Analytic Study of Distributed Optima Coexisting within a Network of Homogeneous Computers

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Analytic Study of Distributed Optima Coexisting within a Network of Homogeneous Computers

Hisao Kameda*, Eitan Altman† and Odile Pourtallier‡

Abstract

The study of equilibria in networks has gained much interest in recent years with the emergence of new application areas. Along with the ongoing research on road traffic equilibria, there has been new interest in understanding competitive situations in telecommunications as well as in parallel and distributed computing. In addition to the classical Wardrop equilibria, which have been the main solution concept for road traffic assignment problems (in which a single player is atomless, i.e. it has a negligible impact on performance of other players), there has been a growing interest in the study of Nash equilibria that allow to handle atomic players (such as service providers). In this paper we study the *combination* of several types of equilibria in a single network. More precisely, we consider the situation in which some finite number of players are atomic; each one of these has a large amount of traffic to ship, and the decisions of these players have a nonnegligible effect on the performance (or on the cost) of other players. At the same time, there may also be some other classes of players; each one of an infinite number of individual players within the latter classes is assumed to be atomless. We study both qualitative as well as quantitative issues in such mixed equilibria occurring in a distributed computer network. We focus on networks that have some symmetry properties and establish the uniqueness of the equilibria and analytic expressions for them.

keywords Distributed decision, Braess paradox, Nash equilibrium, Wardrop equilibrium, performance optimization, parallel queues, load balancing.

1 Introduction

The study of equilibria in networks has gained much interest in recent years with the emergence of new application areas. Along with the ongoing research on road traffic equilibria, there has been new interest in understanding competitive situations in telecommunications as well as in parallel and distributed computing.

The competition in telecommunications has increased with the deregulation of public monopolies. Hence, many optimization issues in telecommunication networks can no more be handled in a framework of a single decision maker, and equilibria notions have to be introduced. In the framework of computing, we have witnessed a rapid increase of the performance of computers and of interconnecting networks, and computers nowadays are

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often part of computer networks. This allows users, or classes of users, to distribute the execution of tasks among several connected processors. In this context too, classical optimization issues often cannot be handled using the framework of a single decision maker, and equilibria notions have to be introduced.

Both road traffic networks as well as computer and communication networks can be modeled as systems that consist of a finite number of facilities and arriving threads or flows of infinitely many customers to be served by the facilities. For example, distributed computer systems have continuing arrivals of infinitely many jobs to be processed by computers, communication networks have flows of infinitely many packets or calls to pass through communication links, and transportation flow networks have incoming threads of infinitely many vehicles to drive through roads, etc. We may have various objectives for distributed optimization of performance for such systems depending on the degree of the distribution of decisions.

(A) [Completely distributed decision scheme]: Each of infinitely many individuals, users, jobs, etc., optimizes its own cost or the expected response time for itself independently of others. In this optimized situation each of infinitely many individuals cannot receive any further benefit by changing its own decision. It is further assumed that each individual is atomless, i.e. the decision of a single individual has a negligible impact on the performance of other individuals. This optimized situation is called the individual optimum, Wardrop equilibrium, or user optimum (by some people). We call it the individual optimum or Wardrop equilibrium here.

(B) [Intermediately distributed decision scheme]: Infinitely many users, jobs, packets, or vehicles are classified into a finite number ($N > 1$) of groups, each of which has its own decision maker and is regarded as one player, user, or class. Each decision maker optimizes non-cooperatively its own cost or the expected response time over only the jobs of the class. Each decision maker is an atomic player, i.e. the decision of a single decision maker of a class has a nonnegligible impact on the performance of other classes. In this optimized situation each of a finite number of users, classes, or players cannot receive any further benefit by changing its decision. This optimized situation is called the class optimum, Nash non-cooperative equilibrium, or user optimum (by some other people). We call it the class optimum or Nash equilibrium here. We may have different levels in the intermediately distributed optimization.

Under quite general conditions we know that (B) approaches (A) when the number of players becomes infinitely many ($N > 1$) [7].

Unlike Wardrop equilibrium, for which the equilibrium is known to be unique under quite general conditions (see e.g. [2, 20] and references therein), the situation in the case of Nash equilibrium is more complicated. Counterexamples are known in which there are several different equilibria [19]. Only in some special cases (for special topologies or special cost functions) the uniqueness of Nash equilibrium has been established [1, 2, 9, 13, 19]. Another complication of Nash equilibrium is that it is harder to compute: it cannot be transformed into a convex optimization problem as is the case for Wardrop equilibrium (see e.g. [20]).

