

**Bounds on Benefits and Harms of  
Adding Connections to  
Noncooperative Networks**

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# Bounds on Benefits and Harms of Adding Connections to Noncooperative Networks

Hisao Kameda\*

## Abstract

In computer networks (and, say, transportation networks), we can consider the situation where each user has its own routing decision so as to minimize noncooperatively the expected passage time of its packet/job given the routing decisions of other users. Intuitively, it is anticipated that adding connections to such a noncooperative network may bring benefits at least to some users. The Braess paradox is, however, the first example of paradoxical cases where it is not always the case. This paper studies the bounds on the degrees of coincident cost improvement (benefits) and degradation (harms) for all users by adding connections to noncooperative networks. For Wardrop networks (noncooperative networks with infinitesimal users), the degree of benefits for all users can increase without bound by adding connections whereas no Wardrop network has been found for which the degree of harms can increase without bound for all users. In contrast, for Nash networks (noncooperative networks with a finite number of users), the degrees of both benefits and harms can increase without bound for all users. On the other hand, we see that, for some category of Wardrop networks, adding connections to them can bring neither benefits nor harms to all users, and that, for some homogeneous networks, adding connections to them can never bring benefits to all users under any static policy including cooperative and noncooperative ones.

*Keywords*— Braess paradox, Wardrop equilibrium, Nash equilibrium, coincident cost degradation and improvement, computer and communication networks, distributed computer systems, network routing, shortest path routing, source routing, static load balancing.

## 1 Introduction

There exist networks and systems wherein a number of independent users share and compete for resources. For example, communication networks like the Internet consist of a finite number of links, and of flows of packets each of which is issued by an independent user and flows through the links. Distributed computer systems like GRIDs [14] consist of computers widely distributed and interconnected by communication networks, and of flows of jobs each of which is issued by an independent user.

We can consider communication networks where each infinitesimal user has its own routing decision so as to minimize the expected passage time of its packet from the origin and the destination given the routing decisions by other users. The situation where every infinitesimal user has attained its optimization given the decision of other users and would not unilaterally deviate its routing decision is called a *Wardrop equilibrium*, a Nash equilibrium with infinitesimal players. We call here such a noncooperative network in a Wardrop equilibrium a *Wardrop networks*. In computer networking, some shortest path routing protocols may bring about situations close to Wardrop equilibria. It would be anticipated that users' benefits would increase by adding connections to a network, and thus increasing the degree of freedom in choices to each user. This is not always the case, however, as first exemplified in the Braess paradox [3].

The famous Braess paradox shows that adding connections (links) to a network may sometimes degrade the cost for all users in a Wardrop equilibrium. The Braess paradox attracted the attention of many researchers and a lot of work has been accumulated, *e.g.*, [2, 4, 8–12, 15, 16, 28, 30, 32, 35, 38]. Coincident cost degradation for all users by adding connections to a network looks paradoxical, and is called the *paradox*.

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It seems that, since its discovery, the paradox has been studied mostly with respect to Wardrop equilibria. It also seems that the topologies of concrete networks examined with respect to the paradox have been similar to or centered on that of the Braess network.

On the other hand, we can consider networks where users are classified into a small number of groups, each of which optimizes its cost non-cooperatively. The situation where, in such a scheme, every user has attained its optimization given the decision of other users and would not unilaterally deviate its decision is called a *Nash equilibrium*. We call here such a noncooperative network in a Nash equilibrium a *Nash network*. In computer networking, some source routing protocols may bring about situations close to Nash equilibria. Examples of paradoxes similar to Braess's in a Nash equilibrium have been found not only for networks of topologies similar to the Braess one [26, 27], but also for a network of another topology [20]. Motivation for studying Wardrop and Nash equilibria in computer networking is discussed in the Appendix.

It seems that, except [19, 31], few studies have emphasized the estimation of how harmful the paradox can be, *i.e.*, the worst-case degree of coincident cost degradation for all users by adding connections to a noncooperative network. Moreover, we have not seen the estimation of how beneficial the addition of connections to a noncooperative network can be, *i.e.*, the best-case degree of coincident cost improvement by adding connections to a noncooperative network.

We consider networks with fixed numbers of vertices. It has been shown that the degree of coincident cost degradation by adding connections to Wardrop networks with one origin-destination pair is bounded [19, 31]. On the other hand, it has been shown that the degree of coincident cost degradation by adding connections to Nash networks can increase without bound [22]. The present paper addresses the estimation of the best-case degrees of coincident cost improvement for all users by adding connections to Wardrop and Nash networks. We show that the degree of coincident cost improvement (benefits) by adding connections to both Wardrop and Nash networks can increase without bound. In contrast, for homogeneous distributed systems, no coincident cost improvement occurs by adding connections under any static load balancing policy including cooperative and noncooperative ones. Furthermore, in Wardrop equilibria for any distributed system, neither coincident cost degradation nor coincident cost improvement occurs by adding connections to the system.

Note, in passing, that each user of a network may have decisions on flow control, in addition to routing. In wireless networks, users may have power control. The concept of a Nash equilibrium is also discussed in flow control [1, 5, 7, 17, 25, 29] and in power control [13, 18, 33, 34, 37], both of which are not addressed in this paper.

The outline of this paper is as follows. Section 2 presents the concepts around Pareto superiority, Braess-like paradoxes, and their degrees of magnitude. Section 3 gives the description of the Wardrop networks investigated, and presents the results. On the other hand, Section 4 examines networks with a topology different from that of the Wardrop networks, in particular, with that of distributed systems. Section 5 concludes this paper.

