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AN OPERATOR THEORETIC METHOD OF ERROR ANALYSIS FOR RELIABILITY QUANTIFICATION

by

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ABSTRACT

This paper develops a method for evaluating the magnitude of errors which may arise in numerical computation of probabilistic parameters for system safety and reliability quantification. Our method is constructed on the basis of the theory of linear operators in functional analysis. Reliability theoretic interpretation for our main mathematical result is presented.

1. INTRODUCTION

Probabilistic evaluation of reliability parameters is essential in system safety and reliability quantification. For coherent systems [1], the computer program KITT basing on the theory by Vesely [2] is widely used to calculate reliability parameters such as availability, expected number of failures, etc. Inagaki & Henley [3] developed recently a new probabilistic method for evaluating reliability parameters of non-coherent systems to which the Vesely's theory does not apply.

It is necessary to obtain reliability parameters with high precision since the parameters are basic information for designing or upgrading systems, inspection scheduling, repair policy determination, etc. The authenticity of the obtained reliability parameters is dependent on (i) the degree of uncertainty contained in data which are used in assessing reliability parameters, and/or (ii) the magnitude of errors which may occur in numerical computation by use of computer.

The problems of data uncertainty are discussed in WASH-1400 [4], in which a statistical approach (Monte Carlo simulation) is adopted for evaluating the effect of error caused by data uncertainty to system reliability characteristics. The error analysis of numerical computation, on the contrary, has not been performed.

This paper develops a method for quantitative evaluation of the magnitude of errors which may arise in computing reliability parameters numerically. We take a linear operator theoretic approach in constructing our method for evaluating errors. The developed method have a potential applicability in the data uncertainty problems. This suggests that we will have a non-statistical (i.e. deterministic) approach as well as a statistical approach which has already been developed in [4] for the problems.

2. INTEGRAL EQUATIONS

System safety and reliability characteristics are completely represented in terms of component reliability parameters whether the system is coherent or non-coherent. It is known that reliability parameters of every component are governed by the following simultaneous system of integral equations (see, e.g. [5]):

$$\left.\begin{aligned} w(t) &= f(t) + \int_0^t f(t-u)v(u)du \\ v(t) &= \int_0^t g(t-u)w(u)du \end{aligned}\right\} \qquad (1)$$

where:

- w(t): unconditional failure intensity at time t; viz. the s-expected[†] number of times the failure of component occurs at time t per unit time
- v(t): unconditional repair intensity at time t; viz. the s-expected number of repairs of component completed at time t per unit time
- f(t): probability density function for first failure of component; viz. f(t)dt is the probability that the first component failure occurs during the small interval [t, t+dt), given that the component was like new at time zero
- g(t): probability density function for repair; viz. g(t)dt is the probability that component repair is completed during [t, t+dt), given that the component failed at time zero

Integral equations in the form of (1) are called "Volterra equations." Let A and B denote integral operators called "Volterra operators" of convolution type defined as follows:

$$A(\cdot) = \int_0^t g(t-u)(\cdot)du, \text{ i.e. } A(w(t)) = \int_0^t g(t-u)w(u)du \quad (2)$$

$$B(\cdot) = \int_0^t f(t-u)(\cdot)du, \text{ i.e. } B(v(t)) = \int_0^t f(t-u)v(u)du \quad (3)$$

† The abbreviation "s-" implies "statistical(ly)".

Then (1) can be written as follows:

$$\begin{pmatrix} w(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} f(t) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & \int_0^t f(t-u)v(u)du \\ \int_0^t g(t-u)w(u)du & 0 \end{pmatrix}$$

$$= \begin{pmatrix} f(t) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & \int_0^t f(t-u)(\cdot)du \\ \int_0^t g(t-u)(\cdot)du & 0 \end{pmatrix} \begin{pmatrix} w(t) \\ v(t) \end{pmatrix}$$

$$= \begin{pmatrix} f(t) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix} \begin{pmatrix} w(t) \\ v(t) \end{pmatrix} \tag{4}$$

By introducing the identity operator I, i.e. $I(w(t)) = w(t)$, $I(v(t)) = v(t)$, we obtain more compact representation for (4); viz.