For further references to the related work on load balancing and paradoxes, see [11, 12, 13, 14, 15, 23] and [3, 4, 5, 6, 8, 9, 10, 16, 17], respectively.

In this paper we study the *combination* of several types of equilibria in a single network. More precisely, we consider the situation in which some finite number of players are atomic; each one of these have a large amount of traffic to ship, and the decisions of these players have a nonnegligible effect on the performance (or on the cost) of other players. At the same time, there may also be some other classes of players; each one of an infinite number of individual players within the latter classes is assumed to be atomless. We study both qualitative as well as quantitative issues in such mixed equilibria occurring in a distributed computer network. We focus on networks that have some symmetry properties. We establish the uniqueness of the equilibria and obtain analytic expressions for them.

The structure of the paper is as follows. In Section 2 we introduce the model for the distributed computing and define the decision variables. Moreover, we define more precisely the mixed-type equilibria in which some classes may seek for an individual (Wardrop-type) equilibrium whereas others may seek for different types of Nash equilibria. In Section 3 we then establish the uniqueness of the equilibria and provide analytical expressions for the mixed-equilibrium. In Section 4 we provide numerical examples, and the paper ends with a Concluding section.

2 The Model and Assumptions

We consider a system with m nodes (host computers or processors) connected with a communication means. The jobs that arrive at each node i, $i = 1, 2, \cdots, m$, are classified into n types k, $k = 1, 2, \cdots, n$. Consequently, we have mn different job classes R_{ik}. Each of class R_{ik} is distinguished by the node i at which its jobs arrive and by the type k of the jobs. We call such a class *local class*, or simply *class.*

We assume that each node has identical arrival and identical processing capacity. That is, the system has multiple nodes that are identical with one another. Jobs of type k arrive at each node with node-independent rate ϕ_k. We denote the total arrival rate to the node by ϕ $(= \sum_k \phi_k)$, and we have the time scale whereby $\phi = 1$.

We also consider what we call *global class* J_k that consists in the collection of local class R_{ik}, *i.e.*, $J_k = \bigcup_i R_{ik}$. J_k thus consists of all jobs of type k. Whereas, for local class R_{ik}, all the jobs arrive at the same node i, the arrivals of the jobs of global class J_k are equally distributed over all nodes i.

The average processing (service) time (without queueing delays) of a type k job at any node is $1/\mu_k$ and is, in particular, node-independent. We denote ϕ_k/μ_k by ρ_k and $\rho = \sum_k \rho_k$.

Out of type k jobs arriving at node i, the ratio x_{ijk} of jobs is forwarded upon arrival through the communication means to another node j $(\neq i)$ to be processed there. The remaining ratio $x_{iik} = 1 - \sum_{j(\neq i)} x_{ijk}$ is processed at node i. Thus $\sum_j x_{ijk} = 1$. That is, the rate $\phi_k x_{ijk}$ of type k jobs that arrive at node i is forwarded through the communication means to node j, while the rate $\phi_k x_{iik}$ of local-class R_{ik} jobs is processed at the arrival node i. We have $0 \leq x_{ijk} \leq 1$, for all i, j, k. Within these constraints, a set of values of $\boldsymbol{x}_{ik}$ $(i = 1, 2, \cdots, m, k = 1, 2, \cdots, n)$ are chosen to achieve optimization, where

$$\boldsymbol{x}_{ik} = (x_{i1k}, \cdots, x_{imk})$$

is an m-dimensional vector and called 'local-class R_{ik} strategy.'

