The Worst Ratio of Paradoxical Cost Degradation in the Braess Network

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Abstract

The Braess paradox is known to be the first example of the paradoxical cases where adding capacity to a network degrades the costs for all users in the Wardrop equilibrium where each user strives to optimize his/her own cost non-cooperatively. The paradox stimulated many researchers to a large number of related studies. This paper investigates the networks of the same topology as the original Braess network further. The measure of cost degradation considered is the ratio of the cost of each path of the network after adding capacity (a link) to that of the network before adding capacity. The Braess paradox shows that there exists a network for which the measure is greater than one. The results given here show that, for each Braess network, there exists a simple and symmetrical network that has the measure of cost degradation not less than the Braess network. Furthermore, the measure of paradoxical cost degradation is, at most, 2 for general Braess networks and 4/3 for Braess networks of linear link costs. The measure greater than 2 cannot be achieved even if the Braess network is embedded within a general network in the way considered. On the other hand, an example of the network is given for which the measure of paradoxical cost degradation can be unlimitedly large in the Nash equilibrium where users are classified into groups and each group of users strives to optimize the cost of the group non-cooperatively.

keywords Braess paradox, Wardrop equilibrium, paradoxical cost degradation, routing, load balancing, computer and communication networks.

1 Introduction

There exist networks that consist of a finite number of links or facilities and of arriving threads or flows of infinitely many users to flow through the networks. For example, communication networks have flows of infinitely many packets or calls to pass through communication links, distributed computer systems have continuing arrivals of infinitely many transactions or jobs to be processed by computers, transportation flow networks have incoming threads of infinitely many vehicles to drive through roads, *etc.*

It would be anticipated that the benefits of users would be increased by adding capacity to a network. This is not always the case, however, as first exemplified in the Braess paradox [1]. The Braess network consists of four nodes: one origin, one destination, and two relay nodes. Before adding capacity (a link), the network has two paths each of which contains two links, the origin to one relay and the relay to the destination. After adding a one-way link connecting two relays to a link, the network has three paths including the new path connecting the origin, one relay, the other relay, and the destination (Fig. 1). Each user flows through one of the paths. In the network considered, the cost of each link is an increasing and/or nondecreasing function of the amount of flow through the link. The cost of a path is the sum of the costs of links in the path. Each user chooses the path of the minimum cost. The choice of a single user has only a negligible impact on the cost of each link. The situation where no user can reduce his/her cost by unilaterally choosing another path is called the *Wardrop equilibrium*. Therefore, in the Wardrop equilibrium the cost of each path used is identical and is not greater than the costs of paths not used. The famous Braess paradox shows that adding capacity (a link) to a network may sometimes degrade the cost of paths for all users in the Wardrop equilibrium.

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The Braess paradox attracted attention of many researchers including Nobel laureate Paul Samuelson [2], and lots of work has been accumulated[1] including a paper [3], published in a scientific journal, *Nature*, that discusses similar phenomena in mechanical and electrical networks, to name a few, [4–9]. Examples where a paradox similar to Braess's appear in a Nash equilibrium have been found for networks of a topology similar to the Braess [10, 11] and for a network of another topology [12].

The description of the model investigated is given in Section 2, and Section 3 presents the results and shows that our measure of paradoxical cost degradation is worse in some reduced networks. It is worst in symmetrical networks and the worst measure is, at most, 2 for general Braess networks. In the Braess networks with linear link cost (the original Braess type [1]), it is 4/3 in the worst. In the Braess networks with link costs of a type of hyperbolic functions, i.e., single-server queueing delays or constants, i.e., simple delays (the Cohen-Kelly type [7]), it is 2 at most. Section 4 shows that no Braess network embedded in a way considered there within a network of general topology cannot have a larger measure of cost degradation than the above. On the other hand, in a KAKH network [12], a network of topology different from the Braess network, the measure of Braess-like paradoxical cost degradation can be unlimitedly large as is shown in Section 5. Section 6 concludes this paper.

2 The Model and Assumptions

The Braess network considered (Fig. 1) consists of four nodes: one origin (numbered 0), two relay nodes (1 and 2), and one destination (3). Before adding capacity (a link), the network has two paths, 0-1-3 and 0-2-3, each of which contains two links, the origin to one relay (0-1 or 0-2) and the relay to the destination (1-3 or 2-3), respectively. After adding capacity, i.e., a one-way link connecting two relays (1-2), the network has three paths including the new path (0-1-2-3) connecting the origin, one relay, the other relay, and the destination. Each user flows through one of the paths. In the original Braess network, the cost of each link is a linear function of the amount of flow through the link (Fig. 6). This paper considers the networks that have nonlinear link cost functions also.

Denote the amounts of the flow through paths 0-1-3 and 0-2-3, respectively, by x and y before adding link 1-2. Denote the amounts of the flow through paths 0-1-3, 0-2-3, and 0-1-2-3, respectively, by u, v, and w after adding link 1-2. Denote by X the total flow, and thus

$$x + y = u + v + w = X. \tag{1}$$

The cost of a path is the sum of the cost of each link in the path. Each user chooses the path of the minimum cost. The choice of a single user has only a negligible impact on the cost of each link. The situation where no user can reduce his/her cost by unilaterally choosing another path is the Wardrop equilibrium. In the Wardrop equilibrium, the cost of each path used is identical. Denote by C_o and C_c, respectively, the costs of the paths that are used before and after adding link 1-2. All the costs of paths used must be equal, which are not greater than the cost of paths not used. The costs, C_o and C_c, of paths used before and after adding the link may be different from each other, respectively. Denote by k the ratio of cost degradation by adding the link, and thus, $k = C_c/C_o$. In this paper, k is considered the measure of cost degradation by adding link 1-2. $k > 1$ means paradoxical cost degradation. $k < 1$ means cost improvement that is naturally expected when a link is added to the network.

Consider a number of types of networks reflecting the degree of specialization in the following subsections.

2.1 General Braess networks

- [Type-G0] or [**General Braess network**] (Fig. 1) The cost of links 0-1, 1-3, 0-2, and 2-3 are, respectively, $a(x)$, $b(x)$, $d(y)$, and $c(y)$ before adding link 1-2. The cost of links 0-1, 1-3, 0-2, 2-3, and 1-2 are, respectively, $a(u+v)$, $b(u)$, $d(v)$, $c(v+w)$, and $t(w)$ after adding the link. a and c are strictly increasing functions. b, d, and t are non-decreasing functions. All the link cost functions

[1] As of March 2001, some 60 references to the Braess paradox are listed on the web page titled Paradoxes on Traffic Flow by Professor Braess (URL, http://homepage.ruhr-uni-bochum.de/Dietrich.Braess/#paradox). But there are more related papers than those shown in the list.

