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THREE-DIMENSIONAL COMPUTER GRAPHICS WITH ORTHOGONAL MATRICES

by

Yasuhiko Ikebe

Toshiyuki Inagaki

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INSTITUTE
OF
INFORMATION SCIENCES AND ELECTRONICS

UNIVERSITY OF TSUKUBA

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Yasuhiko Ikebe

and

Toshiyuki Inagaki

Institute of Information Sciences and Electronics

University of Tsukuba, Ibaraki 305 JAPAN

ABSTRACT

This paper discusses rotations and reflections in three-dimensional (3D) computer graphics as an application of matrix algebra. We give two representation theorems in which a given 3 x 3 real orthogonal matrix L is expressed in terms of direction cosines of the axis of rotation and of the angle of rotation. It is shown that these representation theorems enables us to compute the axis of and the angle of rotation that the given matrix represents, where reflection is combined with rotation in the case of det L = -1. Several applications of our method in 3D computer graphics are included.

1. INTRODUCTION

This paper discusses rotations and reflections in three-dimensional (3D) computer graphics as an application of matrix algebra. It is well-known that a 3 x 3 real orthogonal matrix can be constructed that represents the 3D rotation about a given straight line through the origin by a given angle; see, e.g., Rogers and Adams [1, Chapter 3].

In this paper we are mainly interested in the converse problem: given the initial and the final states of a rigid body with a single fixed point (i.e., the origin), find the 3D rotation of the space which brings the given initial state of the rigid body to the given final state of the rigid body; i.e., find the axis of the rotation and the angle of the rotation in question. We will give a numerical method for computing these quantities.

To this end, we first state and prove two representation theorems (Theorem 1 and Theorem 2) each of which expresses a given real orthogonal matrix L, depending on det L = 1 or det L = -1, in terms of the direction cosines of the axis of rotation and the angle of rotation that the matrix L represents. We agree that the action of L means the mapping $v \to Lv$, where v is a column vector of order 3, regarded as the position vector emanating from the origin. We will show that the solution to the stated problem is easily derived from the representation theorems. We might mention in passing that our main representation theorems are obtained entirely without geometric intuition but in a language of matrix algebra. Applications to 3D computer graphics is then effected by interpreting the main theorems in an appropriate geometric language.

As an application of our approach to 3D computer graphics, we will state a method for mapping a given right-handed orthonormal set of vectors existing at a point in a space onto another right-handed orthonormal set of vectors existing at another point in the space, using translations and rotations.

We will also show that our matrix algebraic approach applies to cases in which 3D reflections are combined with 3D rotations.

2. REPRESENTATION THEOREMS

Let $L = (\ell_{ij})$, i, j = 1, 2, 3, be a 3 x 3 real orthogonal matrix; viz., $LL^T = L^TL = I$ where L^T denotes the transpose of L and I denotes the third order identity matrix. We give a representation theorem for L for each case that det L = 1 and det L = -1.

The first theorem connects a 3D rotation about a straight line with an orthogonal matrix whose determinant equals one.

Theorem 1. Suppose det L = 1. Then we have the following five properties:

1° L has the eigenvalue 1.

2° There exists an orthogonal matrix V with det V = 1 such that

$$(2.1)\quad V^TLV = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

3° Let $n = (n_1, n_2, n_3)^T$ denote an unit eigenvector ($n^T n = 1$) associated with the eigenvalue 1 for L. Then L is represented as follows:

$$L = nn^T(1 - \cos\theta) + I\cos\theta + N\sin\theta \tag{2.2}$$

where

$$N = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}$$

or equivalently,

$$L = \begin{bmatrix} n_1^2(1-\cos\theta)+\cos\theta & n_1n_2(1-\cos\theta)-n_3\sin\theta & n_1n_3(1-\cos\theta)+n_2\sin\theta \\ n_1n_2(1-\cos\theta)+n_3\sin\theta & n_2^2(1-\cos\theta)+\cos\theta & n_2n_3(1-\cos\theta)-n_1\sin\theta \\ n_1n_3(1-\cos\theta)-n_2\sin\theta & n_2n_3(1-\cos\theta)+n_1\sin\theta & n_3^2(1-\cos\theta)+\cos\theta \end{bmatrix} \tag{2.3}$$