We define a global-class J_k strategy as the mm-dimensional vector

$$\boldsymbol{x}_k = (\boldsymbol{x}_{1k}, \boldsymbol{x}_{2k}, \cdots, \boldsymbol{x}_{mk}).$$

We will also denote $\boldsymbol{x}$ the vector of strategies concerning all job classes, called strategy profile, *i.e.*, the vector of length mmn,

$$\boldsymbol{x} = (\boldsymbol{x}_{11}, \boldsymbol{x}_{12}, \cdots, \boldsymbol{x}_{1n}, \boldsymbol{x}_{21}, \cdots, \boldsymbol{x}_{2n}, \cdots, \boldsymbol{x}_{m1}, \cdots, \boldsymbol{x}_{mn}), \quad \text{or} \quad \boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n).$$

For a strategy profile $\boldsymbol{x}$, the load β_i on node i is

$$\beta_i = \beta_i(\boldsymbol{x}) = \sum_{j,k} \rho_k x_{jik}. \tag{1}$$

The contribution $\beta_i^{(k)}$ on the load of node i by the global-class k jobs is

$$\beta_i^{(k)} = \beta_i^{(k)}(\boldsymbol{x}) = \sum_{j} \rho_k x_{jik}, \tag{2}$$

and clearly $\beta_i = \beta_i^{(1)} + \beta_i^{(2)} + \cdots + \beta_i^{(n)}$.

We denote the set of $\boldsymbol{x}$'s that satisfy the constraints (*i.e.*, $\sum_l x_{ilk} = 1, x_{ijk} \geq 0$, for all i, j, k) by $\boldsymbol{C}$. Note that $\boldsymbol{C}$ is a compact set.

We have the following assumptions:

Assumption Π1 *We assume that the expected processing (including queueing) time of a type k job that is processed at node i (or the cost function at node i), is a strictly increasing, strictly convex and continuously differentiable function of β_i, denoted by $\mu_k^{-1} D(\beta_i)$ for all i, k.*

Assumption Π2 *We assume that the mean communication delay (including queueing delay) or the cost for forwarding type k jobs arriving at node i to node j ($i \neq j$), denoted by $G_{ijk}(\boldsymbol{x})$, is a positive, nondecreasing, convex and continuously differentiable function of $\boldsymbol{x}$. We assume that $G_{iik}(\boldsymbol{x}) = 0$. We assume further that each job is forwarded at most once and is not forwarded further from the node to which it is forwarded.*

Example 1 We may consider the following simple functions for the mean processing time and the mean communication delay. For the mean processing time:

$$1/\mu_k D(\beta_i) = \frac{1/\mu_k}{1-\beta_i} \text{ for } \beta_i < 1, \text{ otherwise it is infinite.} \tag{3}$$

For the mean communication delay:

$$G_{ijk}(\boldsymbol{x}) = t. \tag{4}$$

Equation (3) holds, *e.g.*, if we have a simple assumption of the external time-invariant Poisson arrival for each local class, and the mean service time (without queueing delays) for each type k jobs is μ_k^{-1} at each node i. The service discipline is processor sharing or preemptive-resume last-come first-served. When $\mu_k = \mu$ for all k and when no forwarding of jobs occurs, the mean processing time is, simply, $1/(\mu - 1)$.

Equation (4) holds, *e.g.*, if we assume that one communication line is provided separately for sending jobs from one node to another. The line (ij) is used for forwarding a job that arrives at node i to node j ($\neq i$). The expected communication time of a job arriving at node i and being processed at node j ($\neq i$) is expressed simply as t, *i.e.*, independent of the traffic and of the job class, with no queueing delay. □

In particular we assume that the following assumption holds true:

[**Assumption** Π3] *We assume that $G_{ijk}(\boldsymbol{x})$ is one of the following functions, where ω_k are constants, $\sigma_k = \phi_k/\omega_k$ and $\underline{G}(x)$ is a nondecreasing, convex, and differentiable function* *of x with $\underline{G}(0) = 1$.*

Type G-I

$$G_{ijk}(\boldsymbol{x}) = \omega_k^{-1}\underline{G}(\sigma_k x_{ijk})$$

(one dedicated line for each combination of a pair of origin and destination nodes, and a local class: i.e., $m(m-1)n$ lines in total),

Type G-II(a)

$$G_{ijk}(\boldsymbol{x}) = \omega_k^{-1}\underline{G}(\sum_{p,q \neq p} \sigma_k x_{pqk})$$

(one bus line for each global class: i.e., n bus lines in total),

Type G-II(b)

$$G_{ijk}(\boldsymbol{x}) = \omega_k^{-1}\underline{G}(\sum_{p,q(\neq p),k} \sigma_k x_{pqk})$$

