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THE NUMERICAL SOLUTION OF THE INTEGRAL EQUATION FORMULATION OF RELIABILITY QUANTIFICATION

by

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ABSTRACT

This paper gives a method for quantitative evaluation of reliability parameters of components which are essential information for probabilistic evaluation of system reliability and safety. We take a linear operator theoretic approach in constructing our method for solving a linear system of integral equations which govern the time-dependent behavior of the reliability parameters.

1. INTRODUCTION

Probabilistic evaluation of system reliability and safety is essential for designing and upgrading systems, optimization of inspection and repair policy, etc. Reliability characteristics of a system are completely represented in terms of reliability and maintainability parameters of components whether the system is coherent [3, p.6] or non-coherent (a typical example of a non-coherent system can be found in computer-controlled systems with sensor systems; for example, see [4]). It is necessary to compute reliability parameters of components with high precision of accuracy for precise evaluation of system reliability and safety.

This paper gives a method for quantitative evaluation of reliability parameters (more explicitly, unconditional failure and repair intensities) of a system component. We take a linear operator theoretic approach in solving a linear system of Volterra integral equations of the second kind which represents the time-dependent behavior of the reliability parameters of a component. We give a method for solving the system of integral equations numerically, where (i) covergence of a numerical solution of an exact solution, and (ii) convergence of typical iterative methods are proved.

2. PROBLEM STATEMENT

System unavailability $A_s(t)$ at a specified time point t and the expected number of failures $W_s[0,T]$ in the prescribed time interval[0,T] are essential quantities for safety and reliability evaluation of a system. Methodologies for computing $A_s(t)$ and $W_s[0,T]$ are established by Vesely[2] for coherent systems[3,p.6] and by Inagaki and Henley[4] for non-coherent systems. Both of these theories assume as their fundamental information reliability parameters w(t) and v(t) of each of system components where

- w(t) : unconditional failure intensity at time t; viz. w(t)dt is the expected number of failures of a component during time interval (t,t+dt]
- v(t) : unconditional repair intensity at time t; viz. v(t)dt is the expected number of repairs of a component during time interval (t,t+dt].

It is important to evaluate w(t) and v(t) at high precision of accuracy so that $Q_s(t)$ and $W_s[0,T]$ can be obtained accurately.

The following linear system of integral equations relates unknown parameters w(t) and v(t) with known parameters f(t) and g(t) [1,p.193]:

$$(1)\qquad \begin{cases} w(t) - \int_0^t f(t-u)v(u)du = f(t) \\ v(t) - \int_0^t g(t-u)w(u)du = 0 \quad , \quad 0 \le t \le T \end{cases}$$

where f(t) and g(t) are probability densities for the first failure time and the repair time of a component, respectively. It is usually difficult to solve (1) analytically except for the case in which f(t) and g(t) are probability densities for exponential distributions. In this paper we study a method for solving (1) numerically for the case in which f(t) and g(t) are arbitrary probability densities.

3. OPERATOR EQUATION

We reformulate (1) as an operator equation. Let C[0,T] denote the Banach space of real-valued continuous functions on [0,T] with sup-norm

$$(2)\qquad ||\phi|| = \max \{ |\phi(t)| : 0 \le t \le T \}$$

for ϕ in C[0,T]. Let A_f and A_g denote integral operators defined as:

$$(3)\qquad A_f\phi = \int_0^t f(t-u)\phi(u)du = \int_0^t K_f(t,u)\phi(u)du$$

$$(4)\qquad A_g\phi = \int_0^t g(t-u)\phi(u)du = \int_0^t K_g(t,u)\phi(u)du$$

where

$$(5)\qquad K_f(t,u) = \begin{cases} f(t-u), & t \ge u \\ 0 \quad , & \text{otherwise} \end{cases}$$

$$(6)\qquad K_g(t,u) = \begin{cases} g(t-u), & t \ge u \\ 0 \quad , & \text{otherwise} \end{cases}$$

Let $(\phi,\psi)^T$ be a column vector for ϕ and ψ in C[0,T]. The space of all such vectors gives a Banach space X with norm defined by:

$$(7)\qquad ||(\phi,\psi)^T|| = \max \{ ||\phi||, ||\psi|| \}.$$

Let L denote an operator on X defined as:

$$(8) \qquad L = \begin{pmatrix} I & -A_f \\ -A_g & I \end{pmatrix}$$

where I denotes an identity operator for which $I\phi = \phi$. Then (1) is written as

$$(9) \qquad Lx = b$$

where $x = (w(t), v(t))^T$ (unknown) and $b = (f(t), 0)^T$ (known). Ikebe and Inagaki [5] showed that (1) is well-conditioned (i.e. the condition number $\mathrm{cond}(L) = \|L\| \cdot \|L^{-1}\|$ is not very large) irrespective of the value of $G(T) = \int_0^T g(t)dt$ if $F(T) = \int_0^T f(t)dt$ is not close to 1 and $F(T) \ll G(T)$, where the last condition is usually valid.