4° If $L \neq I$, then the eigenvector associated with the eigenvalue 1 for L can be found among vectors c_1, c_2 and c_3 given by:

$$c_1 = \begin{bmatrix} 1+\ell_{11}-\ell_{22}-\ell_{33} \\ \ell_{12}+\ell_{21} \\ \ell_{13}+\ell_{31} \end{bmatrix},\ c_2 = \begin{bmatrix} \ell_{12}+\ell_{21} \\ 1-\ell_{11}+\ell_{22}-\ell_{33} \\ \ell_{23}+\ell_{32} \end{bmatrix},\ c_3 = \begin{bmatrix} \ell_{13}+\ell_{31} \\ \ell_{23}+\ell_{32} \\ 1-\ell_{11}-\ell_{22}+\ell_{33} \end{bmatrix} \tag{2.4}$$

whose representation in terms of n_1, n_2, n_3 and θ are given by:

$$c_1 = 2(1-\cos\theta)n_1 \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix},\ c_2 = 2(1-\cos\theta)n_2 \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix},\ c_3 = 2(1-\cos\theta)n_3 \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \tag{2.5}$$

5° The angle θ satisfies the following equations:

$$\begin{cases} \ell_{11} + \ell_{22} + \ell_{33} = 1 + 2\cos\theta \\ \ell_{21} - \ell_{12} = 2n_3\sin\theta \\ \ell_{13} - \ell_{31} = 2n_2\sin\theta \\ \ell_{32} - \ell_{23} = 2n_1\sin\theta \end{cases}$$

Proof of Theorem 1 will be given in Section 3.

The above theorem states that an orthogonal matrix L with det L = 1 performs a 3D rotation about the axis n of rotation by the positive angle θ. The fact that the straight line specified by vector n is the axis of rotation may be seen by noting that Ln = n.

The second theorem is for the case det L = -1.

Theorem 2. Suppose det L = -1. Then we have the following five properties:

1° L has the eigenvalue -1.

2° There exists an orthogonal matrix V with det V = 1 such that

$$(2.7)\quad V^T L V = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

3° Let $n = (n_1, n_2, n_3)^T$ denote a unit eigenvector ($n^T n = 1$) associated with the eigenvalue -1 for L. Then L is represented as follows:

$$(2.8)\quad L = -nn^T(1 + \cos\theta) + I\cos\theta + N\sin\theta$$

where

$$N = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}$$

or equivalently,

$$(2.9)\ L = \begin{bmatrix} -n_1^2(1+\cos\theta)+\cos\theta & -n_1n_2(1+\cos\theta)-n_3\sin\theta & -n_1n_3(1+\cos\theta)+n_2\sin\theta \\ -n_1n_2(1+\cos\theta)+n_3\sin\theta & -n_2^2(1+\cos\theta)+\cos\theta & -n_2n_3(1+\cos\theta)-n_1\sin\theta \\ -n_1n_3(1+\cos\theta)-n_2\sin\theta & -n_2n_3(1+\cos\theta)+n_1\sin\theta & -n_3^2(1+\cos\theta)+\cos\theta \end{bmatrix}$$

4° The eigenvector associated with the eigenvalue -1 for L can be found among vectors c_1, c_2 and c_3 given by:

$$(2.10)\quad c_1 = \begin{bmatrix} 1-\ell_{11}+\ell_{22}+\ell_{33} \\ -\ell_{12}-\ell_{21} \\ -\ell_{31}-\ell_{13} \end{bmatrix},\quad c_2 = \begin{bmatrix} -\ell_{12}-\ell_{21} \\ 1+\ell_{11}-\ell_{22}+\ell_{33} \\ -\ell_{23}-\ell_{32} \end{bmatrix},\quad c_3 = \begin{bmatrix} -\ell_{13}-\ell_{31} \\ -\ell_{23}-\ell_{32} \\ 1+\ell_{11}+\ell_{22}-\ell_{33} \end{bmatrix}$$

whose representation in terms of n_1, n_2, n_3 and θ are given by:

$$(2.11)\quad c_1 = 2(1+\cos\theta)n_1 \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix},\quad c_2 = 2(1+\cos\theta)n_2 \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix},\quad c_3 = 2(1+\cos\theta)n_3 \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