## 2 A Measure Showing the Degree of the Paradox

This paper uses a single scalar measure that shows the degree of Pareto superiority of a system state before adding connections to that after doing so, or the degree of paradoxes in Nash equilibria.

The Pareto superiority is defined as follows. Consider a system consisting of  $n$  users (or players, decision makers),  $1, 2, \dots, n$ . User  $i$  has its cost  $C_i(S)$ , in the system state  $S$ . Denote by  $S^a$  and  $S^b$  two different states of the system.  $k_i \triangleq C_i(S^a)/C_i(S^b)$ .  $S^b$  is *Pareto superior* to  $S^a$  iff  $k_i > 1$  for some  $i$  and  $k_j \geq 1$  for all other  $j$ . In particular, we say that  $S^b$  is *strongly Pareto superior* to  $S^a$  iff  $k_i > 1$  for all  $i$ . A state to which some other state is Pareto superior is *Pareto inefficient*. Thus, the Pareto superiority depends on the vector  $(k_1, k_2, \dots, k_n)$ . It may, however, be convenient to express the degree of Pareto superiority, using a single scalar measure. It is required that the measure should clearly reflect Pareto superiority. If  $k_{\min} > 1$ , the state  $S^b$  is (strongly) Pareto superior to  $S^a$ , and if  $k_{\min} < 1$ , the state  $S^b$  is Pareto indifferent or superior to  $S^a$ . Thus, the measure  $k_{\min}$  may be used as a primary measure of Pareto superiority. In contrast, for example, a measure based on a certain average or on a product of all  $k_i$  cannot satisfy the above requirement, but may

be used as a secondary measure for tie-breaking the case where  $k_{\min} = 1$ .

It would be anticipated that in a system state where each user has more freedom of choice than in another state, at least one user should enjoy higher utility than in the latter state. But, as the famous *Braess paradox* shows, it is not always the case for noncooperative systems. Thus, a Nash equilibrium of a system with less freedom may be *Pareto superior* to that with more freedom, which is called *paradox*. We, therefore, use the measure of Pareto superiority as the measure of the degree of the paradox, *i.e.*, the coincident cost degradation for all users by adding connections. The measure of the degree of coincident cost improvement for all users after due by adding connections can be considered in a similar way.

### 3 Wardrop Networks

#### 3.1 Assumptions on Wardrop Networks

Wardrop networks considered here consist of one origin and one destination and some relay nodes, some pairs of which a one-way link connects. One of the simplest networks is the general Braess network (Fig. 1) discussed later. There are a number of paths each of which connects the origin and the destination through a different series of links. The cost of a path is the sum of the cost of each link in the path. Infinitely many infinitesimal users send their packets through the network. Each user chooses a path of the minimum cost. The choice of a single infinitesimal user has only a negligible impact on the cost of each link. The situation where no user can reduce his/her cost by unilaterally choosing another path is a Wardrop equilibrium, an infinitesimal-user version of a Nash equilibrium.

It is assumed that the cost of each link is a non-decreasing function of the total flow, *i.e.*, the rate of packets through the link. In a Wardrop equilibrium of the networks with only one pair of origin and destination, the costs of all users are identical.  $C_0$  and  $C_c$ , respectively, denote the costs of users of a Wardrop network before and after adding connections to the network. Define  $k = C_c/C_0$ . Then,  $k$  expresses the degree of cost change for all users by adding the connections.  $k_{\min}$  of Section 2 reduces to  $k$  here.

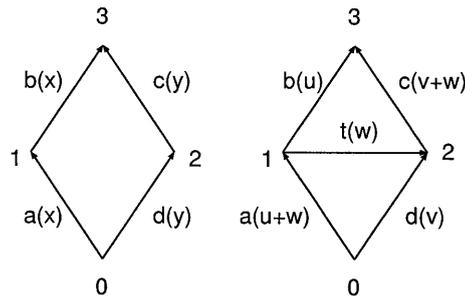


Figure 1: General Braess network. (Type-G0). **Left:** The network before link 1-2 is added. **Right:** The network after link 1-2 is added. Similarly for the subsequent pairs of drawings.

The Braess network consists of four nodes: one origin, one destination, and two relay nodes (Fig. 1). Each user flows through one of the paths. Before adding connections, the network has two paths, 0-1-3 and 0-2-3, each of which contains two links, the origin to one relay (0-1 or 0-2) and the relay to the destination (1-3 or 2-3), respectively. After adding connections, *i.e.*, a one-way link connecting two relays (1-2), the network has three paths including the new path (0-1-2-3) connecting the origin, one relay, the other relay, and the destination. Each user flows through one of the paths. In the original Braess network, the cost of each link is a linear function of the amount of the flow through the link [3].

