j, t_k)$$

$$g_{ik} = g^{\varepsilon}(x_i, t_k)$$

$$\chi_{ijk} = \chi^{\varepsilon}(x_i, y_j, t_k)$$

We define the discretized characteristic function by

$$(11) \qquad \chi_{ijk} = \begin{cases} 1 & j > [g_{ik}/h] \\ \dfrac{1 - \rho_{ik}}{1 + \frac{h^2}{4\varepsilon}\rho_{ik}} & j = [g_{ik}/h] \\ 0 & j < [g_{ik}/h] \end{cases}$$

where [ ] denotes the Gauss symbol and

$$\rho_{ik} = g_{ik}/h - [g_{ik}/h].$$

Then the potential function ϕ_{ijk} satisfies the following equations for any k :

$$(4+\frac{h^2}{\varepsilon}\chi_{ijk})\phi_{ijk}-\phi_{i-1\,j\,k}-\phi_{i+1\,j\,k}-\phi_{i\,j-1\,k}-\phi_{i\,j+1\,k}$$

(12)
$$= \frac{h^2}{\varepsilon} y_j \chi_{ijk}$$

$$(0\leqq i\leqq N \ , \ 0\leqq j\leqq M \ )$$

This system of linear equations is solved by the incomplete Cholesky decomposition combined with conjugate gradient method [11].

The free boundary g_{ik} is obtained by

(13) $$g_{i\,k+1} = g_{ik}+\Delta t F(g_{ik})$$

(14) $$F(g_{ik}) = -1+\frac{h}{\varepsilon}\sum_{j=0}^{M}\chi_{ijk}(\phi_{ijk}-y_j)$$

It should be noted that χ_{ijk} and ϕ_{ijk} are determined by g_{ik}.

3.3 How to choose the penalty parameter ε.

We assume ε is expressed by

(15) $$\varepsilon = h^{\sigma}$$

and try to find an optimal value of σ to minimize the difference of the right sides of equations (6) and (14). For this purpose, we consider a simple test problem (See Fig.3) :

(16)
$$\begin{aligned} \Delta u &= 0 && \text{in} && \Omega_g \\ u &= 1-y && \text{on} && x = 0 \\ u &= 1-x && \text{on} && y = 0 \\ u &= 0 && \text{on} && y = 1-x \end{aligned}$$

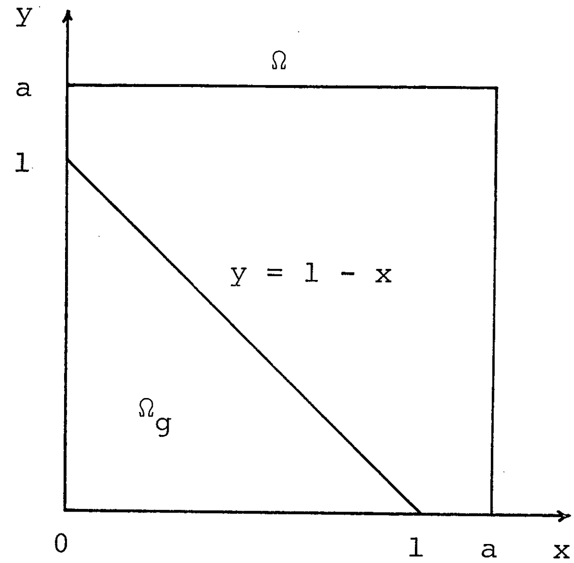

Fig. 3

This problem has an exact solution $u = 1-x-y$. Therfore,

$$(17) \qquad - \sqrt{1+g_x^2}\, \frac{\partial u}{\partial n}\Big|_{y=g(x)} = 2,$$

where $g(x) = 1-x$.

Here we construct the discretized problem (P_h^ε) of the penalized equation :

$$(18) \qquad \Delta u^\varepsilon - \frac{1}{\varepsilon}\chi u^\varepsilon = 0 \quad \text{in} \quad \Omega$$

We investigate the difference between (17) and the integrated penalty ;

$$(IP)_i = \frac{h}{\varepsilon}\sum_{j=0}^{M} \chi_{ij} u_{ij}$$

by varying the value of σ in (15). We get that the optimal value of σ is 3~4 for $1/8 \leqq h \leqq 1/16$.

3.4 Stability condition

Here we study the stability condition of (13). It is well known that the stability condition is

$$(19) \qquad \Delta t\left|\frac{\partial F}{\partial g}\right| \leqq 2,$$

where

$$F(g) = -1+\frac{1}{\varepsilon}\int_0^b \chi^\varepsilon(g)(\phi^\varepsilon(g)-y)dy.$$

If we use the property [12]

$$|\phi^\varepsilon(x,g(x,t))| \leqq C_0\sqrt{\varepsilon},$$

where C_0 depends on ϕ^ε and g^ε, then we have

$$|F(g)-F(\bar{g})| \leqq \frac{C_0}{\sqrt{\varepsilon}}|g-\bar{g}|.$$

If we substitute $C_0/\sqrt{\varepsilon}$ into $|\partial F/\partial g|$ in (19), then we have

$$0 < \Delta t\frac{C_0}{\sqrt{\varepsilon}} \leqq 2.$$

Therefore we may choose

$$(20) \qquad \Delta t = C_1\sqrt{\varepsilon}$$

4. Numerical Example

In this section we present results for the problem (1)-(6), obtained by the method of integrated penalty. Data of the problem are as follows :

$a = 1$

$b = 0.3125$

$k = 0.625$

The tapping hole is located at (0,1/16). The initial surface is given by

$$g(x,0) = 0.25.$$

The parameters used for the numerical calculations are

$$h = 1/16$$

$$\varepsilon = h^3 = 1/4096$$

$$\Delta t = \sqrt{\varepsilon} = 1/64.$$

In Fig.4 we show the results.

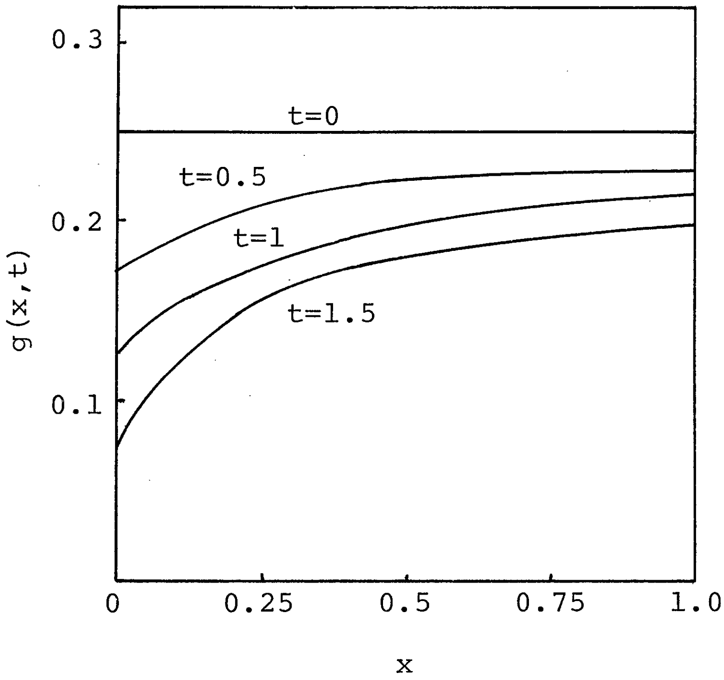

Fig. 4

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SUPPLEMENTARY NOTES