5° The angle θ satisfies the following equations:

$$
(2.12)\begin{cases}
\ell_{11} + \ell_{22} + \ell_{33} = -1 + 2\cos\theta \\
\ell_{21} - \ell_{12} = 2n_3\sin\theta \\
\ell_{13} - \ell_{31} = 2n_2\sin\theta \\
\ell_{32} - \ell_{23} = 2n_1\sin\theta
\end{cases}
$$

The geometric interpretation for L given by (2.8) may be stated as follows: Let $\tilde{L}$ denote an orthogonal matrix that represents a 3D rotation about the axis n of rotation by the positive angle θ. By Theorem 1 $\tilde{L}$ is given by:

$$\tilde{L} = nn^T(1 - \cos\theta) + I\cos\theta + N\sin\theta \ .$$

Now let $H = I - 2nn^T$, which is a Householder transformation [2, p.286]. For any given vector x, Hx gives the mirror image of x relative to the plane that contains the origin and is perpendicular to vector n; because of this, H is also called a reflector. A simple calculation proves that H and $\tilde{L}$ commute (i.e., $H\tilde{L} = \tilde{L}H$) and that

$$H\tilde{L} = \tilde{L}H = -nn^T(1 + \cos\theta) + I\cos\theta + N\sin\theta = L.$$

Thus matrix L given by (2.8) represents:

(1) a 3D rotation about the axis n of rotation by the positive angle θ, followed by a 3D reflection through a plane that contains the origin and is perpendicular to n, or

(2) a 3D reflection through a plane that contains the origin and is perpendicular to n, followed by a 3D rotation about the axis n of rotation by the positive angle θ, where the positive angle θ is measured about the original direction of n.

3. PROOFS OF REPRESENTATION THEOREMS

Properties 1° through 5° in Theorem 1 are proved as follows:

1° We will show that $\det(L - I) = 0$:

$$\begin{aligned}\det(L - I) &= \det(L - LL^T) = \det L \cdot \det(I - L^T) \\ &= \det L \cdot \det(I - L) = \det(I - L) \\ &= (-1)^3 \det(L - I) = -\det(L - I).\end{aligned}$$

2° Let n be an unit eigenvector associated with the eigenvalue 1 for L. It is possible to find vectors a and b so that:

(1) the triple {n, a, b} forms an orthonormal basis in the three-dimensional Euclidean space E^3, and

(2) the 3 x 3 matrix V = [n, a, b] satisfies det V = 1.

Then,

$$LV = L\,[n, a, b] = [Ln, La, Lb]. \tag{3.1}$$

Since La and Lb are expressed as linear combinations of n, a and b and Ln = n, (3.1) reduces to:

$$LV = [n, a, b]\begin{bmatrix} 1 & c & d \\ 0 & e & f \\ 0 & g & h \end{bmatrix} = V\begin{bmatrix} 1 & c & d \\ 0 & e & f \\ 0 & g & h \end{bmatrix}$$

where c, d, e, f, g, and h are scalar constants. Because

$$V^TLV = \begin{bmatrix} 1 & c & d \\ 0 & e & f \\ 0 & g & h \end{bmatrix}$$

is an orthogonal matrix, we have c = d = 0 and thus

$$L_1 = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

is an orthogonal matrix. Moreover, we have det L_1 = 1 since

$$\det L_1 = \det(V^T L V) = \det L = 1.$$

It is well-known that for a 2 x 2 real orthogonal matrix L_1 with det L_1 = 1, there exists θ such that

$$L_1 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

We thus have:

$$V^T L V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

3° For proving 3°, we need the following lemma.

<u>Lemma 1</u>. Let $A = (a_{ij})$ be an n x n real orthogonal matrix with det A = 1. Let A_{ij} denote the cofactor of a_{ij}. Then,

$$a_{ij} = A_{ij} \quad \text{for any } i \text{ and } j.$$

(Proof) By the orthogonality of $A = (a_{ij})$,

$$A^{-1} = A^T = (a_{ji}).$$

On the other hand,

$$A^{-1} = \frac{1}{\det A} (A_{ji}) = (A_{ji}) = \operatorname{adj} A$$

where adj A denotes the adjoint of A. Thus we have:

$$a_{ji} = A_{ji} \quad \text{for any } i \text{ and } j.$$

(End of Proof of Lemma 1).