This paper also considers the networks, called *general Braess networks*, that have nonlinear link cost functions. If we denote by  $\eta$  the flow of each link, the costs of links 0-1, 1-3, 0-2, 2-3, and 1-2 are, respectively, denoted by  $a(\eta)$ ,  $b(\eta)$ ,  $c(\eta)$ ,  $d(\eta)$ , and  $e(\eta)$ . The total flow through the network is denoted by  $X$ . Denote the amounts of the flows through paths 0-1-3 and 0-2-3, respectively, by  $x$  and  $y$  before adding link

1-2. Denote the amounts of the flows through paths 0-1-3, 0-2-3, and 0-1-2-3, respectively, by  $u$ ,  $v$ , and  $w$  after adding link 1-2. Denote by  $X$  the total flow, and, thus,

$$x + y = X = u + v + w. \quad (1)$$

The cost of links 0-1, 1-3, 2-3, and 0-2, are, respectively,  $a(x)$ ,  $b(x)$ ,  $c(y)$ , and  $d(y)$  before adding link 1-2. The cost of links 0-1, 1-3, 2-3, 0-2, and 1-2 are, respectively,  $a(u+v)$ ,  $b(u)$ ,  $c(v+w)$ ,  $d(v)$ , and  $t(w)$  after adding the link.

For the original Braess network,  $a(\eta) = c(\eta) = 10\eta$ ,  $b(\eta) = d(\eta) = \eta + 50$ ,  $t(\eta) = \eta + 10$ , and  $X = 6$ , which result in  $C_o = 83$  and  $C_c = 92$ , and thus  $k = C_c/C_o = 1.1084\dots$  [3]. Recall that  $k$  shows the degree of cost change by adding link 1-2.  $k > 1$  means coincident cost degradation. In the above case, it is about 11 % degradation.

By *general Cohen-Kelly networks*, we mean a subset of general Braess networks for which the costs of links 0-1 and 2-3, are, respectively,  $a(\eta) = \alpha/(a - \eta)$  and  $c(\eta) = \gamma/(a - \eta)$  for  $0 \leq \eta < a$  and for which the costs of links 1-3 and 0-2, are, respectively,  $b(\eta) = d(\eta) = b$  for any  $\eta > 0$  with  $t(\eta) = t$  (constant). Cohen and Kelly [10] considered a network of this type for which  $\alpha = 1$ ,  $a = \phi$ ,  $b = 2$ ,  $t = 1$ , and  $X = 2\lambda$ , which is actually symmetric. They showed that  $C_o = 1/(\phi - \lambda) + 2 < 3 = C_c$ , i.e.,  $1 < k < 3/2$ , assuming that  $2\lambda > \phi - 1 > \lambda > 0$ , which is a paradox. In the above case, it is less than 50 % degradation.

As a general result on the Braess networks, it has been shown that the degree,  $k$ , of coincident cost degradation is bounded to 2 for the general Braess networks for which  $a(\cdot)$  and  $c(\cdot)$  are increasing and  $b(\cdot)$ ,  $d(\cdot)$ , and  $t(\cdot)$  are non-decreasing [19]. Furthermore, as a general result on the Wardrop networks, it has been shown that  $k$  is bounded to  $\lfloor n/2 \rfloor$  for Wardrop networks that consist of  $n$  vertices and have link costs each of which is a nondecreasing function of the flow through the link [31].

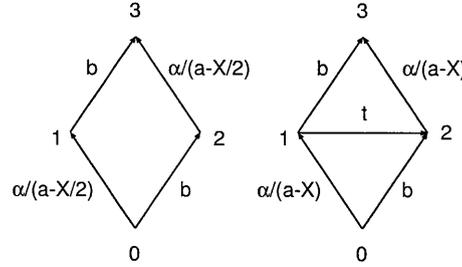


Figure 2: Model Cohen-Kelly network.

We define more special Braess networks that will be used in the next section.

- [*Model Cohen-Kelly network*](Fig. 2) Networks that are the same as general Cohen-Kelly networks with  $b \geq \alpha/(a - X) + t$ . Thus,  $u = v = 0$  and  $w = X$ . Then, the following relations hold.

$$C_o = \frac{\alpha}{a - X/2} + b, \quad (2)$$

$$C_c = \frac{2\alpha}{a - X} + t. \quad (3)$$

We consider the case of  $t = 0$ , in particular. Then, model Cohen-Kelly networks are described by the values of parameters  $\alpha$ ,  $a$ ,  $b$ , and  $X$  that satisfy  $0 < X < a$  and  $0 < \alpha/(a - X) \leq b$ .

- [*Reduced Cohen-Kelly network*] A subset of model Cohen-Kelly networks with  $b = \alpha/(a - X) + t$ . Thus, the following relation holds for  $C_c$  while  $C_o$  is given by (2).

$$C_c = \frac{2\alpha}{a - X} + t = 2b - t = \frac{\alpha}{a - X} + b. \quad (4)$$

We also consider the case of  $t = 0$ , in particular.

### 3.2 The Results

**Proposition 1** For every value of  $k$ , s.t.  $0 < k < 2$ , there exist model Cohen-Kelly networks for which the measure  $k$  is that value.

[Proof] The outline of the proof is given as follows.

Step 1) shows that  $k$  depends only on  $\rho$  ( $= X/a$ ) and  $Z$  ( $= b(a - X)/\alpha - 1$ ), i.e.,  $k = k(\rho, Z)$ .

Step 2) sees that  $k = k(\rho, 0)$  monotonically increases in  $\rho$  with the range  $1 < k < 2$  and the domain  $0 < \rho < 1$ . Thus, given  $k$ , s.t.  $1 < k < 2$ , the corresponding value of  $\rho$  can be obtained and thus, with  $Z = 0$ , the corresponding combinations of values of  $\alpha$ ,  $a$ ,  $X$ , and  $b$ , can be obtained, which describe model Cohen-Kelly networks that are also reduced Cohen-Kelly networks.