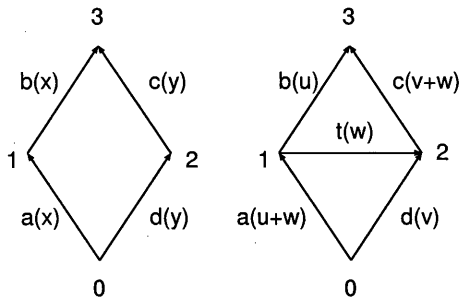

Figure 1: General Braess network. (Type-G0). **Left**: The network before link 1-2 is added. **Right**: The network after link 1-2 is added. Similarly for the subsequent pairs of figures.

are differentiable. If all of x, y, u, v, and w are positive, the following relations hold.

$$C_o = a(x) + b(x) = c(y) + d(y), \tag{2}$$

$$C_c = a(u+w) + b(u) = c(v+w) + d(v) \tag{3}$$

$$= a(u+w) + t(w) + c(v+w). \tag{4}$$

- [Type-G1] (Fig. 2) A subset of Type-G0 networks for which the cost of link 1-2 is independent of

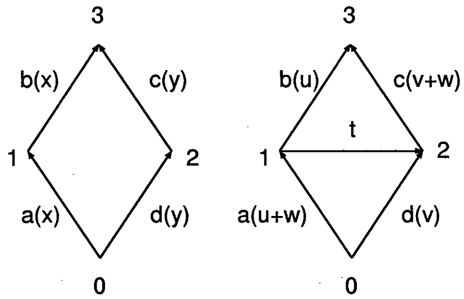

Figure 2: Type-G1 network.

the flow of the link, i.e., $t(\bullet) = t$.

- [Type-G2] (Fig. 3) A subset of Type-G1 networks for which the costs of links 1-3 and 0-2 are

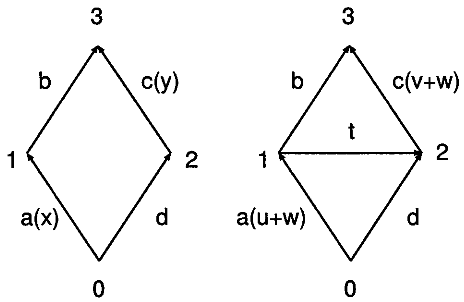

Figure 3: Type-G2 network.

independent of the link flow, and, respectively, $b(\bullet) = b$ and $d(\bullet) = d$. If all of x, y, u, v, and w are positive, the following relations hold.

$$C_o = a(x) + b = c(y) + d, \tag{5}$$
$$C_c = a(u{+}w) + b = c(v{+}w) + d = b + d - t$$
$$= a(u{+}w) + t + c(v{+}w). \tag{6}$$

- [Type-G3] or [**Reduced Braess network**] (Fig. 4) A subset of Type-G2 networks for which

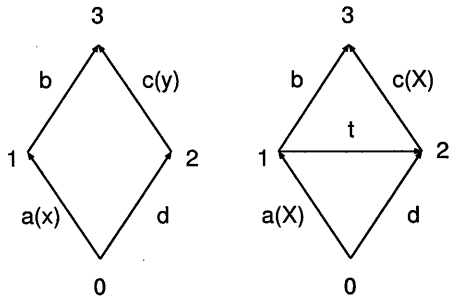

Figure 4: Reduced Braess (Type-G3) network.

$u = v = 0$, $w = X$, and the following relations hold after the link 1-2 is added.

$$C_o = a(x) + b = c(y) + d, \tag{7}$$
$$C_c = a(X) + b = c(X) + d = b + d - t$$
$$= a(X) + t + c(X), \tag{8}$$

which imply

$$a(X) + t = d \text{ and } c(X) + t = b. \tag{9}$$

Since the cost of any unused path cannot be less than the cost of used paths, $0 < x, y < X$.

- [Type-G4] or [**Symmetric reduced Braess network**] (Fig. 5) A subset of Type-G3 networks for

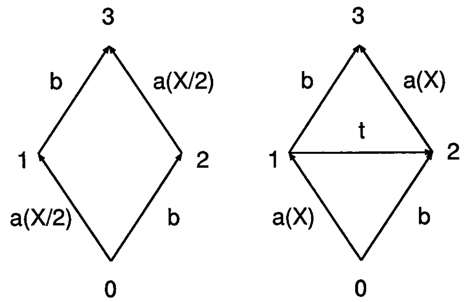

Figure 5: Symmetric Braess (Type-G4) network.

which $a = c$, $b = d$, and $x = y = X/2$ hold.

2.2 Braess networks with linear link cost

The original Braess paradox was discovered for the network that has linear functions for link costs [1].

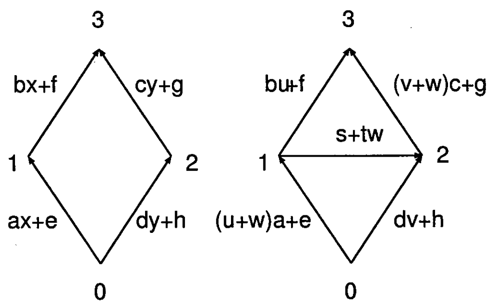

Figure 6: linear Braess (Type-L0) network.

- [Type-L0] or [Linear Braess network] (Fig. 6) A subset of general Braess (Type-G0) networks for which the link costs are linear functions [1]. The networks of this type are called **linear Braess networks** or Type-L0 networks here. Denote the costs of links 0-1, 1-3, 2-3, 0-1, and 1-2 by $a\lambda_{01}+e$, $b\lambda_{13}+f$, $c\lambda_{23}+g$, $d\lambda_{01}+h$, and $t\lambda_{12}+s$, respectively where λ_{ij} denotes the flow of the link i-j.

  In the original Braess network [1], $a = c = 10$, $e = g = 0$, $b = d = 1$, $f = h = 50$, $t = 1$, $s = 10$, and $X = 6$. In that case, $C_o = 83$, $C_c = 92$, and the measure of paradoxical cost degradation is $k = C_c/C_o = 1.1084\ldots$.

