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APPLICATION OF HOMOTOPY METHOD TO TWO POINT BOUNDARY VALUE PROBLEMS OF FIRST ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT

The degree theory and the homotopy method related to it are applied to nonlinear two point boundary value problems of ordinary differential equations. The system is described by first order differential equations which can not necessarily be brought into a second order system. Two types of the applications are considered. First, existence of the solution of a class of the boundary value problems is considered by using the Leray-Schauder degree theory. The differential system is reduced to a nonlinear integral equation, which is imbedded into a homotopy with compact operators. Sufficient conditions for the existence of the solution are given. Secondly, an algorithm for calculating a fixed point of a differentiable map suggested by Watson is applied to the boundary value problem. The algorithm follows a homotopy curve from an initial value to the fixed point. A sufficient condition on which the homotopy algorithm converges globally is discussed. Matrix differential equation to be solved in the algorithm is derived. Moreover, the property of finite arc length of the homotopy is proved.

1. Introduction

Nonlinear boundary value problems have been extensively and widely studied, because of their importance in physical processes. Theoretical studies have been mainly concentrated on boundary value problems of the second order differential equations [1]:

(1) $$x'' = f(t,x,x'),$$

Many problems in engineering such as those of optimal control are, however, reduced to the solution of first order differential equations

(2) $$\begin{aligned} x' &= F(t,x,y) \\ y' &= G(t,x,y) \end{aligned} \qquad 0 < t < T$$

with the boundary condition

$$x(0) = x_0, \qquad y(T) = y_T,$$

which can not be written in a form of the second order systems such as (1). For the latter form of the boundary value problems, techniques based on local linearization have been studied, but global studies are still rare.

On the other hand, recent studies on topological properties of continuous maps have proved that the homotopy techniques are important both as a theoretical tool and as a method giving algorithms useful in applications. Both features are typically exhibited in the study of fixed point theorems. Namely, the Brower fixed point theorem is proved by the degree theory [2]. Furthermore, methods for the computation of a fixed point are based on the homotopy from a known initial value to the solution [3],[4].

This paper is concerned with the application of the homotopies to the study of the boundary value problem of the first order system (2). The content is divided into two parts. The first part discusses the existence of a solution of a class of nonlinear boundary value problems by using the Leray-Schauder degree theory [2]; the second part concerns the computation of the solution based on a fixed point algorithm.

2. Existence of a solution of a class of two point boundary value problems

2.1 Preliminaries

Let E^n be n-dimensional Euclid space and a system in E^n is considered:

(3) $$y' = F(y,t), \qquad 0 < t < 1$$

(4) $$M\,y(0) + N\,y(1) = c\,,$$

where $y(t) \in E^n$; M, N : $n \times n$ constant matrices, $c \in E^n$: constant vector. Moreover we assume that $F(y,t)$ is a C^1 map.

It is necessary to consider a linear two point boundary value problem associated with the above system:

(5) $$z' = V(t)z + f(t), \qquad 0 < t < 1\,,$$

(6) $$M\,z(0) + N\,z(1) = c\,,$$

where $V(t)$: $n \times n$ matrix, $f(t)$: n vector.

Let the fundamental solution for

$$z' = V(t)z$$

be $\Phi(t,s)$. Then the following three propositions are true [5].

<u>Prop. 1</u>([5], p.61)

Suppose

$$\det[\, M + N\ \Phi(1,0)\,] \neq 0 \,. \tag{7}$$

Then, there exists a unique solution $z(t)$ for (5),(6) satisfying

$$z(t) = H(t)c + \int_0^1 G(t,s)f(s)ds \,, \tag{8}$$

where

$$H(t) = \Phi(t,0)[\, M + N\ \Phi(1,0)]^{-1} \tag{9}$$

$$G(t,s) = \begin{cases} \Phi(t,0)[M+N\ \Phi(1,0)]^{-1}M\ \Phi(0,s), & 0 \leq s < t \\ -\Phi(t,0)[M+N\ \Phi(1,0)]^{-1}N\ \Phi(1,s), & t < s \leq 1 \end{cases} \tag{10}$$

Prop. 2([5],p.62)

A necessary and sufficient condition that there is a $V(t)$ such that $\det[M + N\,\Phi(1,0)] \neq 0$ is that the $n \times 2n$ matrix $[M\ N]$ have full rank n.