Step 3) sees that, given  $\rho$ , s.t.  $0 < \rho < 1$ ,  $k = k(\rho, Z)$  monotonically decreases in  $Z$  with the range  $0 < k \leq 1$  and the domain  $Z_\rho \leq Z$  where  $Z_\rho = \rho/(2 - \rho) > 0$ . Thus, given  $k$ , s.t.  $0 < k \leq 1$ , the corresponding value of  $Z$  can be obtained and thus, with given  $\rho$ , the corresponding combinations of values of  $\alpha$ ,  $a$ ,  $X$ , and  $b$ , can be obtained, which describe model Cohen-Kelly networks.

Steps 4) shows, by combining 2) and 3), that, for every value of  $k$ , s.t.  $0 < k < 2$ , the corresponding combinations of values of  $\alpha$ ,  $a$ ,  $X$ , and  $b$ , can be obtained, which describe model Cohen-Kelly networks. Therefore, for every value of  $k$ , s.t.  $0 < k < 2$ , there exist model Cohen-Kelly networks that have the value of  $k$ .

The details of the proof are as follows.

1) Consider a model Cohen-Kelly network. Then,

$$\begin{aligned} k &= \frac{C_c}{C_o} = \frac{\frac{2\alpha}{a-X}}{\frac{\alpha}{a-X/2} + \frac{\alpha}{a-X} + b - \frac{\alpha}{a-X}} \\ &= \frac{2}{Z+1 + \frac{a-X}{a-X/2}} = \frac{2}{Z+3 - \frac{2}{2-\rho}}, \end{aligned}$$

where  $\rho = X/a$  ( $0 < \rho < 1$ ) and  $Z = b(a - X)/\alpha - 1$  ( $Z \geq 0$ ).

2) Consider the combinations of values of  $\alpha$ ,  $a$ ,  $X$ , and  $b$  such that  $Z = 0$ . Note that, with  $Z = 0$ ,  $k$  is a continuous and strictly increasing function of  $\rho$  with the domain  $0 < \rho < 1$  and the range  $1 < k < 2$ . Therefore, for every value of  $k$ , s.t.  $1 < k < 2$ , the corresponding value of  $\rho$  is found, and then, combinations of values of  $\alpha$ ,  $a$ ,  $X$ , and  $b$  that satisfy  $\rho = X/a$  and  $Z = b(a - X)/\alpha - 1 = 0$  can be obtained, which describe model Cohen-Kelly networks that are also reduced Cohen-Kelly networks. Thus, it is seen that, for every value of  $k$ , s.t.  $1 < k < 2$ , there exist model Cohen-Kelly networks that have the value of  $k$ . It is also seen that the worst-case value, 2, of the measure  $k$  of coincident cost degradation due to adding link 1-2 is asymptotically reached in certain reduced Cohen-Kelly networks as  $\rho$  approaches 1.

3) Next, consider the case where  $a$ ,  $X$ , and thus  $\rho$  ( $0 < \rho < 1$ ) are given. Then, a value of  $Z$ ,  $Z_\rho = \rho/(2 - \rho) > 0$ , gives that  $k = 1$ . Note that  $k$  is continuous and strictly decreasing in  $Z$  with the domain  $Z \geq Z_\rho$  and the range  $0 < k \leq 1$ . It is seen therefore that, given  $a$ ,  $X$ , and thus  $\rho$  ( $0 < \rho < 1$ ), for every value of  $k$ , s.t.  $0 < k \leq 1$ , the corresponding value of  $Z \geq Z_\rho$  is found, and then, combinations of values of  $\alpha$  and  $b$  that satisfy  $Z = b(a - X)/\alpha - 1$  can be obtained, which describe model Cohen-Kelly networks along with the values of  $a$  and  $X$  given at the beginning of 3). Thus, it is seen that, for every value of  $k$ , s.t.  $0 < k \leq 1$ , there exist model Cohen-Kelly networks that have the value of  $k$ .

4) Combining 2) and 3) above, it is seen, therefore, that for any value of  $k$ , s.t.  $0 < k < 2$ , there exist model Cohen-Kelly networks that have the value of  $k$ . Furthermore, from 2), it is seen that the worst-case value, 2, of the measure  $k$  of coincident cost degradation due to adding link 1-2 is asymptotically reached in certain reduced Cohen-Kelly networks as  $\rho$  approaches 1.  $\square$

**Remark 1** In the above proof, it is seen, in 3), that the degree of coincident cost improvement can increase without bound ( $k \approx 0$ ) in some model Cohen-Kelly networks, with  $b \geq \alpha/(a - X)$  or  $Z \geq 0$ , as, e.g.,  $b \rightarrow \infty$  with  $\alpha$ ,  $a$ , and  $X$  fixed. On the other hand, in 2), in reduced Cohen-Kelly networks with  $t = 0$ , the ratio of the paradox approaches 2 as  $\rho \rightarrow 1$ , which means asymptotically infinite link costs.

Define the degree  $\kappa = \log_2 k$ . Then, adding link 1-2 leads to the cost improvement of  $2^{|\kappa|}$  times if  $\kappa \leq 0$ , and leads to the cost degradation of  $2^{|\kappa|}$  times if  $\kappa \geq 0$ , and unlimited large values of  $|\kappa|$  can be considered. The above result shows, however, that  $-\infty < \kappa < 1$ .  $\square$

From Proposition 1, we see that there exist Wardrop networks for which the degree of coincident cost improvement (benefits) for all users by adding connections to the networks can increase without bound. Therefore, we have the following property.