- [Type-L1] (Fig. 7) Networks that are both Type-G3 and Type-L0, and for which the costs of links

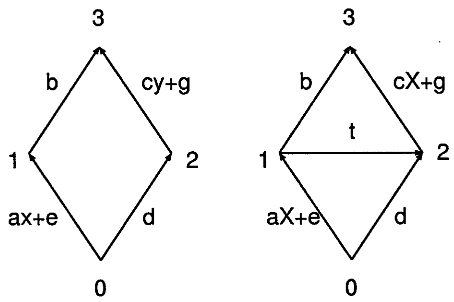

Figure 7: Type-L1 network.

  0-1 and 2-3 are linear functions and are, respectively, $a(\lambda_{01}) = a\lambda_{01} + e$ and $c(\lambda_{23}) = c\lambda_{23} + g$. If both x and y are positive, the following relations hold.

$$C_o = ax + e + b = cy + g + d, \tag{10}$$

$$\begin{aligned} C_c &= aX + e + b = cX + d = b + d - t \\ &= aX + e + t + cX + g. \end{aligned} \tag{11}$$

- [Type-L2] or [**Reduced linear Braess network**] (Fig. 8) A subset of Type-L1 networks for which the costs of links 0-1 and 2-3 are, respectively, $a(\lambda_{01}) = a\lambda_{01}$, $c(\lambda_{23}) = c\lambda_{23}$, i.e., $e = g = 0$. Note that, as in Type-G3, x and y are positive. The following relations hold.

$$C_o = ax + b = cy + d, \tag{12}$$

$$\begin{aligned} C_c &= aX + b = cX + d = b + d - t \\ &= aX + t + cX. \end{aligned} \tag{13}$$

- [Type-L3] or [**Symmetric reduced linear Braess network**](Fig. 9) A subset of Type-L2 networks for which $a = c$, $b = d$, and thus $x = y = X/2$, that is, symmetric networks.

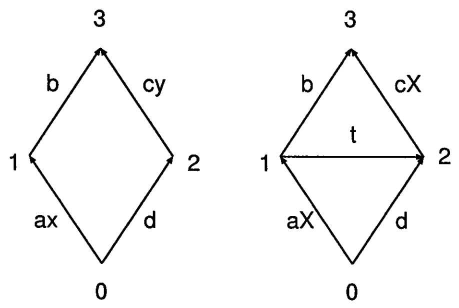

Figure 8: Reduced linear Braess (Type-L2) network.

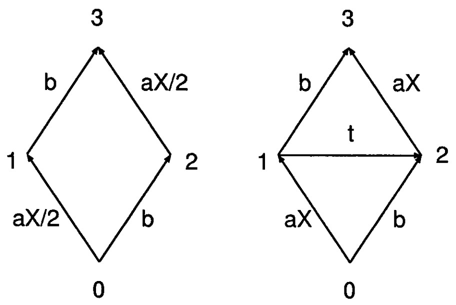

Figure 9: Symmetric reduced linear Braess (Type-L3) network.

2.3 Cohen-Kelly networks

Cohen and Kelly [7] considered a Braess network that has a type of hyperbolic functions expressing exponential single-server queueing delays and constants expressing simple delays for link costs.

- [Type-N0] or [**Cohen-Kelly network**] (Fig. 10) A subset of Type-G2 networks for which the costs of links 0-1 and 2-3, are, respectively, $a(\lambda_{01}) = \alpha/(a-\lambda_{01})$ and $c(\lambda_{23}) = \gamma/(c-\lambda_{23})$ for $0 \le \lambda_{01} < a$ and $0 \le \lambda_{23} < c$. Otherwise, $a(\lambda_{01})$ and $c(\lambda_{23})$ are infinite. As in Type-G3 networks, both x and y

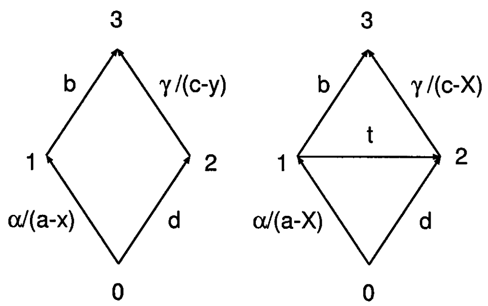

Figure 10: Cohen-Kelly (Type-N0) network.

are positive and less than X, and the following relations hold.

$$C_o = \frac{\alpha}{a-x} + b = \frac{\gamma}{c-y} + d, \tag{14}$$

$$\begin{aligned} C_c &= \frac{\alpha}{a-X} + b = \frac{\gamma}{c-X} + d = b + d - t \\ &= \frac{\alpha}{a-X} + t + \frac{\gamma}{c-X}. \end{aligned} \tag{15}$$

Cohen and Kelly [7] considered a network of this type for which $\alpha = \gamma = 1$, $a = c = \phi$, $b = d = 2$, $t = 1$, and $X = 2\lambda$, which is actually symmetric. They showed that $C_c = 3$ and $C_o = 1/(\phi-\lambda)+2 < 3$ assuming that $2\lambda > \phi - 1 > \lambda > 0$, which is a Braess paradox.

- [Type-N1] or [**Symmetric Cohen-Kelly network**] (Fig. 11) A subset of Cohen-Kelly (Type-N3)

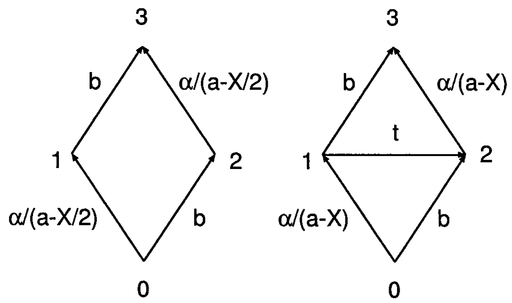

Figure 11: Symmetric Cohen-Kelly (Type-N1) network.

networks for which $a = c$, $b = d$, and thus $x = y = X/2$, that the networks are symmetric.

3 The Results

3.1 General Braess networks

Lemma 1 For any Type-G0 network, there exists a Type-G1 network that has the same measure k as the Type-G0 network has.