$$\begin{pmatrix} I & -B \\ -A & I \end{pmatrix} \begin{pmatrix} w(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} f(t) \\ 0 \end{pmatrix} \tag{5a}$$

or

$$Lx = b \tag{5b}$$

where

$$L = \begin{pmatrix} I & -B \\ -A & I \end{pmatrix}, \quad x = \begin{pmatrix} w(t) \\ v(t) \end{pmatrix}, \quad b = \begin{pmatrix} f(t) \\ 0 \end{pmatrix}$$

3. POSSIBLE ERRORS IN NUMERICAL COMPUTATION

It is difficult to obtain an exact solution x of (5b) because of:

1) data uncertainty which arises by shortage of enough amount of field data for assessing the true f(t) or g(t)

2) truncate error in representing real f(t) or g(t) in terms of tractable analytic functions

3) round-off error in computation

4) use of approximate formulae for numerical integration, e.g. trapezoidal rule.

Errors 1) and 2) occur in data representation, and 3) and 4) occur in numerical computation. Because of these errors, parameters in (5b) are actually $L+\Delta L$ and $b+\Delta b$ instead of L and b, respectively. Thus the integral equation which we actually solve is written in the form:

$$(L+\Delta L)(x+\Delta x) = b+\Delta b \tag{6}$$

where $x+\Delta x$ is the exact solution of (6).

An essential problem is to assess the magnitude of the deviation Δx from the exact solution x of the ideal equation (5b) in terms of ΔL and Δb. By expanding (6), we obtain:

$$\Delta x = (L+\Delta L)^{-1} \; (-(\Delta L)x + \Delta b)$$

$$= (I + L^{-1} \Delta L)^{-1} L^{-1} (-(\Delta L)x + \Delta b) \qquad (7)$$

Thus

$$\|\Delta x\| \leq \frac{\|L^{-1}\|}{1 - \|L^{-1} \Delta L\|} (\|\Delta L\| \cdot \|x\| + \|\Delta b\|)$$

$$\leq \frac{\|L^{-1}\|}{1 - \|L^{-1}\| \cdot \|\Delta L\|} (\|\Delta L\| \cdot \|x\| + \|\Delta b\|) \qquad (8)$$

Therefore

$$\frac{\|\Delta x\|}{\|x\|} \leq \frac{\|L^{-1}\| \cdot \|L\|}{1 - \|L^{-1}\| \cdot \|\Delta L\|} \left(\frac{\|\Delta L\|}{\|L\|} + \frac{\|\Delta b\|}{\|L\| \cdot \|x\|} \right)$$

$$\leq \frac{\text{cond}(L)}{1 - \text{cond}(L) \cdot \frac{\|\Delta L\|}{\|L\|}} \left(\frac{\|\Delta L\|}{\|L\|} + \frac{\|\Delta b\|}{\|b\|} \right) \qquad (9)$$

where cond(L) is the "condition number of L" which is defined as follows [6]:

$$\text{cond}(L) = \|L\| \cdot \|L^{-1}\| \qquad (10)$$

(9) tells us the relationship among norms of relative errors

$\|\Delta x\|/\|x\|$, $\|\Delta L\|/\|L\|$ and $\|\Delta b\|/\|b\|$. Also says (9) that cond(L) governs the precision of the solution; viz. if cond(L) is small, then solution x is insensitive to the parameter deviation ΔL or Δb; if cond(L) is large, on the contrary, then x may be sensitive to ΔL or Δb and it may be hard to obtain a high quality solution by use of any sophisticated solution method even if the integral equation (5b) has slight parameter errors.

4. EVALUATION OF cond(L)

4.1 Upper Bound for cond(L)

Now let us evaluate $\|L\|$ and $\|L^{-1}\|$, and thus cond(L). Operator norm $\|L\|$ is defined as:

$$\|L\| = \sup_{\|z\|=1} \|Lz\| \tag{11}$$

where vector norm $\|z\| = \|\text{col.}(z_1, z_2)\|$ is taken as the "maximum norm" which is computationally tractable; viz.