**Theorem 1** The degree of coincident cost improvement for all users by adding connections to Wardrop networks can increase without bound.

## 4 Noncooperative Load Balancing in Distributed Systems

The previous section has shown that by adding connections to Wardrop networks with fixed numbers of vertices, the degree of coincident cost improvement can increase without bound, although the degree of coincident cost degradation cannot be over some bound.

This section gives a category of noncooperative networks [20], for which it has been shown that the degree of the Braess-like coincident cost degradation can increase without bound in Nash equilibria [22]. This category of networks presents models of distributed systems, like GRIDs [14]. Load balancing of jobs among nodes in distributed systems are regarded as routing in the equivalent networks [21, 24, 39].

### 4.1 Assumptions on the Networks Equivalent to Distributed Systems

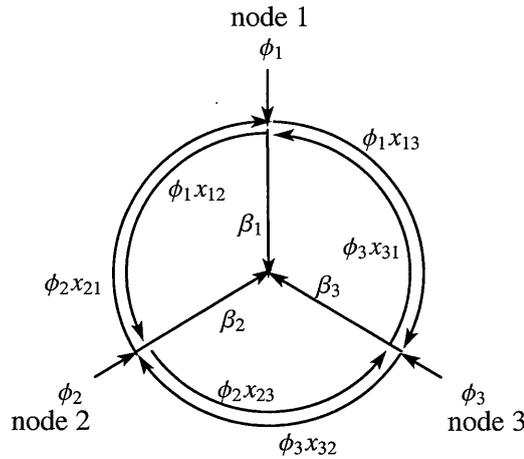


Figure 3: The model of a distributed system for  $m = 3$ . The symbols near each arrow express the rate of jobs that flow through the arrow.  $\beta_i = \phi_1 x_{1i} + \phi_2 x_{2i} + \phi_3 x_{3i}$ ,  $i = 1, 2, 3$ .

A network equivalent to a distributed system consists of  $m$  origins and one destination, with each origin being connected to the destination through one separate link, which is often called ‘node’ here. Denote the total flow of O-D pair  $i$  is  $\phi_i$ .

Before adding connections, there is only one path for each O-D payer. After adding connections (see Fig. 3), some flow passes through the connections to forward to other nodes, out of the total flow  $\phi_i$  that arrives at the origin  $i$ . Let the flow from origin  $i$  to node  $j$  be  $\phi_i x_{ij}$ ,  $i \neq j$ . The amount of flow,  $\phi_i x_{ii}$ , goes to

node  $i$ . Then,  $0 \leq x_{ij} \leq 1$ ,  $i, j = 1, 2, \dots, m$ ,  $\sum_p x_{ip} = 1$ . Denote the vector  $(x_{i1}, x_{i2}, \dots, x_{im})$  by  $\mathbf{x}_i$  and the vector  $(x_1, x_2, \dots, x_m)$  by  $\mathbf{x}$ . The load balancing policies considered determine (only)  $\mathbf{x}$ , and, thus, are static in nature. Denote by  $\mathcal{C}$  the set of  $\mathbf{x}$ 's that satisfy the above constraints. The resulting flow  $\beta_i$  through node  $i$ ,  $i = 1, 2, \dots, m$ , is

$$\beta_i = \sum_p \phi_p x_{pi}. \quad (5)$$

Thus, the cost for the entire flow associated with the O-D pair  $i$  is

$$T_i(\mathbf{x}) = \sum_p x_{ip} T_{ip}(\mathbf{x}), \quad (6)$$

where

$$T_{ii}(\mathbf{x}) = D_i(\beta_i), \quad \text{and} \quad (7)$$

$$T_{ij}(\mathbf{x}) = D_j(\beta_j) + G_{ij}(\mathbf{x}), \quad \text{for } j \neq i. \quad (8)$$

$D_i(\beta_i)$  denotes the cost of processing a packet/job at node  $i$  in the case where the load on node  $i$  is  $\beta_i$ .  $G_{ij}(\mathbf{x})$  denotes the cost of sending a packet/job from origin  $i$  to node  $j$  in the case where the strategy profile is  $\mathbf{x}$ . It is assumed that, for all  $i, j, k$  ( $i \neq j \neq k \neq i$ ),

$$G_{ij}(\mathbf{x}) < G_{ik}(\mathbf{x}) + G_{kj}(\mathbf{x}). \quad (9)$$

Thus, a job forwarded from a node is not to be forwarded again to another node.

## 4.2 Wardrop Networks of This Category

We consider an individual optimization scheme for the networks of this category. Each infinitesimal user seeks its own optimization and chooses the path of the minimum cost to send its jobs through. The situation where every infinitesimal user attains unilaterally its optimization given the decisions of other users is a Wardrop equilibrium. We call the network at a Wardrop equilibrium a *Wardrop network* as before. A Wardrop network in question is characterized as follows: Infinitesimal users choose the path of the smallest cost for each O-D pair. Thus, the Wardrop equilibrium  $\bar{\mathbf{x}}$  after adding connections satisfies the following for all  $i, j$ :

$$T_{ij}(\bar{\mathbf{x}}) = \min_k \{T_{ik}(\bar{\mathbf{x}})\} \text{ for } \bar{x}_{ij} > 0, \quad (10)$$

$$T_{ij}(\bar{\mathbf{x}}) \geq \min_k \{T_{ik}(\bar{\mathbf{x}})\} \text{ for } \bar{x}_{ij} = 0. \quad (11)$$

That is, all the paths used have the equal cost for each O-D pair. We have the following property.