[Proof] C_o is independent of the cost of link 1-2. Thus even if the cost of link 1-2 is replaced by the new constant cost $t = t(w)$, the value of measure k does not change. ∎

Lemma 2 For any Type-G1 network, there exists a Type-G2 network that has the value of measure k that is not less than that of the Type-G1 network.

[Proof] 1) It can be shown that $x \geq u$, $y \geq v$, $x \leq u + w$, and $y \leq v + w$, by contradiction as follows: Assume $x < u$. Then $b(x) \leq b(u)$ and $x < u + w$, from which follows $a(x) < a(u+w)$. Therefore,
$C_o = a(x) + b(x) < a(u+w) + b(u) = C_c$.
On the other hand, from (1) and $x < u$ by assumption, it follows that $y > v + w$, from which follows $c(y) > c(v + w)$. $y > v + w$ implies $y > v$, from which follows $d(y) > d(v)$. Therefore
$C_o = c(y) + d(y) > c(v + w) + d(v) = C_c$, which contradicts with the above. Thus it must hold that $x \geq u$. In a similar way, it holds that $y \geq v$, $x \leq u + w$, and $y \leq v + w$.
2) Note that if the link costs $b(\bullet)$ and $d(\bullet)$ are replaced by constants b (= $b(u)$) and d (= $d(v)$), respectively, the equilibrium flows of the network with link 1-2 are the same as u, v, and w before this replacement.
Then, make a Type-G2 network by replacing the link costs $b(\bullet)$ and $d(\bullet)$ by constants b and d, respectively. Let the flows of paths 0-1-3 and 0-2-3 be $\hat{x}$ and $\hat{y}$ before adding link 1-2 to the Type-G2 network, respectively. Note that
$\hat{x} + \hat{y} = x + y = X$. Therefore, either $\hat{x} \leq x$ or $\hat{y} \leq y$ holds. Then,
$a(\hat{x}) \leq a(x)$ or $c(\hat{y}) \leq c(y)$.
3) Denote by $\hat{C_o}$ the value of C_o of the Type-G2 network. By noting the above 2) and by recalling that $b = b(u) \leq b(x)$ and $d = d(v) \leq d(x)$ it holds that

$$\begin{aligned}\hat{C_o} &= a(\hat{x}) + b = c(\hat{y}) + d \\ &\leq a(x) + b(x) = c(y) + d(y) = C_o.\end{aligned}$$

4) Recall that the value of C_c is the same for both the Type-G1 and Type-G2 networks. It has thus been shown that the measure k of the Type-G2 network is not less than that of the Type-G1 network. ■

Lemma 3 In the Type-G2 network, C_o is increasing in b and d.

[Proof] From $a(x) + b = c(y) + d$ (the same as (7)) and $x + y = X$ (1), it follows that

$$\frac{da}{dx}\frac{dx}{db} - \frac{dc}{dy}\frac{dy}{db} = -1, \quad \frac{dx}{db} = -\frac{dy}{db}$$
$$\text{and} \quad \frac{dx}{db} = -1/(\frac{da}{dx} + \frac{dc}{dy}).$$

Then, by noting that $a(\bullet)$ and $c(\bullet)$ are increasing, it follows that,

$$\begin{aligned} \frac{dC_o}{db} &= \frac{d}{db}[a(x) + b] = \frac{da}{dx}\frac{dx}{db} + 1 \\ &= \frac{dc}{dy}/(\frac{da}{dx} + \frac{dc}{dy}) > 0. \end{aligned} \tag{16}$$

Thus C_o is increasing in b. Similarly C_o is increasing in d. ■

Lemma 4 For any Type-G2 network, there exists a Type-G3 network that has the value of measure k that is not less than that of the Type-G2 network.

[Proof] 1) Consider the case where $b > c(X) + t$. Then $u = 0$ and $v + w = X$. Thus, if b is reduced to $c(X) + t$, u, v, w and C_c remain unchanged. On the other hand, C_o decreases as, by Lemma 3, C_o is increasing in b. Similarly for the case where $d > a(X) + t$.

2) Next consider the case where $b < c(X) + t$ and $d < a(X) + t$. Then, it must hold that $u > 0$, $v > 0$, $u + w < X$, $v + w < X$, $b = c(v+w) + t$ and $d = a(u+w) + t$.

Then, define $\xi < 1$ and $\eta < 1$ as follows: $u+w = \xi X$ and $v+w = \eta X$. Consider a network that has the link cost functions $a_\xi(\lambda_{01}) = a(\xi\lambda_{01})$ and $c_\eta(\lambda_{23}) = c(\eta\lambda_{23})$ in place of $a(\lambda_{01})$ and $c(\lambda_{23})$, respectively. In this new network, $u = v = 0$, $w = X$, $b = c_\eta(X) + t$, and $d = a_\xi(X) + t$, and this is a Type-G3 network, but C_c is the same as the previous Type-G2 network. The flows $\hat{x}$ and $\hat{y}$ of links 0-1 and 2-3 in this Type-G3 network before adding link 1-2 satisfy the following:

$$\begin{aligned} a_\xi(\hat{x}) + b &= c_\eta(\hat{y}) + d, \\ \text{i.e.,} \quad a(\xi\hat{x}) + b &= c(\eta\hat{y}) + d, \\ \hat{x} + \hat{y} &= X. \end{aligned} \tag{17}$$

Then, by making derivation with respect to ξ,
$a'(\xi\hat{x})(\xi\frac{d\hat{x}}{d\xi} + \hat{x}) = c'(\eta\hat{y})\eta\frac{d\hat{y}}{d\xi}$, and $\frac{d\hat{x}}{d\xi} + \frac{d\hat{y}}{d\xi} = 0$, from which follows $\frac{d\hat{y}}{d\xi} = \frac{a'\hat{x}}{a'\xi + c'\eta}$. Then,

$$\frac{dC_o}{d\xi} = \frac{d}{d\xi}[c(\eta\hat{y}) + b] = c'\eta\frac{d\hat{y}}{d\xi} = \frac{a'c'\hat{x}\eta}{a'\xi + c'\eta} > 0.$$

This implies that C_o decreases with decrease in ξ. Similarly, C_o decreases with decrease in η.

Therefore, the new Type-G3 network ($\xi, \eta < 1$) has the same value of C_c but a smaller value of C_o compared to the Tyep-G2 network ($\xi = \eta = 1$).