Prop. 3([5],p.68)

Assume that $\det[M + N\,\Phi(1,0)] \neq 0$. Then the nonlinear two point boundary value problem (3),(4) has an equivalent representation

$$(11) \qquad y(t) = H(t)c + \int_0^1 G(t,s)[F(y(s),s) - V(s)y(s)]ds ,$$

where $H(t)$ and $G(t,s)$ are given by (9) and (10), respectively.

On the other hand, we need an important result based on the Leray-Schauder degree theory.

Prop. 4(Schaefer; see [2],p.71)

Let X be a Banach space and $\phi : X \to X$ be a compact operator which is not necessarily linear. If the set

$$(12) \qquad S = \{ u \mid u = \lambda\phi(u) , \text{ for some } \lambda \in [0,1) \}$$

is bounded, then ϕ has a fixed point u: $u = \phi(u)$.

2.2 The existence of the solution

Let us define the norm $|z|$, $z = (z_1, z_2, \ldots, z_n)^T \varepsilon E^n$ by

$$|z| = \max_{1 \leq i \leq n} |z_i|$$

and let $|A|$ denote the matrix norm of nxn matrix A, associated with the vector norm $|z|$. Assume that there exists a monotone nondecreasing function $k(x)$: $\mathbf{R} \to \mathbf{R}$ and constants $M_1 > 0$, $M_2 > 0$ such that

$$(13) \qquad \begin{gathered} |F(y(s),s) - V(s)y(s)| < k(|y(s)|) \\ |H(t)| < M_1, \qquad |G(t,s)| < M_2 \\ 0 \leq t,s \leq 1 . \end{gathered}$$

where $H(t)$ and $G(t,s)$ are bounded by the previous result (9), (10). Then we have the following theorem.

Th. 1

Suppose that $\det[M + N\,\Phi(1,0)] \neq 0$ and the relation (13) holds. If the solution of the inequality

$$x < M_1 + M_2 k(x), \qquad x \in \mathbf{R}$$

satisfies $x < m$ for a constant $m > 0$. Then there exists a solution of the two point boundary value problem (3),(4).

(Proof) We consider the equation (11), which is equivalent to (3),(4), according to Prop. 3. Let

$$\phi(y) = H(t)c + \int_0^1 G(t,s)[F(y(s),s) - V(s)y(s)]ds .$$

Since $G(t,s)$ is bounded and continuous when $t \neq s$, it is easy to see that the map $u(t) \to \int G(t,s)u(s)ds$ is compact. (See [6].) Seeing that $F(y,s)$ represents a bounded operator, it follows that ϕ is a compact operator.

Let us see that if $y = \lambda\phi(y)$ for some $\lambda \varepsilon [0, 1)$, then there exists $m > 0$ independent of λ such that $\|y\| < m$. Let $X = C[0, 1]$ in Prop. 4 and $\|y\| = \|y\|_\infty = \sup_t |y(t)|$. Then,

$$|y(t)| < M_1 + M_2 \int_0^1 k(|y(s)|)ds$$

from (13). Hence it follows that

$$\|y\| < M + M k(\|y\|),$$

which means that $\|y\| < m$ from the assumption. That is, $y(t)$ is bounded, independent of λ. Therefore there exists a solution $y(t)$ of (11) by referring to Prop. 4. Q.E.D.

2.3 Examples

An important class of the boundary conditions in application takes the form

$$y_1(0) = a_1, \quad y_2(0) = a_2, \quad \dots, \quad y_r(0) = a_r,$$
$$y_{r+1}(1) = b_{r+1}, \quad \dots\dots\dots\dots, \quad y_n(1) = b_n.$$

for $y(t) = (y_1(t), y_2(t), \dots, y_n(t))^T$.