We now prove property 3°. From (2.1) we have:

$$L = V \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} V^T = [n, a, b] \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} n^T \\ a^T \\ b^T \end{bmatrix}$$

which yields:

$$(3.2) \quad L = nn^T + (aa^T + bb^T) \cos\theta + (ba^T - ab^T) \sin\theta \ .$$

Since V is an orthogonal matrix,

$$I = VV^T = [n, a, b] \begin{bmatrix} n^T \\ a^T \\ b^T \end{bmatrix} = nn^T + aa^T + bb^T$$

and thus:

$$(3.3) \quad aa^T + bb^T = I - nn^T.$$

A direct calculation gives:

$$(3.4)\quad ba^T - ab^T = \begin{bmatrix} b_1a_1-a_1b_1 & b_1a_2-a_1b_2 & b_1a_3-a_1b_3 \\ b_2a_1-a_2b_1 & b_2a_2-a_2b_2 & b_2a_3-a_2b_3 \\ b_3a_1-a_3b_1 & b_3a_2-a_3b_2 & b_3a_3-a_3b_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} = N$$

where the second equality is obtained by applying Lemma 1 to $V = [n, a, b]$; viz.,

$$n_1 = \det \begin{bmatrix} a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} = a_2b_3 - a_3b_2,$$

$$n_2 = -\det \begin{bmatrix} a_1 & b_1 \\ a_3 & b_3 \end{bmatrix} = a_3b_1 - a_1b_3,$$

$$n_3 = \det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = a_1b_2 - a_2b_1, \text{ etc.}$$

Substitution of (3.3) and (3.4) into (3.2) yields (2.2). Eqn. (2.3) is obtained by expanding (2.2).

4° Equating the matrix elements ℓ_{ij} with the corresponding elements of L in (2.3) and using them in (2.4), we find the vectors c_1, c_2 and c_3 may be expressed in the form of (2.5). Eqn. (2.5) proves that each of c_1, c_2 and c_3 is a scalar multiple of n. It may be verified that $[c_1, c_2, c_3] = \mathrm{adj}(L - I)$.

5° Eqn. (2.6) is obtained from (2.3) by noting that $n^T n = 1$. This completes the Proof of Theorem 1.

Remark 1. Representation of L in (2.2) or (2.3) contains vector n but is independent of vectors a and b that are introduced to form with given n an orthonormal basis in E^3.

Remark 2. Eqn. (2.2) in Property 2° in Theorem 1 is the canonical form of a real orthogonal matrix L with det L = 1, which is obtained by applying Schur's theorem [2, p.302].

Theorem 2 can be proved in a similar manner to the case of Theorem 1, and we will omit the proof. We just note that the following lemma is required instead of Lemma 1.

<u>Lemma 2</u>. Let $A = (a_{ij})$ be an n x n real orthogonal matrix with det A = -1. Let A_{ij} denote the cofactor of a_{ij}. Then,

$$a_{ij} = - A_{ij} \quad \text{for any } i \text{ and } j.$$

(Proof) Similar to the proof of Lemma 1, and is omitted.

4. APPLICATION TO 3D COMPUTER GRAPHICS

This section shows how Theorems 1 and 2 are applied to 3D computer graphics. Some possible situations are stated below in the form of problem-solution.

<u>Problem 1</u>. Find a real orthogonal matrix L that rotates a rigid body about a given axis through the origin by the positive angle θ,

where the axis of rotacion is specified by a unit vector n.

Solution: The solution to this problem is well-known as already stated.

<u>Problem 2</u>. (Converse problem of Problem 1) Suppose we are given a real orthogonal matrix L with det L = 1. Find the unit vector n along the axis of the rotation that L represents and the corresponding positive angle θ of the rotation.

Solution: The procedure consists of three steps:

(1) We compute vectors c_1, c_2 and c_3 from eqn. (2.4). Among them choose a vector c for which

$$\|c\| = \max\{\|c_1\|, \|c_2\|, \|c_3\|\},$$

where $\|a\|$ denotes a norm of vector a that is defined here as $\|a\| = (a^T a)^{1/2}$. For this, we note the following two properties;

i) $\mathrm{adj}(L - I) = [c_1, c_2, c_3]$ is symmetric (see (2.4)), and

ii) the multiplicity of the eigenvalue 1 for matrix L ($L \neq I$) is one, and thus any non-zero vector among c_1, c_2 and c_3 can be expressed as a scalar multiple of another non-zero vector in c_1, c_2 and c_3, as demonstrated by (2.5).