**Theorem 2** *The costs of all users neither degrade nor improve coincidentally by adding connections to any Wardrop network of this category.*

**[Proof]** Similarly as [39], [21, 40], after adding connections to a Wardrop network in question, nodes are one of the following:

- (1) idle source ( $R_d$ ): The node sends jobs and does not process any jobs. That is,  $\beta_i = 0$ .
- (2) active source ( $R_d$ ): The node sends jobs and does not receive any jobs. But, the node processes a part of the jobs that arrive at the node. That is,  $\phi_i > \beta_i > 0$ .
- (3) neutral ( $N$ ): The node processes jobs locally without sending or receiving jobs. That is,  $\beta_i = \phi_i$ .
- (4) sink ( $S$ ): The node receives jobs from other nodes but does not send out any jobs. That is,  $\beta_i > \phi_i$ .

That is, there does not exist such a node that both sends and receives jobs. Indeed, suppose that node  $i$  sends jobs to node  $j$  and receives jobs from node  $k$ . Then,

$$D_i(\beta_i) \geq D_j(\beta_j) + G_{ij}(\mathbf{x}), \quad (12)$$

$$D_k(\beta_k) \geq D_i(\beta_i) + G_{ki}(\mathbf{x}) = T_{ki}. \quad (13)$$

It is because, if (12) does not hold, processing jobs at node  $i$  incurs the smaller cost than sending jobs to node  $j$  from node  $i$ . Similarly for (13).

Then, from (12) and (9), we have  $T_{ki} = D_i(\beta_i) + G_{ki}(\mathbf{x}) \geq D_j(\beta_j) + G_{ij}(\mathbf{x}) + G_{ki}(\mathbf{x}) > D_j(\beta_j) + G_{kj}(\mathbf{x}) = T_{kj}$ . That is,  $T_{ki} > T_{kj}$ . Then, from (10) and (11),  $x_{ki} = 0$ . That is, node  $i$  does not receive jobs from node  $k$ , which contradicts the supposition.

Denote by  $\bar{T}_i$  and  $\hat{T}_i$ , respectively, the cost of O-D pair  $i$  before and after adding connections to the network in question. Recall that  $D_i$  is nondecreasing for all  $i$ .

1) Assume that adding connections to a network in question brings about coincident cost degradation to all O-D pairs.

Suppose that there exists an idle or active node  $i$  after adding connections, then  $\bar{T}_i = D_i(\phi_i) \geq D_i(\beta_i) \geq \hat{T}_i$ . That is, the O-D pair  $i$  suffers no cost degradation. Thus, we see that there exists neither idle nor active node.

Then, since there exists neither idle node nor active node, there must exist no sink node. Only remain neutral nodes. That is, no coincident cost degradation occurs for all O-D pairs by adding connections to the network in question.

2) Assume that adding connections to a network in question brings about coincident cost improvement to all O-D pairs.

Suppose that there exists a sink node  $i$ , then  $\bar{T}_i = D_i(\phi_i) \leq D_i(\beta_i) = \hat{T}_i$ , which is no cost improvement for the O-D pair  $i$ . Thus, we see that there exists no sink node.

Then, since there exists no sink node, then there must exist neither idle node nor active node. Only remain neutral nodes. That is, no coincident cost improvement occurs for all O-D pairs by adding connections to the network in question.  $\square$

### 4.3 The Best-Case Degree of Coincident Cost Improvement by Adding Connections to Nash Networks

A Nash network in question is described as follows: For each O-D pair, there is one decision maker, or a player, that strives to minimize the cost for the O-D pair, *i.e.*, decision maker  $i$  for O-D pair  $i$ ,  $i = 1, 2, \dots, m$ . Before adding connections, each decision maker has no choice since there is only one path for each O-D payer. After adding connections, decision maker  $i$  ( $i = 1, 2, \dots, m$ ) chooses the amount of the flow to pass through the connections to forward to other nodes, out of the total flow  $\phi_i$  that arrives at the origin  $i$ . Thus, within these constraints, decision maker  $i$  ( $i = 1, 2, \dots, m$ ) chooses the values of  $x_{ij}$ ,  $j = 1, 2, \dots, m$ , to achieve optimization. Assume that each decision maker strives to optimize non-cooperatively the cost associated only with the corresponding O-D pair.

A Nash equilibrium is given by such  $\tilde{\mathbf{x}}$  as satisfies the following for all  $i$ ,

$$T_i(\tilde{\mathbf{x}}) = \min_{x_i} T_i(x_i; \tilde{\mathbf{x}}_{-(i)}), \quad \text{such that } (x_i; \tilde{\mathbf{x}}_{-(i)}) \in \mathcal{C},$$

where  $(x_i; \tilde{\mathbf{x}}_{-(i)})$  denotes an  $m$ -dimensional vector in which the element corresponding to  $\tilde{x}_i$  has been replaced by  $x_i$ .  $\tilde{\mathbf{x}}$  is called a *solution* for the above non-cooperative optimization.

**Theorem 3** *The degree of coincident cost improvement for all users by adding connections to Nash networks can increase without bound.*

**[Proof]** 1) We first show this property even if  $D_i(\cdot)$  and  $G_{ij}(\cdot)$  are nondecreasing,  $i, j (\neq i) = 1, 2, \dots, m$ .