Then

$$(14) \qquad M = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \qquad N = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix} \qquad c = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ b_{r+1} \\ \vdots \\ b_n \end{pmatrix}$$

The above condition is assumed in the sequel. Assume also that

(15) $$F(y,t) = A(t)y + B(y,t)$$

(16) $$|B(y,t)| < K_1 |y|^{\alpha} + K_2 ,$$

$$0 < \alpha < 1, \quad K_1 > 0, \quad K_2 > 0.$$

The component $A(t)y$ is the linear part of $F(y,t)$, whereas $B(y,t)$ represents the nonlinear part of it. The relation (16) means that the nonlinearity grows slowly with $|y|$. (See Fig. 1.)

Suppose first that $V(t) = 0$,
then $\Phi(t,s) = I$, whence
$M + N\,\Phi(1,0) = I$, i.e.,
condition (7) is satisfied.
Moreover,

$$H(t) = I,$$

$$G(t,s) = \begin{cases} M, & 0 \le s < t \\ -N, & t < s \le 1, \end{cases}$$

therefore $|G(t,s)| \le 1$, $|H(t)| = 1$.

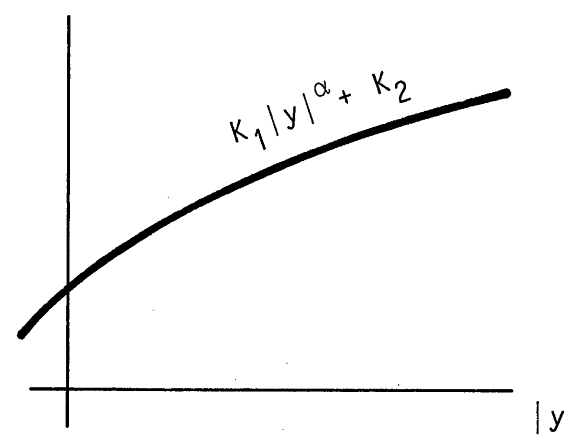

Fig. 1

Th. 2

Under the hypotheses (14), (15), and (16), the following statement is valid. If $|A(t)| < \beta$ for $\beta < 1$, then there exists a solution of the two point boundary value problem (3),(4).

(Proof) It is sufficient to verify that the condition in Th. 1 holds. Since

$$|F(y,t) - V(t)y| < \beta|y| + K_1|y|^{\alpha} + K_2$$

and $M_1 = M_2 = 1$, it follows that

$$M_1 + M_2 k(x) = \beta x + K_1 x^{\alpha} + K_2 .$$

The inequality

$$x < \beta x + K_1 x^{\alpha} + K_2$$

is transformed into

$$x < (K_1 x^{\alpha} + K_2)/(1 - \beta), \qquad 1 - \beta > 0 ,$$

from which it follows that x is bounded. Therefore Th. 1 applies and the proof is finished. Q. E. D.

Next, suppose that $V(t) = A(t)$. In this case it is not obvious that whether $\det[M + N\Phi(1,0)] \neq 0$ or not: the condition (7) should be examined individually.

Th. 3

Under the hypotheses of (14), (15), and (16), the following statement is valid. If $\det[M + N\Phi(1,0)] = 0$ for $V(t) = A(t)$, then there exists a solution of the two point boundary value problem (3),(4).

(Proof) From the equation

$$|F(y,t) - V(t)y| = |B(y,t)| < K_1 |y|^{\alpha} + K_2 ,$$

$$M_1 + M_2 k(x) = M_1 + M_2 K_2 + M_2 K_1 x^{\alpha} .$$

It is clear that the solution of

$$x < M_2 K_1 x^{\alpha} + M_1 + M_2 K_2$$

satisfies $x < m$ for some constant $m > 0$. Therefore Th. 1 shows the existence of the solution. Q. E. D.

3. A fixed point algorithm applied to a boundary value problem of first order differential equations

3.1 Results on the traceability of the zero curve of a homotopy

This section is independent of the previous one; here the consideration is devoted to an algorithmic feature.