3) Then, make a Type-G3 network as follows:

If $b > c(X) + t$ in the Type-G2 network, make the cost of link 2-3 $c(X) + t = b$ (b reduced) in the Type-G3 network.

If $d > a(X) + t$ in the Type-G2 network, make the cost of link 0-1 $a(X) + t = d$ (d reduced) in the Type-G3 network.

After the above replacement if $b \leq c(X)+t$ and $d \leq a(X)+t$, obtain $\xi = (u+w)/X$ and $\eta = (v+w)/X$, and make the costs of links 0-1 and 2-3, respectively, $a(\xi\lambda_{01})$ and $c(\eta\lambda_{23})$ in the Type-G3 network.

From the above 1) and 2), it is shown that the Type-G3 network has the value of measure k that is not less than that of the Type-G2 network. ∎

From Lemmas 1, 2, and 4, the following proposition is derived.

Proposition 1 For any general Braess (Type-G0) network, there exists a reduced Braess (Type-G3) network that has the value of measure k that is not less than that of the general Braess network.

From Proposition 1, the following corollary follows:

Corollary 1 For any linear Braess (Type-L0) network, there exists a Type-L1 network that has the value of measure k that is not less than that of the linear Braess network.

Proposition 2 The Braess paradox, i.e., the paradoxical cost degradation, always occurs for reduced Braess (Type-G3) networks.

[Proof] The measure of paradoxical cost degradation k is, by noting that $b = c(X) + t$ and $d = a(X) + t$,

$$k = \frac{C_c}{C_o} = \frac{a(X) + c(X) + t}{a(x) + c(X) + t} = \frac{a(X) + c(X) + t}{c(y) + a(X) + t}. \tag{18}$$

If $x = X$, then $a(x) = a(X)$, and from (18), it follows that $k = 1$ and $y = X$, which is impossible. Thus, by noting that $x < X$ and $y < X$, paradoxical cost degradation always occurs by adding the link 1-2 to the reduced Braess network. ∎

Proposition 3 The measure of paradoxical cost degradation of general Braess (Type-G0) networks is not more than 2.

[Proof] From (18),

$$a(x) + c(X) = c(y) + a(X). \tag{19}$$

Eq. (19) implies that if $a(X) \gg a(x)$ then $c(X) \gg a(x)$ and $a(X) \lesssim c(X)$. Then, $\frac{a(X) + c(X) + t}{a(x) + c(X) + t}$ is less than but can be close to $k_a = \frac{a(X) + c(X)}{c(X)} \lesssim 2$ when $t = 0$ and $a(X) \gg a(x)$. $k_a = 2$ when $a(X) = c(X)$. The situation where both $a(X) = c(X)$ and $a(X) \gg a(x)$ hold is achieved when $c(X) \gg c(y)$ by noting (19). Similarly for $\frac{a(X) + c(X) + t}{a(X) + c(y) + t}$. Therefore, k is less than but can be close to 2 in the situation where $t = 0$, $a(X) = c(X)$, $a(X) \gg a(x)$ and $c(X) \gg c(y)$. Such a situation will be achieved by a symmetric network where $a(\bullet) = c(\bullet)$, $b = d$, and thus $x = y$ as an example will be shown in subsection 3.3. In any case, the worst ratio of paradoxical cost degradation by adding the link is at most 2. ∎

This is in contrast to the KAKH network [12] where the measure of paradoxical cost degradation can be unlimitedly large as is given in Section 5.

3.2 Braess networks with linear link cost

This subsection discusses the Braess networks that have linear functions for link costs, i.e., linear Braess networks, here [1]. Recall Corollary 1 that for any linear Braess (Type-L0) network, there exists Type-L1 network that has the value of measure k that is not less than that of the linear Braess network.

Lemma 5 For any Type-L1 network, there exists a reduced linear Braess (Type-L2) network that has the value of measure k that is not less than that of the Type-L1 network.

[Proof] Consider the case where $g > 0$ in a Type-L1 network. Note that $b \geq g$ since, from the definition of Type-G3 networks, $b = cX + g + t$ holds in the network with link 1-2. Then consider a new network that has, as the costs of links 1-3 and 2-3, $b - g$ and $c(\lambda_{23})$ replacing b and $c(\lambda_{23}) + g$ of the Type-L1 network, respectively. It is clear that the link flows of the network before and after adding link 1-2 remain the same in both the Type-L1 and new networks.

By noting $x < X$, however, it holds that

$$k^{new} = \frac{aX + e + b - g}{ax + e + b - g} > \frac{aX + e + b}{ax + e + b} = k^{Type-L1}.$$

Thus the new network has the value of measure k not less than the old Type-L1 network. Similar arguments hold for the links 0-1 and 0-2.

Therefore, if a Type-L2 network is made out of the Type-L1 network by replacing the costs of links 1-3, 2-3, 0-2, and 0-1 by $b - g$, $c(\lambda_{23})$, $d - e$, and $a(\lambda_{01})$, respectively, the Type-L2 network has the value of measure k not less than the old Type-L1 network. ∎

Lemma 6 For any Type-L2 network, there exists a symmetric reduced linear Braess (Type-L3) network that has the value of measure k that is not less than that of the Type-L2 network.

[Proof] Recall that the following relations hold for Type-L2 networks.

$$\begin{aligned} C_o &= ax + b = cy + d, \text{ (eq. (12))} \\ C_c &= aX + b = cX + d = b + d - t \\ &= aX + t + cX. \text{ (eq. (13))} \end{aligned}$$

Consider the group of Type-L2 networks for which C_c, X and t are the same. Then, $b+d$ is also the same for the group. From (13), it follows that $b = cX + t$ and $d = aX + t$, from which follows $a + c$ is also the same for the group. Then define $A = a+c$ and $B = b+d$. Again from (13), it follows that $b-d = (c-a)X$. From (12), it follows that $ax - cy = d - b$, from which follows $[A(x-y) + X(a-c)]/2 = d - b$. Combining the above two relations, it follows that

$$A(x - y) = X(a - c) = d - b. \tag{20}$$

Then, by noting (20),

$$\begin{aligned} 2C_o &= (ax + b) + (cy + d) = ax + cy + B \\ &= B + \frac{AX}{2} + \frac{(a-c)(x-y)}{2} \\ &= \frac{AX}{2} + B + \frac{X(a-c)^2}{2A}, \end{aligned} \tag{21}$$

which is smallest for $a = c$, and thus $b = d$ and $x = y$.