Consider the following network with  $m = 2$ ,  $\phi_1 = \phi$ ,  $\phi_2 = \phi + \epsilon$  ( $\phi > 0$ ,  $\epsilon > 0$ ),  $\delta/D = D/\Delta \triangleq \kappa$  ( $0 < \delta < D < \Delta$ ),

$$\begin{aligned} D_1(\beta_1) &= D \text{ (constant),} \\ D_2(\beta_2) &= \begin{cases} \delta, & \text{for } 0 \leq \beta_2 \leq \phi, \\ \Delta, & \text{for } \beta_2 > \phi, \end{cases} \\ G_{12} &= G_{21} = 0. \end{aligned}$$

$D_1, D_2, G_{12}$ , and  $G_{21}$  are nondecreasing. Then, clearly,  $\tilde{\mathbf{x}} = (\tilde{x}_{11}, \tilde{x}_{12}, \tilde{x}_{21}, \tilde{x}_{22}) = (0, 1, 1, 0)$  is a Nash equilibrium after adding connections, and

$$T_1(\tilde{\mathbf{x}}) = \delta, \quad T_2(\tilde{\mathbf{x}}) = D.$$

Note, however, that, before adding connections,

$$T_1(\tilde{\mathbf{x}}) = D, \quad T_2(\tilde{\mathbf{x}}) = \Delta.$$

Therefore,  $k_1 = k_2 = \kappa < 1$ , and both users 1 and 2 have coincident cost improvements by adding connections, and the best-case degree can increase without bound as  $\kappa \rightarrow 0$ .

2) We can easily show this property if we relax the condition that  $D_i(\cdot)$  and  $G_{ij}(\cdot)$  are nondecreasing,  $i, j (\neq i) = 1, 2, \dots, m$ .

Consider the following network that is different from the above network only in the following.  $0 < \delta < D = \Delta$ ,

$$\begin{aligned} D_1(\beta_1) &= \begin{cases} D, & \text{for } 0 \leq \beta_1 \leq \phi, \\ \delta, & \text{for } \beta_1 > \phi, \end{cases} \\ D_2(\beta_1) &= D \text{ (constant).} \end{aligned} \tag{14}$$

$D_1, D_2, G_{12}$ , and  $G_{21}$  are nonincreasing. Then, clearly,  $\tilde{\mathbf{x}} = (\tilde{x}_{11}, \tilde{x}_{12}, \tilde{x}_{21}, \tilde{x}_{22}) = (1, 0, 1, 0)$  is a Nash equilibrium after adding connections, and

$$T_1(\tilde{\mathbf{x}}) = \delta, \quad T_2(\tilde{\mathbf{x}}) = \delta.$$

Note, however, that, before adding connections,

$$T_1(\tilde{\mathbf{x}}) = D, \quad T_2(\tilde{\mathbf{x}}) = D,$$

Therefore,  $k_1 = k_2 = \delta/D < 1$ , and both users 1 and 2 have coincident cost improvements by adding connections, and the best-case degree can increase without bound as  $\delta/D \rightarrow 0$ .

3) Thus, we have seen that there exist Nash networks for which the degree of coincident cost improvement (benefits) for all users by adding connections to the networks can increase without bound. Therefore, we can say that the degree of coincident cost improvement for all users by adding connections to Nash networks can increase without bound.  $\square$

#### 4.4 Homogeneous Distributed Systems

As previously noted, it has been shown that, for homogeneous distributed systems, the degree of coincident cost degradation by adding connections can increase without bound in Nash equilibria [22]. That is, for any value of the ratio  $k$ , s.t.  $k > 1$ , of coincident cost degradation by adding connections, there exists a homogeneous distributed system that has that value of the ratio in a Nash equilibrium. However, as to coincident cost improvement by adding connections to homogeneous distributed systems, we have the following property.

**Theorem 4** *No static load balancing policy in homogeneous distributed systems after adding connections can bring about coincident cost improvement for all users over the situations before adding connections, if node cost functions are nondecreasing and convex.*

**[Proof]** Before adding connections, each user has the identical mean response time for its job,  $\bar{T}_i = \bar{T} = D(\phi)$ , for all  $i$ . Denote by  $\hat{x}$  the strategy profile that all users take after adding connections under some static load balancing policy which includes an overall optimal policy. Then, user  $i$  has the mean response time for its job,

$$\begin{aligned}\hat{T}_i &\geq \sum_p \hat{x}_{ip} D(\beta_p), \text{ for all } i, \text{ Then,} \\ \phi \sum_p \hat{T}_p &\geq \sum_p \beta_p D(\beta_p).\end{aligned}\tag{15}$$

Note that  $\sum_p \beta_p = m\phi$ . We can show by induction, for  $m > 0$ ,

$$\sum_p \beta_p D(\beta_p) \geq m\phi D(\phi).\tag{16}$$

**Induction steps:** (i) Relation (16) clearly holds for  $m = 1$ .

(ii) Suppose that (16) holds for  $m = M (\geq 1)$ . Denote  $M\phi_M = \sum_{p=1}^M \beta_p$ . Then, we have, for some  $\xi$ ,  $\phi_M = \phi + \xi/M$ , and  $\beta_{M+1} = \phi - \xi$ .