4. CONVERGENCE OF AN APPROXIMATE SOLUTION

Let P_n $(n=1,2,...)$ denote a bounded linear projection $(P_n^2 = P_n)$ of $C[0,T]$ onto an n-dimensional subspace S_n of $C[0,T]$, where

$$(10) \qquad \|P_n\phi - \phi\| \to 0 \qquad (n \to \infty)$$

for every ϕ in $C[0,T]$. We discretize (1) as:

$$(11) \qquad \begin{pmatrix} I & -P_nA_f \\ -P_nA_g & I \end{pmatrix} \begin{pmatrix} w_n \\ v_n \end{pmatrix} = \begin{pmatrix} P_nf \\ 0 \end{pmatrix}$$

Then we obtain the following lemma.

LEMMA 1. Assume that P_n satisfies (10), then

(a) $\sup_n \|P_n\| < +\infty$

(b) $\|P_nA_f\phi - A_f\phi\| \to 0$ and $\|P_nA_g\phi - A_g\phi\| \to 0$

for every $\phi \in C[0,T]$

(c) $\|P_nA_f - A_f\| \to 0$ and $\|P_nA_g - A_g\| \to 0$

(d) $(I - P_nA_fP_nA_g)^{-1}$ exists for sufficiently large n.

For the proof of LEMMA 1, see Appendix 1.

The last property (d) means that (11) is uniquely solvable for sufficiently large n:

$$(12)\quad \begin{pmatrix} w_n \\ v_n \end{pmatrix} = \begin{pmatrix} (I-P_nA_fP_nA_g)^{-1} & P_nA_f(I-P_nA_gP_nA_f)^{-1} \\ P_nA_g(I-P_nA_fP_nA_g)^{-1} & (I-P_nA_gP_nA_f)^{-1} \end{pmatrix} \begin{pmatrix} P_nf \\ 0 \end{pmatrix}$$

Then we have the following theorem of convergence.

THEOREM 1. Suppose that P_n satisfies (10). Then,

$$w_n \in S_n, \qquad v_n \in S_n,$$

and

$$(13)\quad \begin{aligned} &w_n \to w \text{ in } C[0,T] \\ &v_n \to v \text{ in } C[0,T] \end{aligned}$$

where (w,v) is the solution of the original equation (9).

For the proof of THEOREM 1, see Appendix II.

5. MATRIX EQUATION

Let $\{e_i: i=1,...,n\}$ be a Schauder basis for subspace S_n of $C[0,T]$. Then P_n can be written as

$$(14)\quad P_n = e_1e_1^* + \cdots + e_ne_n^*, \qquad e_i^*e_j = \delta_{ij}$$

where $e_1^*, \cdots, e_n^*$ are linear functionals defined by

$$(15)\quad e_1^*(\phi)e_1 + \cdots + e_n^*(\phi)e_n = P_n\phi$$

for any ϕ in $C[0,T]$ (for detail, see [7]).

For every n, $n=1,2,\cdots$, we have

$$(16)\quad w_n = \sum_{i=1}^{n} c_ie_i, \qquad v_n = \sum_{i=1}^{n} d_ie_i$$

where c_i and d_i are real numbers depending on w and v, respectively. By substituting (14) and (16) to (11), we obtain the following linear system of equations with 2n unknowns $\{c_i: i=1,\cdots,n\}$ and $\{d_i: i=1,\cdots,n\}$.

$$(17\text{a})\quad c_i - \sum_{j=1}^{n} e_i^*A_fe_j\cdot d_j = e_i^*f, \qquad i=1,\cdots,n$$

$$(17b) \qquad d_i - \sum_{j=1}^{n} e_i{*}A_g e_j \cdot c_j = 0, \qquad i=1,\cdots,n$$

In matrix form, (17) can be written as

$$(18) \qquad \begin{aligned} &\begin{pmatrix} c_1 \\ \vdots \\ c_i \\ \vdots \\ c_n \end{pmatrix} - \begin{pmatrix} & & \vdots & & \\ e_i{*}A_f e_1 & \cdots & e_i{*}A_f e_j & \cdots & e_i{*}A_f e_n \\ & & \vdots & & \end{pmatrix} \begin{pmatrix} d_1 \\ \vdots \\ d_j \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} e_1{*}f \\ \vdots \\ e_i{*}f \\ \vdots \\ e_n{*}f \end{pmatrix} \\ &\begin{pmatrix} d_1 \\ \vdots \\ d_i \\ \vdots \\ d_n \end{pmatrix} - \begin{pmatrix} & & \vdots & & \\ e_i{*}A_g e_1 & \cdots & e_i{*}A_g e_j & \cdots & e_i{*}A_g e_n \\ & & \vdots & & \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_j \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix} \end{aligned}$$

It is easy to solve (18) by the Gaussain elimination method or by an iterative method.