$$\|z\| = \max\{\|z_1\|, \|z_2\|\} \tag{12}$$

Then $\|L\|$ is evaluated as follows:

$$\| L \| = \sup_{\|z\|=1} \| \mathrm{col.}(z_1 - Bz_2, z_2 - Az_1) \|$$

$$= \sup_{\|z\|=1} \max\{\|z_1 - Bz_2\|, \|z_2 - Az_1\|\}$$

$$\leq \sup_{\|z\|=1} \max\{\|z_1\| + \|Bz_2\|, \|z_2\| + \|Az_1\|\}$$

$$\leq \sup_{\|z\|=1} \max\{\|z_1\| + \|B\|\cdot\|z_2\|, \|z_2\| + \|A\|\cdot\|z_1\|\}$$

$$= \max\{ 1 + \|B\| , 1 + \|A\| \} \qquad (13)$$

Let us proceed to evaluating $\|L^{-1}\|$. The inverse operator L^{-1} is given by:

$$L^{-1} = \begin{pmatrix} (I - BA)^{-1} & B(I - AB)^{-1} \\ A(I - BA)^{-1} & (I - AB)^{-1} \end{pmatrix} \qquad (14)$$

as is easily verified, where $LL^{-1} = L^{-1}L = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$. Then $\|L^{-1}\|$ is evaluated as follows:

$$\|L^{-1}\| = \sup_{\|z\|=1} \| L^{-1}z \|$$

$$= \sup_{\|z\|=1} \left\| \begin{pmatrix} (I - BA)^{-1}z_1 + B(I - AB)^{-1}z_2 \\ A(I - BA)^{-1}z_1 + (I - AB)^{-1}z_2 \end{pmatrix} \right\|$$

$$= \sup_{\|z\|=1} \max\{ \|(I-BA)^{-1}z_1+B(I-AB)^{-1}z_2\|, \|A(I-BA)^{-1}z_1+(I-AB)^{-1}z_2\|\}$$

$$\leq \sup_{\|z\|=1} \max\{ \|(I-BA)^{-1}z_1\| + \|B(I-AB)^{-1}z_2\|, \|A(I-BA)^{-1}z_1\| + \|(I-AB)^{-1}z_2\|\}$$

$$\leq \sup_{\|z\|=1} \max\{ \|(I-BA)^{-1}\| \cdot \|z_1\| + \|B(I-AB)^{-1}\| \cdot \|z_2\| , \|A(I-BA)^{-1}\| \cdot \|z_1\| + \|(I-AB)^{-1}\| \cdot \|z_2\| \}$$

$$= \max\{ \|(I-BA)^{-1}\| + \|B(I-AB)^{-1}\|, \|A(I-BA)^{-1}\| + \|(I-AB)^{-1}\|\} \quad (15)$$

By applying the following inequalities

$$\begin{aligned}
&\|(I-BA)^{-1}\| \leq 1/(1 - \|BA\|) \leq 1/(1 - \|B\| \cdot \|A\|) \\
&\|(I-AB)^{-1}\| \leq 1/(1 - \|AB\|) \leq 1/(1 - \|A\| \cdot \|B\|) \\
&\|A(I-BA)^{-1}\| \leq \|A\|/(1 - \|BA\|) \leq \|A\|/(1 - \|B\| \cdot \|A\|) \\
&\|B(I-AB)^{-1}\| \leq \|B\|/(1 - \|AB\|) \leq \|B\|/(1 - \|A\| \cdot \|B\|)
\end{aligned} \quad (16)$$

to (15), $\|L^{-1}\|$ is further evaluated as:

$$\|L^{-1}\| \leq \max\left\{ \frac{1 + \|B\|}{1 - \|A\| \cdot \|B\|}, \frac{1 + \|A\|}{1 - \|A\| \cdot \|B\|} \right\} \quad (17)$$

Thus cond(L) is evaluated as follows:

$$\text{cond}(L) = \|L\| \cdot \|L^{-1}\|$$

$$\leq [\max\{1+\|B\|,\ 1+\|A\|\}] \cdot [\max\{\frac{1+\|B\|}{1-\|A\|\cdot\|B\|},\ \frac{1+\|A\|}{1-\|A\|\cdot\|B\|}\}]$$

$$= \max\{\frac{(1+\|A\|)^2}{1-\|A\|\cdot\|B\|},\ \frac{(1+\|B\|)^2}{1-\|A\|\cdot\|B\|}\} \qquad (18)$$