$$\begin{aligned}\sum_{p=1}^{M+1} \beta_p D(\beta_p) &= \sum_{p=1}^M \beta_p D(\beta_p) + \beta_{M+1} D(\beta_{M+1}) \\ &\geq M\phi_M D(\phi_M) + \beta_{M+1} D(\beta_{M+1}) \\ &\text{(by the assumption of the induction),} \\ &= (M\phi + \xi) D(\phi + \frac{\xi}{M}) + (\phi - \xi) D(\phi - \xi) \\ &= \phi \left[ MD(\phi + \frac{\xi}{M}) + D(\phi - \xi) \right] \\ &\quad + \xi \left[ D(\phi + \frac{\xi}{M}) - D(\phi - \xi) \right] \\ &\text{(Note that } D \text{ is convex and nondecreasing,} \\ &\text{as to the first and second terms, respectively.)} \\ &\geq (M+1)\phi D(\phi).\end{aligned}$$

We, thus, see that (16) holds for any  $m > 0$ , and, therefore, from (15),

$$\frac{1}{m} \sum_p \hat{T}_p \geq D(\phi) = \bar{T},$$

which means that it is impossible that all  $\hat{T}_i$  are not greater than  $\bar{T}$  with some  $\hat{T}_i$  are strictly less than  $\bar{T}$ . That is, no coincident cost improvement for all users can occur by adding connections to the networks.  $\square$

**Remark 2** *From the above, we see that under any static load balancing policy (any policy that determines  $x$ , not only noncooperatively but also cooperatively), it is impossible that all users have benefits coincidentally from adding connections to homogeneous distributed systems. This result would give some insight into scheduling in GRID computing.*  $\square$

## 5 Concluding Remarks

The present paper has examined Wardrop and Nash networks. The results imply the following: For Wardrop networks, the degree of coincident cost improvement (benefits) can increase without bound by adding connections whereas there has been found no Wardrop network for which the degree of coincident cost degradation (harms) can increase without bound. On the other hand, for Nash networks, the degrees of both benefits

and harms can increase without bound. In contrast, we have seen that some Wardrop network can have neither cost improvement nor cost degradation coincidentally for all users by adding connection to it, and that some homogeneous network cannot have cost improvement coincidentally for all users by adding connection to it under any static policy including cooperative and noncooperative ones.

## Appendix. Motivation for Studying Wardrop / Nash equilibria in Computer Networking

Consider a packet-switched computer network like the Internet (see, *e.g.*, [6, 23, 36]). The network consists of nodes, *i.e.*, computers (hosts) and routers, connected by communication lines (links). Each packet is associated with a unique pair of its origin and destination hosts. Packets originated at a host flow through a path that consists of a series of interconnected routers to their destination host. Each router may keep routing information in some form, *e.g.*, of a routing table. Such information tells each arriving packet to which adjacent router the packet is to be forwarded. Another possibility is what is called ‘source routing’ where the path of each packet is specified at the outset. Then, each packet carries this information about its path while passing through the network. Source routing will be discussed later.

There exists a routing protocol that provides routing information at each router, so that each packet may be guided through a path of the shortest cost among the paths that connect the same pair of origin and destination nodes, given the cost of each link. Thus, another routing protocol can be conceived, one that assigns to each link the cost that reflects the estimated communication delay through the link. The communication delay and availability of each link may vary from time to time, and so routers need to exchange packets to update routing information. So-called ‘dynamic routing protocols’ will do such exchange. Such exchange of packets, however, cannot be done too frequently. Otherwise, links would be flooded with such packets and performance degradation would result. Thus, the information at each router is to be updated at some regular (but not too short) intervals, and the cost optimization process cannot be truly ‘dynamic’ but rather ‘quasi-static.’ Such an interval is called an *update interval* here.

Packet generation is regarded as a stochastic process. It is probable that this process is stable, *i.e.*, in a stochastic equilibrium, during a time period that contains a large number of update intervals. In quasi-static control, the shortest path routing that reflects communication delays as link costs may cause oscillations in the amount of packets that flow through each link, and thus, oscillations in the communication delay of each link or path. Such oscillations could be avoided by means of suitably forecasting the expected delay of each link for the next update interval, and/or by employing, if needed, an adequate mixing strategy of using more than one possible path, each used at a certain ratio or frequency. Thus, if suitably controlled, the shortest path routing may bring about situations wherein each packet flows through one of the paths of the shortest cost (communication delay) among the paths that connect its origin and destination nodes, and the situation must be very close to a *Wardrop equilibrium*. Thus, shortest cost path routing may have problems like that of the Braess paradox. That is, adding new links to the network may cause performance degradation to all the packets that have the same pair of origin and destination nodes.

On the other hand, the above mentioned source routing may provide situations quite close to a *Nash equilibrium* as seen in the following. Consider a situation where the information on the estimated delay of each link, and thus each path, is available at each origin. An autonomous system, *e.g.*, a local telephone network run by an Internet-service provider, or an autonomous network system belonging to an independent organization, may be connected to the network at an origin.

Then, the manager or the administrator of the autonomous system would like to minimize the overall cost or mean delay of the packets that are sent from the origin into the computer communication network. Again, such oscillations as mentioned above could be avoided by means of suitably forecasting the expected delay of each link in the next update interval, and/or by employing, if needed, an adequate mixing strategy of using more than one possible path, each used at a certain ratio or frequency. Thus, if suitably controlled, such source routing may bring about situations wherein the mean delay of the packets of an autonomous system is minimum, given the routing decisions on the packets of the other autonomous systems that are

connected to the network. In such cases, the situation must be very close to a *Nash equilibrium*.

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