Because of the above two characteristics, we can find vector c that gives the greatest norm among $\|c_1\|$, $\|c_2\|$ and $\|c_3\|$ by comparing the absolute values of elements of any one of vectors c_1, c_2 and c_3. Let c_1 be taken and let c_1 be denoted in an element-wise form as $(c_{11}, c_{21}, c_{31})^T$. An algorithm for finding vector c is stated as follows:

If $|c_{i1}| = \max\{|c_{11}|, |c_{21}|, |c_{31}|\}$,

then $\|c_i\| = \max\{\|c_1\|, \|c_2\|, \|c_3\|\}$.

(2) The unit eigenvector n that represents the axis of rotation is computed as $n = c / \|c\|$.

(3) From (2.6) we have:

$$\cos\theta = (\ell_{11} + \ell_{22} + \ell_{33} - 1) / 2.$$

An algorithm for choosing one of relations for $\sin\theta$ in (2.6) is as follows:

If $n = c_1 / \|c_1\|$, then $\sin\theta = (\ell_{32} - \ell_{23}) / (2n_1)$;

If $n = c_2 / \|c_2\|$, then $\sin\theta = (\ell_{13} - \ell_{31}) / (2n_2)$;

If $n = c_3 / \|c_3\|$, then $\sin\theta = (\ell_{21} - \ell_{12}) / (2n_3)$.

The angle θ of rotation is obtained by a standard algorithm for computing $\arccos x$ and $\arcsin x$.

Remark 3. The angle θ of rotation generally falls in the range of $0 \leq \theta \leq 2\pi$. In case of $\pi \leq \theta \leq 2\pi$, one can represent the same rotation by another angle $\theta' = 2\pi - \theta$ of rotation for which $0 \leq \theta' \leq \pi$ provided the direction of the original axis n of rotation is reversed, i.e., n is replaced by -n.

<u>Problem 3</u>. Suppose we are given:

(1) a right-handed orthonormal set of vectors (u_1, u_2, u_3) with its origin at point 0 in the three-dimensional Euclidean space E^3,

(2) another right-handed orthonormal set of vectors (v_1, v_2, v_3) that shares the origin with (u_1, u_2, u_3).

Find the rotation of the space about a fixed axis that maps (u_1, u_2, u_3) onto (v_1, v_2, v_3).

Solution: Without any loss of generality, we assume that the point O is the origin of a fixed coordinate system in E^3; i.e., the position vector of point O is a zero vector.

Let U and V be orthogonal matrices defined by:

$$U \equiv [u_1, u_2, u_3], \quad V \equiv [v_1, v_2, v_3]$$

where det U = det V = 1. Then the real orthogonal matrix L with det L = 1 can be found such that:

$$LU = V$$

or equivalently,

$$(4.1) \quad L = VU^T.$$

By Theorem 1, L represents a rotation of the space about a fixed axis. Let $n = (n_1, n_2, n_3)^T$ and θ denote, respectively, the unit vector along the axis of rotation and the angle of rotation such that L represents. The vector n and the angle θ are determined by the same procedure as given in Problem 2.

Application. We will provide here an example of direct application of Problem 3. Suppose we are given right-handed orthonormal sets of vectors (u_1, u_2, u_3) and (v_1, v_2, v_3) at two distinct points in E^3, say P and Q, respectively. Let us assume that a smooth three-dimensional curve C connects P and Q and is represented parametrically; viz., the end point R(t) of the position vector $\vec{OR}(t) = r(t) \equiv (r_1(t), r_2(t), r_3(t))^T$, $\alpha \leq t \leq \beta$, emanating from the origin 0 of a fixed coordinate system in E^3 describes the curve C, where $\vec{OP} = r(\alpha)$ and $\vec{OQ} = r(\beta)$ (see Figure 1).

Let $L = VU^T$ denote the same matrix as that was computed in Problem 3; viz.,

$$L = VU^T =$$

$$\begin{bmatrix} n_1^2(1-\cos\theta)+\cos\theta & n_1n_2(1-\cos\theta)-n_3\sin\theta & n_1n_3(1-\cos\theta)+n_2\sin\theta \\ n_1n_2(1-\cos\theta)+n_3\sin\theta & n_2^2(1-\cos\theta)+\cos\theta & n_2n_3(1-\cos\theta)-n_1\sin\theta \\ n_1n_3(1-\cos\theta)-n_2\sin\theta & n_2n_3(1-\cos\theta)+n_1\sin\theta & n_3^2(1-\cos\theta)+\cos\theta \end{bmatrix}$$

where

$$U = [u_1, u_2, u_3], \quad V = [v_1, v_2, v_3],$$

$$\det U = \det V = 1.$$

We compute $n = (n_1, n_2, n_3)^T$ and θ as in Problem 2.