Thus, for any Type-L2 network with arbitrary values of C_c, X and t, there exists a symmetric reduced Braess (Type-L3) network that has the same values of C_c, X and t and for which the measure of paradoxical cost degradation is not less than the Type-L2 network. ∎

From Corollary 1, Lemmas 5 and 6, the following proposition follows.

Proposition 4 For any linear Braess network, there exists a symmetric reduced linear Braess (Type-L3) network that has the value of measure k that is not less than the linear Braess network.

Proposition 5 Among linear Braess networks, the largest value of measure k is achieved by a symmetric reduced linear Braess (Type-L3) network, and it is 4/3.

[Proof] Consider a Type-L3 network. Then, $x = X/2$, $b = aX + t$. Thus,

$$k = \frac{C_c}{C_o} = \frac{aX + b}{ax + b} = \frac{2aX + t}{3aX/2 + t} \leq \frac{4}{3}.$$

The equality of the above holds when $t = 0$. Therefore, it is seen that the worst ratio of paradoxical cost degradation by adding the link 1-2 in the linear Braess networks is at most 4/3, which is achieved by a symmetrical network. ∎

3.3 Cohen-Kelly networks

This subsection discusses Cohen-Kelly networks, which have a type of hyperbolic functions and/or constants for their link costs [7].

Proposition 6 Among Cohen-Kelly networks, the worst value, 2, of the measure k of paradoxical cost degradation due to adding link 1-2 is asymptotically achieved by symmetric Cohen-Kelly (Type-N1) networks.

[Proof] As in the proof of Proposition 3, the measure of paradoxical cost degradation for general Braess networks can be close to the worst value, 2, in the situation where $t = 0$, $a(X) \simeq c(X)$, $a(X) \gg a(x)$ and $c(X) \gg c(y)$. That is, in Cohen-Kelly networks, $t = 0$, $1/(a - X) \gg 1/(a - x)$, $1/(c - X) \gg 1/(c - y)$, $1/(a - X) = 1/(c - X)$, and thus $a = c$ to which X is close. Therefore, since $b = 1/(a - X) + t$ and $d = 1/(c - X) + t$, it follows that $b = d$ and, thus, $x = y = X/2$. Then

$$k = \frac{C_c}{C_o} = \frac{2}{1 + \dfrac{a - X}{a - X/2}}.$$

It can be easily seen, therefore, that, in the Cohen-Kelly network, the measure of paradoxical cost degradation k is less than but close to 2 as X approaches $a = c$ with $t = 0$. ∎

4 Braess network embedded in networks of general topology

Dafermos and Nagurney [6] showed the following properties on general networks in the Wardrop equilibrium under the condition that every link cost function is non-decreasing in the flow through the link.

[Property I] If one link in the network is improved while the rest remain unchanged, the load on the link cannot decrease and the incurred link cost cannot increase.

[Property II] If only the total flow associated with a particular O-D pair is increased, while the rest remain the same, then the cost of the paths associated with the O-D pair can never decrease.

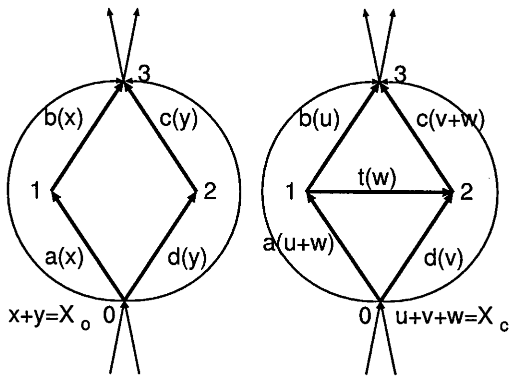

Figure 12: Braess network embedded in a network of general topology. (Type-G0). **Left**: The network before link 1-2 is added. **Right**: The network after link 1-2 is added.

On the basis of these properties, the estimate of k is derived for the Braess network embedded within a general network as follows (Fig. 12). The following two ways in which the Braess network is embedded as a sub-network within a general network are considered.

1] Some paths of other O-D pairs share the paths of the Braess network as their links.

2] Some other sub-networks share only the O-D pair with the Braess network.

Proposition 7 Consider a network in which there are one or more other sub-networks that share only the O-D pair with the Braess network embedded and in which one or more other O-D pairs share the paths of the Braess network as their links. Suppose that the Braess paradox occurs in the embedded Braess network, i.e., $C_c \geq C_o$. Then, there exists an independent Braess network that has no such sharing but for which the measure k of paradoxical cost degradation is not less than the original embedded Braess network that has such sharing.

[Proof] Denote the resulting total flows through the Braess network before and after adding link 1-2 by X_o and X_c, respectively.

1) In the general network, the collection of the paths of the embedded Braess network can be regarded as a united link whose cost is $C_o(X_o)$ and $C_c(X_c)$ before and after adding link 1-2, respectively. Since $C_o(X_o) \leq C_c(X_c)$, then, from Property I by Dafermos-Nagurney, the total flow X_o of the united link is not less than the total flow X_c after adding link 1-2, i.e., $X_o \geq X_c$. Then, from the Property II by Dafermos-Nagurney, $C_c(X_o) \geq C_c(X_c)$.

2) Consider an independent Braess network that is cut off from the general network and for which the total flow is X_o. Then, since $C_c(X_o) \geq C_c(X_c)$ from 1), the independent Braess network has the value of k that is not less than that of the embedded Braess network. ∎

Then from Propositions 3 and 7, the following proposition is derived.

Proposition 8 The measure of paradoxical cost degradation of general Braess (Type-G0) networks embedded in the way considered is not more than 2.

5 KAKH Network

The previous sections showed that the measure of paradoxical cost degradation for Braess networks can be of certain magnitude, but that it cannot be over some finite limit. This section gives an example of networks, a symmetric KAKH network [12], for which the measure of the Braess-like paradoxical cost degradation can be unlimitedly large.

5.1 Description of a symmetric KAKH network

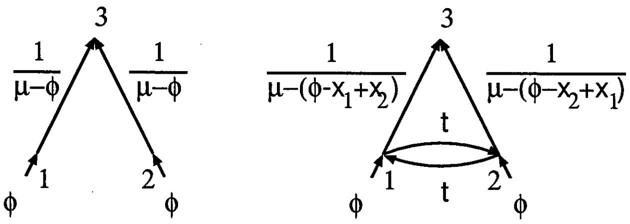

Figure 13: Symmetric KAKH network.