4.2 Evaluating $\|A\|$ and $\|B\|$

We now need norms $\|A\|$ and $\|B\|$ of Volterra operators A and B defined by (2) and (3), respectively. To this end, we briefly turn to another class of integral operators called "Fredholm operators." We have the following well-known fact [7]: "If the Fredholm operator

$$C(\cdot) = \int_a^b K(t,u)(\cdot)du, \text{ i.e. } C(y(t)) = \int_a^b K(t,u)y(u)du \qquad (19)$$

has kernel K(t,u) which is continuous in t and u, then the norm of operator C is given by:

$$\|C\| = \max_{a \leq t \leq b} \int_a^b |K(t,u)|du \text{ "} \qquad (20)$$

It is noted that the Volterra operator

$$D(\cdot) = \int_a^t K(t,u)(\cdot)du, \text{ i.e. } D(y(t)) = \int_a^t K(t,u)y(u)du \quad (21)$$

can be transferred to a Fredholm operator by redefining the kernel K(t,u); viz. if we introduce

$$\tilde{K}(t,u) = \begin{cases} K(t,u), & a \leq u \leq t \\ 0, & t < u \leq b \end{cases} \quad (22)$$

then

$$D(\cdot) = \int_a^b K(t,u)(\cdot)du, \quad t \leq b \quad (23)$$

Thus derived Fredholm operator D has kernel $\tilde{K}(t,u)$ with discontinuity on t = u. In the similar manner to the case of the Fredholm operator C with continuous kernel, it is shown that the norm of operator D with non-negative kernel K(t,u) is given by:

$$\| D \| = \max_{a \leq t \leq b} \int_a^b \tilde{K}(t,u)du \quad (24)$$

(for the proof, see APPENDIX).

Thus we can determine $\|A\|$ and $\|B\|$ as follows:

$$\| A \| = \max_{0 \leq t \leq T} \int_0^T \tilde{g}(t-u)du \quad (25)$$

$$\| B \| = \max_{0 \leq t \leq T} \int_0^T f(t-u)du \tag{26}$$

where

$$\tilde{g}(t-u) = \begin{cases} g(t-u), & 0 \leq u \leq t \\ 0, & t < u \leq T \end{cases} \tag{27}$$

$$\tilde{f}(t-u) = \begin{cases} f(t-u), & 0 \leq u \leq t \\ 0, & t < u \leq T \end{cases} \tag{28}$$

and T is a constant such that $t \leq T$.

It is useful to note that:

$$\int_0^T \tilde{g}(t-u)du = \int_0^t g(t-u)du = G(t) \tag{29}$$

$$\int_0^T \tilde{f}(t-u)du = \int_0^t f(t-u)du = F(t) \tag{30}$$

where

G(t): probability distribution function for repair; viz. G(t) is the probability that component repair is completed by time t, given that the component failed at time zero

F(t): probability distribution function for first failure of component; viz. F(t) is the probability that the first failure of component occurs by time t, given that the component was like new at time zero

Since G(t) and F(t) are monotonically increasing in t, we obtain:

$$\| A \| = \max_{0 \le t \le T} G(t) = G(T) \qquad (31)$$

$$\| B \| = \max_{0 \le t \le T} F(t) = F(T) \qquad (32)$$

The above (31) and (32) give reliability theoretic interpretations for the norms of Volterra operators A and B; i.e. $\| A \|$ is the probability that component repair completes in T units of time , and $\| B \|$ is the probability that the first failure of component occurs in T units of time. Substituting (31) and (32) into (18), we obtain:

$$\mathrm{cond}(L) \le \max \left\{ \frac{(1 + G(T))^2}{1 - G(T)F(T)} , \frac{(1 + F(T))^2}{1 - G(T)F(T)} \right\} \qquad (33)$$

If we evaluate numerical errors in solution x of (5b) for large t, we may usually expect that G(T) >> F(T) holds and that (33) reduces to:

$$\mathrm{cond}(L) \le \frac{(1 + G(T))^2}{1 - G(T)F(T)} \qquad (34)$$

5. CONCLUSION

Linear operator theoretic method was presented for evaluating the magnitude of errors in numerical computation of reliability parameters. It was shown that our method has an easily understandable interpretation from the viewpoint of reliability theory though the derivation of the method might be rather mathematical. Applications of the presented method to practical problems will appear in the succeeding paper.