Let $\overline{\theta}(t)$ and $\overline{L}(t)$ be defined as follows:

$$\overline{\theta}(t) \equiv \frac{\theta}{\beta - \alpha}(t - \alpha)$$

$$\overline{L}(t) \equiv$$

$$\begin{bmatrix} n_1(1-\cos\overline{\theta}(t))+\cos\overline{\theta}(t) & n_1n_2(1-\cos\overline{\theta}(t))-n_3\sin\overline{\theta}(t) & n_1n_3(1-\cos\overline{\theta}(t))+n_2\sin\overline{\theta}(t) \\ n_1n_2(1-\cos\overline{\theta}(t))+n_3\sin\overline{\theta}(t) & n_2^2(1-\cos\overline{\theta}(t))+\cos\overline{\theta}(t) & n_2n_3(1-\cos\overline{\theta}(t))-n_1\sin\overline{\theta}(t) \\ n_1n_3(1-\cos\overline{\theta}(t))-n_2\sin\overline{\theta}(t) & n_2n_3(1-\cos\overline{\theta}(t))+n_1\sin\overline{\theta}(t) & n_3^2(1-\cos\overline{\theta}(t))+\cos\overline{\theta}(t) \end{bmatrix}$$

It is easily checked that:

$$(4.2) \quad \overline{\theta}(\alpha) = 0,\ \overline{\theta}(\beta) = \theta; \quad \overline{L}(\alpha) = I,\ \overline{L}(\beta) = L.$$

In a commomly used homogeneous coordinate expression (e.g., see Rogers and Adams [1] where points in space are represented by row vectors instead of our column vector representation), the transformation $Z(t)$ that maps (u_1, u_2, u_3) at point P onto an orthonormal set of vectors $(w_1(t), w_2(t), w_3(t))$ with its origin at point $R(t)$ on curve C (see Figure 1) is described by:

$$Z(t) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & r_1(t) \\ 0 & 1 & 0 & r_2(t) \\ 0 & 0 & 1 & r_3(t) \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} & & & 0 \\ & \overline{L}(t) & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & -r_1(\alpha) \\ 0 & 1 & 0 & -r_2(\alpha) \\ 0 & 0 & 1 & -r_3(\alpha) \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

If we let $\tilde{w}_i(t)$, $\tilde{u}_i$ and $\tilde{v}_i$ be column vectors of order 4 defined by:

$$\tilde{w}_i(t) \equiv \begin{bmatrix} w_i(t) + r(t) \\ \hdashline 1 \end{bmatrix}, \quad \tilde{u}_i \equiv \begin{bmatrix} u_i + r(\alpha) \\ \hdashline 1 \end{bmatrix}, \quad \tilde{v}_i \equiv \begin{bmatrix} v_i + r(\beta) \\ \hdashline 1 \end{bmatrix}$$

then we have:

(4.3) $\tilde{w}_i(t) = Z(t)\, \tilde{u}_i\ , \quad \alpha \leq t \leq \beta,$

where

(4.4) $\tilde{w}_i(\alpha) = \tilde{u}_i\ , \ \tilde{w}_i(\beta) = \tilde{v}_i\ , \quad i = 1, 2, 3,$

which follows from (4.2). Roughly speaking, (4.3) and (4.4) give the mapping of the orthonormal set of vectors (u_1, u_2, u_3) onto (v_1, v_2, v_3) by continuous rotation about the fixed direction (n) combined with the continuous translation of the origin of the orthonormal set of vectors along the given curve C.

The following problems describe further cases that may happen in handling 3D reflections combined with 3D rotations.

Problem 4a. Find a real orthogonal matrix L that rotates a rigid body about the axis n of rotation by the positive angle θ and then reflects the rigid body relative to the plane that contains the origin and is perpendicular to n.

Problem 4b. Find a real orthogonal matrix L that reflects a rigid body relative to the plane that contains the origin and is perpendicular to n and then rotates the mirror image of the rigid body about the axis n of rotation by the positive angle θ.