The symmetric KAKH network considered here consists of three nodes: two origins (numbered 1 and 2) and one destination (numbered 3), i.e., two O-D pairs (1,3) and (2,3). Before adding capacity, the network has one path for each O-D pair: 1-3 for (1,3) and 2-3 for (2,3). After adding capacity, i.e., a pair of links connecting two origins (1-2 and 2-1), the network has two paths, 1-3 and 1-2-3, for the O-D pair (1,3), and two paths, 2-3 and 2-1-3 for the O-D pair (2,3). See Fig. 13. Assume that the costs of links 1-3 and 2-3 are $D(\beta_1)$ and $D(\beta_2)$, respectively, where β_i denotes the flow of link i-3, $i = 1, 2$, and

$$D(\beta_i) = \frac{1}{\mu - \beta_i} \text{ for } \beta_i < \mu \text{ (otherwise infinite).} \tag{22}$$

(The situation of this assumption can be achieved by simply assuming the external time-invariant Poisson arrival for each node with rate ϕ, and the exponentially distributed service time for each user with identical service rate μ at both links 1-3 and 2-3.) Assume that the costs of links 1-2 and 2-1 are t and independent of the flow through the link. Denote the total flow of each O-D pair is ϕ.

There is one decision maker, or a player, that strives to minimize the cost of the flows of each O-D pair, i.e., decision makers 1 and 2 for O-D pairs (1,3) and (2,3), respectively. Before adding links there is no choice to each decision maker.

After adding links, decision makers 1 and 2 choose non-cooperatively the amount of flows of each two paths out of the total flow ϕ of each of the origins 1 and 2, respectively. Denote the flows of links 1-2 and 2-1 by x_1 and x_2, respectively. $0 \leq x_i \leq \phi$, $i = 1, 2$. Then the decision makers 1 and 2 send the remaining amounts of flows, $\phi - x_1$ and $\phi - x_2$, to paths 1-3 and 2-3, respectively. The resulting flow β_i through link i-3, $i = 1, 2$, is

$$\beta_i = \phi - x_i + x_j, \ i \neq j. \tag{23}$$

Assume that each decision maker strives non-cooperatively to optimize the cost of the flows associated only with the corresponding O-D pair. Denote the vector (x_1, x_2) by $\boldsymbol{x}$. Denote the set of $\boldsymbol{x}$'s that satisfy the constraints by $\boldsymbol{C}$. Thus, within these constraints, the value of x_i $(i = 1, 2)$ are chosen to achieve optimization non-cooperatively by decision maker i.

Thus the cost of the flows associated with the O-D pair (i,3) is

$$T_i(\boldsymbol{x}) = \frac{1}{\phi}\{(\phi - x_i)T_{ii}(\boldsymbol{x}) + x_i T_{ij}(\boldsymbol{x})\}, \ j \neq i, \tag{24}$$

where

$$T_{ii}(\boldsymbol{x}) = D(\beta_i), \quad \text{and} \tag{25}$$

$$T_{ij}(\boldsymbol{x}) = D(\beta_j) + t, \ \text{for } j \neq i. \tag{26}$$

(The above expressions hold, again, only for positive values of denominators, and are otherwise infinite.)

The situation where every decision maker has attained his/her own objective given the decision of the other decision maker is what is to be called *Nash equilibrium.* The Nash equilibrium is given by such $\tilde{\boldsymbol{x}}$ as satisfies the following for all i, k,

$$T_i(\tilde{\boldsymbol{x}}) = \min_{x_i} T_i(\tilde{\boldsymbol{x}}_{-(i)}; x_i), \quad \text{such that } (\tilde{\boldsymbol{x}}_{-(i)}; x_i) \in \boldsymbol{C},$$

where $(\tilde{\boldsymbol{x}}_{-(i)}; x_i)$ denotes the 2-dimensional vector in which the element corresponding to $\tilde{x}_i$ has been replaced by x_i. $\tilde{\boldsymbol{x}}$ is called *solution* for the above non-cooperative optimization.

5.2 The solution

The Nash equilibrium of the symmetric KAKH network is given as follows:

(A) The case where $t > \phi/\{(\mu - \phi)^2\}$: The solution $\tilde{\boldsymbol{x}}$ is unique and given as follows:

$$\tilde{\boldsymbol{x}} = \boldsymbol{0}, \text{ i.e., } \tilde{x}_1 = \tilde{x}_2 = 0.$$

Then,

$$T(\tilde{\boldsymbol{x}}) = T_i(\tilde{\boldsymbol{x}}) = \frac{1}{\mu - \phi}, \quad i = 1, 2, \quad k = 1, 2, \cdots, n.$$

(B) The case where $t \leq \phi/\{(\mu - \phi)^2\}$: The solution $\tilde{\boldsymbol{x}}$ is unique and given as follows:

$$\tilde{x}_1 = \tilde{x}_2 = \frac{1}{2}\{\phi - t(\mu - \phi)^2\}. \tag{27}$$

In that case,

$$\begin{aligned} T(\tilde{\boldsymbol{x}}) &= T_1(\tilde{\boldsymbol{x}}) = T_2(\tilde{\boldsymbol{x}}) \\ &= \frac{1}{\mu - \phi} + \frac{t}{2}\{\phi - t(\mu - \phi)^2\}. \end{aligned} \tag{28}$$

[Proof] From definitions (23), (22), and (24),

$$\phi\frac{\partial T_i}{\partial x_i} = -\frac{\mu - x_j}{(\mu-\phi+x_i-x_j)^2} + \frac{\mu-\phi+x_j}{(\mu-\phi-x_i+x_j)^2} + t \quad (i \neq j). \tag{29}$$

By simple inspection of (29), it is seen that $\frac{\partial T_i}{\partial x_i}$ is monotonically increasing with the increase in x_i with feasible $\boldsymbol{x} \in \boldsymbol{C}$. Thus if there exists a set of such values of $\tilde{\boldsymbol{x}}$ that satisfies

$$\frac{\partial T_i}{\partial x_i}(\tilde{\boldsymbol{x}}) = 0, \text{ for all } i, \tag{30}$$

then the set of values is a solution of the Nash equilibrium. From (29) and by defining $d = x_1 - x_2$,

$$\begin{aligned}\phi(\frac{\partial T_1}{\partial x_1} - \frac{\partial T_2}{\partial x_2}) &= \frac{2\mu-(\phi+d)}{(\mu-\phi-d)^2} - \frac{2\mu-(\phi-d)}{(\mu-\phi+d)^2} \\ &= \{\frac{2d}{(\mu-\phi)^2-d^2}\}\{\frac{2\mu(\mu-\phi)}{(\mu-\phi)^2-d^2}+1\},\end{aligned} \tag{31}$$