Let us consider a subclass of the two point boundary value problems considered in the previous section:

$$(17)\qquad \begin{aligned} x' &= F(t,x,y) \\ y' &= G(t,x,y)\ , \\ x(0) &= x_0, \qquad y(T) = 0, \qquad 0 < t < T\ , \end{aligned}$$

where $x(t),\ y(t) \in E^n$; F and G are of C^2 class.

A shooting method of the solution assume a variable v for the initial value of y:

$$(18)\qquad \begin{aligned} x' &= F(t,x,y) \\ y' &= G(t,x,y)\ , \\ x(0) &= x_0, \qquad y(0) = v\ . \end{aligned}$$

Let $y(T) = f(v)$, then the equation

(19) $\qquad f(v) = 0$

must hold. Namely, the boundary value problem (17) is reduced to the solution of a nonlinear equation (19).

The continuation method [7] to solve (19) use the homotopy

(20) $$H_w(\lambda, v) = (1 - \lambda)(v - w) + \lambda f(v)$$

$$(0 \leq \lambda \leq 1) .$$

Zero curve $(\lambda, v(\lambda))$ of $H(\lambda, v(\lambda)) = 0$ should be followed from $\lambda = 0$, $v = w$ to $\lambda = 1$, $v = \bar{v}$, where it is easy to see that $f(\bar{v}) = 0$. The following proposition is a modified version of the theorem by Chow, Mallet-Paret, Yorke [8].

Prop. 5

Let $D \subset E^n$ be an open convex set isomorphic to an open sphere in E^n. Put $g(v) = -f(v) + v$. If $g(\bar{D}) \subset \bar{D}$, then the following statements (i) - (iv) are valid.

(i) There exists a $\bar{v} \in \bar{D}$ such that $f(\bar{v}) = 0$.

(ii) For almost all $w \in D$ (= Int D), the solution $(\lambda, v(\lambda))$ of $H_w(\lambda, v(\lambda)) = 0$ represents curves, each connected component of which is isomorphic to a line segment or a circle. In other words, for the solution (λ, v) of $H_w(\lambda, v) = 0$, the Jacobian $D_{[\lambda, v]} H_w$ has full rank.

(iii) A component Γ_w of $H_w(\lambda, v(\lambda)) = 0$ connects $(0, w)$ to $(1, \bar{v})$.

(iv) If $Df(\bar{v})$ is not singular, Γ_w has finite arc length.

A condition for applying the above proposition to the boundary value problem (17) is given in the following.

Th. 4

Suppose that $|y|$ denotes an arbitrary norm in E^n. Assume that there exists a constant $M > 0$ such that for arbitrary initial value $y(0) = v$ satisfying

$$(21) \qquad |v| < M,$$

the inequality

$$(22) \qquad \left| \int_0^1 G(t,x(t),y(t))dt \right| < M$$

holds, where $(x(t),y(t))$ is the solution of the initial value problem (18) with $(x(0),y(0)) = (x_0, v)$. Then $g(\bar{D}) \subset \bar{D}$ for $D = \{ v \mid |v| < M \}$ and the conclusion in Prop. 4 are valid.

(Proof)

$$|g(v)| = |f(v) - v| = |y(T) - y(0)|$$
$$= \left| \int_0^1 G(t,x(t),y(t))dt \right| < M,$$

which means $g(\bar{D}) \subset \bar{D}$. Q. E. D.

Sufficient conditions for different norms such as $|\cdot|_\infty$ and $|\cdot|_2$ so that the inequality (22) holds are as follows.

(A) $|G(t,x(t),y(t))|_\infty < M$, for any $t \in [0, 1]$

$$\left| \int_0^1 G(t,x(t),y(t))dt \right|_\infty < M .$$

(B) $|G(t,x(t),y(t))|_\infty < M/\sqrt{n}$, for any $t \in [0, 1]$

$$\left| \int_0^1 G(t,x(t),y(t))dt \right|_2 < M .$$

(C) $|G(t,x(t),y(t))|_2 < M$, for any $t \in [0, 1]$

$$\left| \int_0^1 G(t,x(t),y(t))dt \right|_2 < M .$$

Remark Watson [8] has shown a result analogous to Th. 4 for a second order differential system. If we apply his method of the proof in our case, the condition (21) can be made somewhat weaker: $|v| = M$. In application, however, the both conditions make no difference.