Suppose we solve (18) iteratively by the point Jacobi method or by the Gauss-Seidel method. Then we have the following theorem.

THEOREM 2. The point Jacobi method and the Gauss-Seidel method converges, if

$$(19) \qquad \|P_n\|^2 F(T)G(T) < 1.$$

For the proof, see Appendix III.

APPENDIX I: Proof of LEMMA 1.

Property (a) follows from the uniform boundedness theorem [6].

The proof of (b) is obvious from the assumption on P_n, since $A_f\phi \in C[0,T]$.

Next, since A_f and A_g are completely continuous, the argument in [7, p.470] can be applied and we have (c).

Finally, the relation (c) means that there exists an $N > 0$ such that

$$(A.1) \qquad \|P_n A_f\| \leq \alpha < 1 \quad \text{and} \quad \|P_n A_g\| \leq \beta < 1$$

for all $n \geq N$, since $\|A_f\| = F(T) < 1$, $\|A_g\| = G(T) < 1$ (see Ikebe and Inagaki [5]), and

$$\text{(A.2)}\qquad \begin{aligned} \Big| \, \|P_nA_f\| - \|A_f\| \, \Big| &< \|P_nA_f - A_f\| \to 0 \\ \Big| \, \|P_nA_g\| - \|A_g\| \, \Big| &< \|P_nA_g - A_g\| \to 0 \end{aligned}$$

as $n \to \infty$. Therefore,

$$\text{(A.3)}\qquad \|(I - P_nA_fP_nA_g)^{-1}\| \leq (1 - \|P_nA_fP_nA_g\|)^{-1} \leq (1 - \alpha\beta)^{-1}.$$

End of proof of LEMMA 1.

APPENDIX II: Proof of THEOREM 1.

First property follows immediately from

$$w_n = P_nA_fv_n + P_nf \in S_n$$
$$v_n = P_nA_gw_n \in S_n$$

Next, Suppose that n is sufficiently large and (11) is solvable, then

$$\text{(A.4)}\qquad \begin{pmatrix} w_n - w \\ v_n - v \end{pmatrix} = \begin{pmatrix} (I-A_fA_g)^{-1}f-(I-P_nA_fA_g)^{-1}P_nf \\ A_g(I-A_fA_g)^{-1}f-P_nA_g(I-P_nA_fP_nA_g)^{-1}P_nf \end{pmatrix}$$

Hence

$$\begin{aligned} \text{(A.5)}\qquad \|w_n - w\| &\leq \|(I-A_fA_g)^{-1}\| \, \|P_nf-f\| \\ &\quad + \|(I-A_fA_g)^{-1}- (I-P_nA_fA_g)^{-1}\| \, \|P_nf\| \\ &\leq \|(I-A_fA_g)^{-1}\| \, \|P_nf-f\| \\ &\quad + \|(I-P_nA_fP_nA_g)^{-1}\| \, \|P_nA_fP_nA_g-A_fA_g\| \, \|(I-A_fA_g)^{-1}\| \, \|P_nf\|. \end{aligned}$$

Application of (A.3), $\|A_f\| = F(T)$, $\|A_g\| = G(T)$, and the inequality

$$\text{(A.6)}\qquad \|P_nA_fP_nA_g-A_fA_g\| \leq \|P_nA_f-A_f\| \, \|P_nA_g\| + \|P_nA_g-A_g\| \, \|A_f\|$$

leads to

$$\begin{aligned} \text{(A.7)}\qquad \|w_n-w\| &\leq (1-F(T)G(T))^{-1}(\|P_nf-f\|+(1-\alpha\beta)^{-1}\|P_nf\| \, (\beta\|P_nA_f-A_f\| \\ &\quad + G(T)\|P_nA_g-A_g\|)). \end{aligned}$$

Thus $w_n \to w$ in $C[0,T]$ as $n \to \infty$.