If condition (30) holds, then from (31), $d = 0$. Then from (29),

$$\phi\frac{\partial T_i}{\partial x_i} = \frac{2x_i - \phi}{(\mu-\phi)^2} + t = 0, \quad (i \neq j) \text{ for all } i. \tag{32}$$

Therefore

$$x_i = \frac{1}{2}\{\phi - t(\mu-\phi)^2\} \text{ for all } i \quad \text{if } t \leq \frac{\phi}{(\mu-\phi)^2}. \tag{33}$$

From the above derivation, it is clear that this is a unique solution (in case (B)).

If $t > \frac{\phi}{(\mu-\phi)^2}$ (in case (A))), from (32) when $x_i = 0$, for all i,

$$\phi\frac{\partial T_i}{\partial x_i} = t - \frac{\phi}{(\mu-\phi)^2} > 0, \qquad \text{for all } i. \tag{34}$$

Since $\frac{\partial T_i}{\partial x_i}$ is monotonically increasing in x_i, $\tilde{\boldsymbol{x}} = \boldsymbol{0}$, i.e., $\tilde{x}_i = 0$, for every i, is a Nash equilibrium solution. Uniqueness of this solution in the case (A) can be shown by contradiction as follows. Suppose $\tilde{x}_1 > 0$. From definitions on d and by (29),

$$\phi\frac{\partial T_1}{\partial x_1}\bigg|_{x_1=\tilde{x}_1} = -\frac{\mu+d-\tilde{x}_1}{(\mu-\phi+d)^2} + \frac{\mu-\phi-d+\tilde{x}_1}{(\mu-\phi-d)^2} + t = 0. \tag{35}$$

Then from the above and condition on t,

$$\begin{aligned}0 &< \tilde{x}_1\{\frac{1}{(\mu-\phi+d)^2} + \frac{1}{(\mu-\phi-d)^2}\} \\ &= -t - \frac{2d}{(\mu-\phi)^2-d^2} + \frac{\phi}{(\mu-\phi+d)^2} \\ &< -\frac{\phi}{(\mu-\phi)^2} - \frac{2d}{(\mu-\phi)^2-d^2} + \frac{\phi}{(\mu-\phi+d)^2} \qquad (36) \\ &= -d\{\frac{\phi(2\mu-2\phi+d)}{(\mu-\phi+d)^2(\mu-\phi)^2} + \frac{2}{(\mu-\phi)^2-d^2}\}. \qquad (37)\end{aligned}$$

This implies $d < 0$ for which there must exist nonzero x_2. Then by using the argument similar to the above on x_2, it is derived that $d > 0$, which is a contradiction. Thus $\tilde{\boldsymbol{x}} = \boldsymbol{0}$ is the unique Nash equilibrium solution. ■

In Nash equilibrium with the case (A), users arriving at each origin flow through only one path, and thereby the network has no cost improvement or degradation due to adding the links.

On the other hand, in the Nash equilibrium with the case (B), each decision maker forwards a part of its flow through the link to the other origin node to flow through the other link to the destination, and thereby has degradation in its own cost. *The ratio of such degradation can become unlimitedly large* as the total arrival rate ϕ at each O-D pair approaches the capacity μ of each link that is directly connected to the destination, as seen below.

Consider the case (B). It can be easily seen that $T_i(\tilde{\boldsymbol{x}})(= T(\tilde{\boldsymbol{x}}))$, for every i, has its maximum $\tilde{T}(\mu, \phi)$ (i.e., the worst cost) for given μ and ϕ.

$$\tilde{T}(\mu, \phi) = \frac{1}{\mu - \phi}\{1 + \frac{\phi}{8(\mu - \phi)}\}, \tag{38}$$

when

$$t = \frac{\phi}{2(\mu - \phi)^2}. \tag{39}$$

Thus after adding the links with link cost t $(= \phi/\{2(\mu-\phi)^2\})$ to the network the cost of each decision maker, $T_i(\tilde{\boldsymbol{x}})$, increases in the amount of $\frac{\phi}{8(\mu - \phi)^2}$ (i.e., the cost degrades). This is a Braess-like paradox. Denote the measure or the *worst ratio of the paradoxical cost degradation* for given μ and ϕ by $\Delta(\mu, \phi)$. Then

$$\Delta(\mu, \phi) = \frac{\tilde{T}(\mu, \phi) - T_0(\mu, \phi)}{T_0(\mu, \phi)}, \tag{40}$$

where $T_0(\mu, \phi) = 1/(\mu - \phi)$ is the cost of each decision maker for given μ and ϕ when the network has neither link 1-2 nor link 2-1. Then,

$$\Delta(\mu, \phi) = \frac{\phi}{8(\mu - \phi)}. \tag{41}$$

Relation (41) shows that the measure of paradoxical cost degradation after adding links can be unlimitedly large as μ approaches ϕ. For example,

$\Delta(1.01, 1) = 12.50$ (i.e., 1250% degradation)
$\Delta(1.001, 1) = 125$ (i.e., 12500% degradation),
$\Delta(1.00001, 1) = 12500$ (i.e., 1250000% degradation), etc.

6 Concluding Remarks

The present paper has examined mainly 4-node networks of the same topology as the original Braess network and has shown that the measure of paradoxical cost degradation due to adding capacity (a link) to the networks can be up to 2. That is, in the worst case, adding capacity to a network makes the users suffer the cost twice as much as the cost before adding the capacity. The measure greater than 2 cannot be achieved even if the Braess network is embedded within a general network in the way considered.

Even though some people may not think the ratio 2 a big figure, the finding of the Braess paradox is surprising and many researchers may share the recognition of its significance. As is shown in the later section of this paper, there exist cases of Braess-like paradoxical cost degradation the measure of which can be unlimitedly large. Therefore, it is important to continue research on this type of paradoxes that have been first exemplified by the Braess paradox.

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