3.2 Computation of the derivative

Watson [9] has proposed an algorithm to compute the solution of $f(v) = 0$. He made λ a dependent variable by introducing an indepdendent variable s of arc length and considered the equation

$$\lambda(s)f(v(s)) + (1 - \lambda(s))(v(s) - w) = 0.$$

Thus the zero curve of $H_w(\lambda, v)$ is the solution of the initial value problem

$$\frac{d}{ds} H_w(\lambda(s), v(s)) = 0$$

$$\lambda(0) = 0$$

(23) $$v(0) = w$$

$$\left\| \left(\frac{d\lambda}{ds}, \frac{dv}{ds} \right) \right\|_2 = 1 .$$

In order to use standard ordinary equation solvers, the differential equation in (23) must be put in the explicit form $d\lambda/ds = H_1(s, \lambda, v)$, $dv/ds = H_2(s, \lambda, v)$. For this purpose the first equation in (23):

$$[f(v)-v+w \ , \ (1-\lambda)I + \lambda Df(v)] \begin{bmatrix} \frac{d\lambda}{ds} \\ \frac{dv}{ds} \end{bmatrix} = 0$$

must be computed and the kernel of the matrix should be found. After the matrix is computed, the algorithm of Watson [9] works.

In this computation of the matrix the most difficult part is the calculation of $Df(v)$. Let us show that $Df(v)$ is a solution of a matrix differential equation which is called sensitivity equation [10] in the control engineering. Denote the Frechet derivative of x with respect to v as $\delta x/\delta v$. Then, from (18) it follows that

$$\frac{\delta}{\delta v}\left(\frac{dx}{dt}\right) = \frac{d}{dt}\left(\frac{\delta x}{\delta v}\right) = \frac{\partial F}{\partial x}\frac{\delta x}{\delta v} + \frac{\partial F}{\partial y}\frac{\delta y}{\delta v}$$

(24)

$$\frac{\delta}{\delta v}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{\delta y}{\delta v}\right) = \frac{\partial G}{\partial x}\frac{\delta x}{\delta v} + \frac{\partial G}{\partial y}\frac{\delta y}{\delta v}$$

$$\frac{\delta}{\delta v}(\ x(0)\ ) = 0\ , \quad \frac{\delta}{\delta v}(\ y(0)\ ) = I_n$$

and

$$Df(v) = \frac{\delta}{\delta v}(\ y(t)\ )\Big|_{t=T}\ .$$

Note that $\partial F/\partial x$, $\partial F/\partial y$, $\partial G/\partial x$, and $\partial G/\partial y$ are computed along the solution $(x(t),y(t))$. Let

$$Z(t) = \begin{bmatrix} \frac{\delta x}{\delta v}(t) \\ \\ \frac{\delta y}{\delta v}(t) \end{bmatrix}, \quad L(t) = \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{bmatrix}$$

for simplicity, then

(25) $$\frac{dZ}{dt} = L(t)Z\ , \quad Z(0) = \begin{bmatrix} 0 \\ --- \\ I_n \end{bmatrix}$$

Let the fundamental solution of this system by $\Psi(t,s)$, then whence it follows that

(26)
$$Df(v) = \frac{\delta}{\delta v}(y(t))\Big|_{t=T} = (0 \vdots I_n)\Psi(T,0)\begin{bmatrix} 0 \\ --- \\ I_n \end{bmatrix}$$

Thus, the derivative Df(v) is calculated by using the sensitivity equation (24) or (25). Further, nonsingularity of $\Psi(t,s)$ (See [11].) shows that Df(v) is nonsingular. Therefore we have

Th. 5

The arc length of Γ_W in Prop. 5 for the solution (17) is finite.

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SUPPLEMENTARY NOTES