Similarly, we have

(A.8) $$\|v_n - v\| \le (1-F(T)G(T))^{-1}(\|P_nA_g - A_g\|\ \|f\| + \beta(\|p_nf-f\| + (1-\alpha\beta)^{-1}\|P_nf\|(\|P_nA_f - A_f\| + G(T)\|P_nA_g - A_g\|))).$$

Hence $v_n \to v$ in $C[0,T]$ as $n \to \infty$. End of proof of THEOREM 1.

APPENDIX III: Proof of THEOREM 2.

The linear system of equations (18) $\underline{c} = (c_1, \ldots, c_n)^T$ and $\underline{d} = (d_1, \ldots, d_n)^T$ can be rewritten as follows.

(A.9) $$\begin{pmatrix} I & -F_n \\ -G_n & I \end{pmatrix} \begin{pmatrix} \underline{c} \\ \underline{d} \end{pmatrix} = \begin{pmatrix} \underline{b} \\ 0 \end{pmatrix}$$

where F_n and G_n are $n \times n$ square matrix the (i,j)-th element of which are $e_i{*}A_fe_j$ and $e_i{*}A_ge_j$, respectively, and $\underline{b} = (e_1{*}f, \ldots, e_n{*}f)^T$. The point Jacobi method for (A.9) is represented as:

(A.10) $$\begin{pmatrix} \underline{c}^{(k+1)} \\ \underline{d}^{(k+1)} \end{pmatrix} = \begin{pmatrix} 0 & F_n \\ G_n & 0 \end{pmatrix} \begin{pmatrix} c^{(k)} \\ d^{(k)} \end{pmatrix} + \begin{pmatrix} \underline{b} \\ 0 \end{pmatrix} = B \begin{pmatrix} \underline{c}^{(k)} \\ 0 \end{pmatrix} + \begin{pmatrix} \underline{b} \\ 0 \end{pmatrix}$$

Let λ be an eigenvalue of the point Jacobi matrix B. Then

(A.11) $$0 = \det \begin{pmatrix} -\lambda I & F_n \\ G_n & -\lambda I \end{pmatrix} = \det \begin{pmatrix} -\lambda I & F_n \\ 0 & -\lambda I + \lambda^{-1} G_n F_n \end{pmatrix}$$
$$= (-1)^n \det(G_nF_n - \lambda^2 I)$$

Let us consider a particular norm of an n-vector x $=(x_1, x_2, \ldots, x_n)^T$ defined by $\|x\| = \|\sum_i x_ie_i\|$ = a norm of $\sum_i x_ie_i$ as a function in $C[0,T]$. Then the matrix norm of F_n and G_n are given by

$$\|F_n\| \doteq \sup_{\|x\|=1} \|F_nx\|.$$

It is obvious from (A.11) that

(A.12) $$|\lambda^2| \le \|G_nF_n\| \le \|G_n\|\ \|F_n\|,$$

where

(A.13) $$\|F_n\| \le \|P_nA_f\|, \qquad \|G_n\| \le \|P_nA_g\|.$$

The first inequality, for example, is proved as follows.

$$\|F_n\| = \sup_{\|x\|=1} \|F_n x\| = \sup_{\|\sum_i x_i e_i\|=1} \|(P_n A_f)(\sum_i x_i e_i)\|$$

$$= \sup_{\|P_n\phi\|=1} \|P_n A_f P_n \phi\| \leq \sup_{\|\phi\|=1} \|P_n A_f \Phi\| = \|P_n A_f\|.$$

By (A.11) and (A.13),

$$\text{(A.14)} \qquad |\lambda| \leq \|P_n\| \sqrt{\|A_f\| \|A_g\|} = \|P_n\| \sqrt{G(T)F(T)} .$$

If the last quantity is less than one, then $|\lambda| < 1$, which means that the point Jacobi method converges.

For the Gauss-Seidel method, iteration scheme is represented as

$$\underline{c}^{(k+1)} = F_n \underline{d}^{(k)} + \underline{b}$$

$$\underline{d}^{(k+1)} = G_n \underline{c}^{(k+1)}$$

Then the eigenvalue of the iteration matrix is given by

$$\det \begin{pmatrix} -\lambda I & F_n \\ 0 & -\lambda I + G_n F_n \end{pmatrix} = 0.$$

Hence it follows that the eigenvalues of the iteration matrix of the Gauss-Seidel methods are exactly the square of those of the Jacobi method and n zeros:

$$|\lambda| \leq \|G_n F_n\| \leq \|P_n\|^2 \|A_f\| \|A_g\| = \|P_n\|^2 F(T) G(T).$$

From the last relation it is clear that the condition (19) is sufficient for the convergence of the Gauss-Seidel method. End of proof of THEOREM 2.

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SUPPLEMENTARY NOTES