Solutions to the above problems are obtained in a similar manner to the cases of Problems 1 and 2, and are omitted here.

5. CONCLUSIONS

We have given a matrix algebraic method suitable for studying 3D rotations and reflections in 3D computer graphics applications. In particular, a numerical method for finding the axis of the rotation that a given orthogonal matrix represents and the corresponding angle of the rotation is given.

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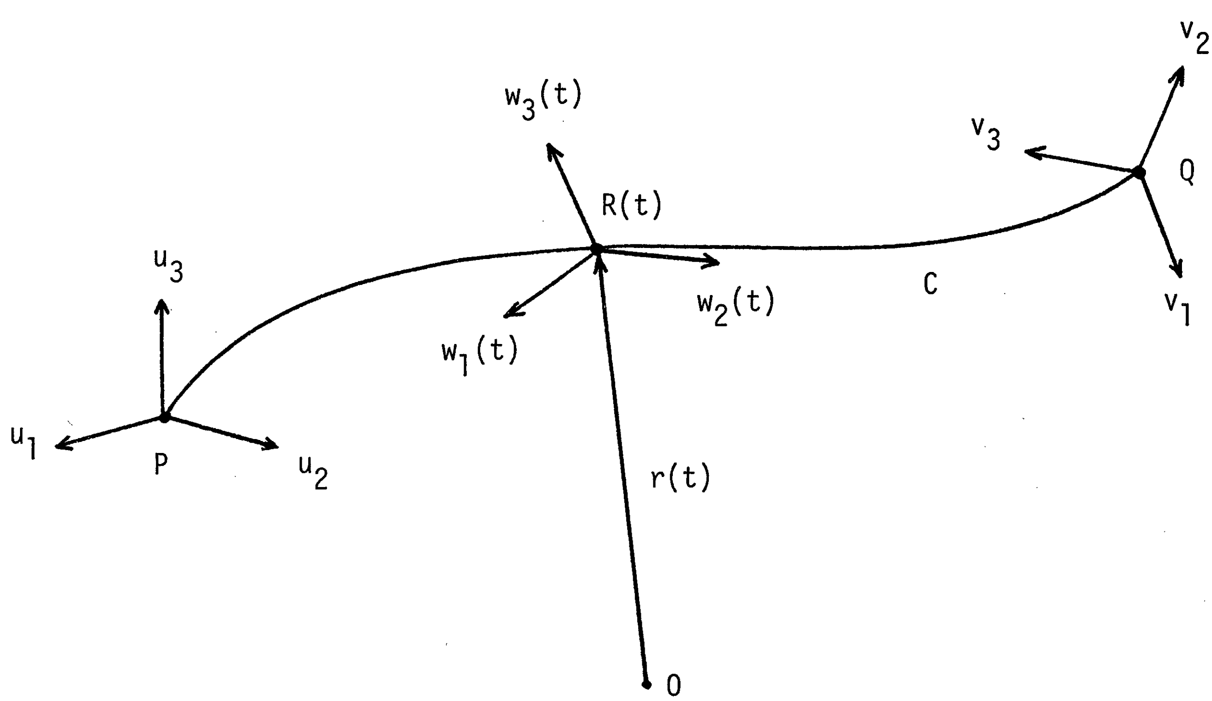

Figure 1. Orthonormal sets of vectors with their origins on curve C

INSTITUTE OF INFORMATION SCIENCES AND ELECTRONICS
UNIVERSITY OF TSUKUBA
SAKURA-MURA, NIIHARI-GUN, IBARAKI 305 JAPAN

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Three-Dimensional Computer Graphics with Orthogonal Matrices

AUTHOR(S)

Yasuhiko Ikebe (Institute of Information Sciences and Electronics)
Toshiyuki Inagaki (Institute of Information Sciences and Electronics)

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ABSTRACT

This paper discusses rotations and reflections in three-dimensional (3D) computer graphics as an application of matrix algebra We give two representation theorems in which a given 3 x 3 real orthogonal matrix L is expressed in terms of direction cosines of the axis ot rotation and of the angle of rotation. It is shown that these representation theorems enables us to compute the axis of and the angle of rotation that the given matrix represents, where reflection is combined with rotation in the case of det L = -1. Several applications of our method in 3D computer graphics are included.

SUPPLEMENTARY NOTES