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APPENDIX: Derivation of norm D

We will determine the norm of operator

$$D(\cdot) = \int_a^b \tilde{K}(t,u)(\cdot)du \qquad \text{(A-1)}$$

with non-negative kernel $\tilde{K}(t,u)$. We will follow the procedure given in [7, pp. 83-84].

Putting

$$Dy = \int_a^b \tilde{K}(t,u)y(u)du, \qquad \text{(A-2)}$$

there results

$$\begin{aligned}
\| Dy \| &= \sup_t \left| \int_a^b \tilde{K}(t,u)y(u)du \right| \\
&\leq \sup_u |y(u)| \sup_t \int_a^b |\tilde{K}(t,u)|du \\
&= \max_u |y(u)| \sup_t \int_a^b \tilde{K}(t,u)du \\
&= \max_u |y(u)| \max_t \int_a^b \tilde{K}(t,u)\ du
\end{aligned}$$

where the last equality holds since $\int_a^b \tilde{K}(t,u)du$ is continuous in t. We thus know that:

$$\|Dy\| \leq \|y\| \max_t \int_a^b \tilde{K}(t,u)du \qquad \text{(A-3)}$$

or

$$\|D\| \leq \max_t \int_a^b \tilde{K}(t,u)du \qquad \text{(A-4)}$$

Since $\int_a^b \tilde{K}(t,u)du$ is continuous in t, $\int_a^b \tilde{K}(t,u)du$ must assume its maximum at a point t_0 of the interval [a, b]; viz.

$$\int_a^b \tilde{K}(t_0,u)du = \max_t \int_a^b \tilde{K}(t,u)du$$

We put

$$z_0(u) = \operatorname{sgn} \tilde{K}(t_0,u) \qquad \text{(A-5)}$$

Let $y_n(u)$ be continuous functions approximating $z_0(u)$ such that always $|y_n(u)| \leq 1$ and in addition everywhere except on a set M_{ε_n} of measure ε_n, $y_n(u) = z_0(u)$ holds. Here $\varepsilon_n \leq 1/(2Mn)$ where

$$M = \max_{t,u} \tilde{K}(t,u) \qquad \text{(A-6)}$$

Then on M_{ε_n}, $|y_n(u) - z_0(u)| \leq 2$. Furthermore

$$\left|\int_a^b \tilde{K}(t,u)z_0(u)du - \int_a^b \tilde{K}(t,u)y_n(u)du\right|$$

$$\leq \int_a^b |\tilde{K}(t,u)|\,|z_0(u) - y_n(u)|du$$

$$= \int_a^b \tilde{K}(t,u)\,|z_0(u) - y_n(u)|\,du$$

$$= \int_{M_{\varepsilon_n}} \tilde{K}(t,u)\,|z_0(u) - y_n(u)|\,du$$

$$\leq 2 \max_{t,u} K(t,u) \frac{1}{2Mn} = \frac{1}{n} \qquad \text{(A-7)}$$

for every $t \in [a, b]$.

Consequently, for all $t \in [a, b]$,

$$\int_a^b \tilde{K}(t,u) z_0(u)\,du \leq \int_a^b \tilde{K}(t,u) y_n(u)\,du + \frac{1}{n} \leq \|D\| \cdot \|y_n\| + \frac{1}{n}$$

Now putting $t = t_0$, we obtain:

$$\int_a^b \tilde{K}(t,u)\,du \leq \|D\| \cdot \|y_n\| + \frac{1}{n} \qquad \text{(A-8)}$$

Since $\|y_n\| \leq 1$, there results from (A-8) as $n \to \infty$

$$\int_a^b \tilde{K}(t_0,u)\,du \leq \|D\| \qquad \text{(A-9)}$$

i.e.

$$\max_t \int_a^b \tilde{K}(t,u)\,du \leq \|D\| \qquad \text{(A-10)}$$

It follows from (A-4) and (A-10):

$$\| D \| = \max_{t} \int_a^b \tilde{K}(t,u)du \tag{A-11}$$

(End of APPENDIX).

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SUPPLEMENTARY NOTES