(one common bus line for the entire system: i.e., 1 bus line),

Remark 2. 1 *ω_k^{-1} can be regarded as the mean communication time (without queueing delays) for forwarding a type k job from the arrival node to another processing node. $\sigma_k x_{ijk} (j \neq i)$ is the traffic intensity of the communication line for the local-class R_{ik} jobs being forwarded to node j.*

Example 2 We use the same definition (3) for the mean processing time as in Example 1. We define $G_{ijk}(\boldsymbol{x})$ for the mean communication delay as follows. We assume $\omega_k = \theta$ for all k and thus $\sigma_k = \phi_k/\theta$, and set

$$G_{ijk}(\boldsymbol{x}) = \frac{1/\theta}{1 - \sum_{p,q(\neq p),k} \sigma_k x_{pqk}} \text{ for } \sum_{p,q(\neq p),k} \sigma_k x_{pqk} < 1, \text{ and otherwise infinite.} \tag{5}$$

This is identical to:

$$G_{ijk}(\boldsymbol{x}) = \frac{1}{\theta - \sum_{p,q(\neq p),k} \phi_k x_{pqk}} \text{ for } \sum_{p,q(\neq p),k} \phi_k x_{pqk} < \theta, \text{ and otherwise infinite.} \tag{6}$$

This delay is obtained in particular if we assume that one bus-type communication line is provided commonly for all the nodes to be used for forwarding of jobs to other nodes in the same way as in Example 1, whereas the transmission time without queueing delay is exponentially distributed with mean θ^{-1} and the scheduling discipline is First-Come-First-Served. Thus, the expected communication time of a job arriving at node i and being processed at node j ($\neq i$) is expressed as $1/(\theta - \sum_{p,q(\neq p),k} \phi_k x_{pqk})$, *i.e.*, independent of the job class and the origin and destination nodes. □

We refer to the length of time between the instant when a job arrives at a node and the instant when it leaves one of the nodes after all processing and communication, if any, are over as *the response time* for the job. The expected response time of a local-class R_{ik} job that arrives at node i, $T_{ik}(\boldsymbol{x})$, is expressed as,

$$T_{ik}(\boldsymbol{x}) = \sum_j x_{ijk} T_{ijk}(\boldsymbol{x}), \tag{7}$$

where

$$T_{iik}(\boldsymbol{x}) = \mu_k^{-1} D(\beta_i(\boldsymbol{x})), \quad \text{and} \tag{8}$$

$$T_{ijk}(\boldsymbol{x}) = \mu_k^{-1} D(\beta_j(\boldsymbol{x})) + G_{ijk}(\boldsymbol{x}), \text{ for } j \neq i. \tag{9}$$

The expected response time of a global-class J_k jobs is

$$T_k(\boldsymbol{x}) = \frac{1}{m}\sum_i T_{ik}(\boldsymbol{x}). \tag{10}$$

The overall expected response time of a job that arrives at the system is

$$\begin{aligned} T(\boldsymbol{x}) &= \sum_k \phi_k T_k(\boldsymbol{x}) = \frac{1}{m}\sum_{i,k} \phi_k T_{ik}(\boldsymbol{x}), \\ &= \frac{1}{m}\{\sum_i \beta_i(\boldsymbol{x}) D(\beta_i(\boldsymbol{x})) + \sum_{i,j(\neq i),k} \phi_k x_{ijk} G_{ijk}(\boldsymbol{x})\}. \end{aligned} \tag{11}$$

Remark 2. 2 *Note that as a consequence of Assumptions* $\Pi 1$ *and* $\Pi 2$*, the functions* $T(\cdot)$*,* $T_{ik}(\cdot)$ *and* $T_k(\cdot)$ *are strictly convex and differentiable with respect to the strategy profile* $\boldsymbol{x}$*.*

We consider several decision strategies for different job types. That is, each type of jobs may have a distinct decision strategy.

(A) In the *individual optimization* strategy for type k jobs, we consider that each single job of type k chooses the node to be processed. Thus for global-class J_k there exist infinitely many decision makers. The resulting optimal ratio of jobs of local class R_{ik} that choose the node j to be processed will be $\hat{x}_{ijk}$. This optimized situation is the *individual optimum* for type k jobs. We denote the individually optimal strategy profile for type k jobs by $\hat{\boldsymbol{x}}_k$.

(B-I) In the *local-class optimization* strategy for type k jobs, each local class R_{ik} has its own decision maker (ik). The amount of forwarding for local-class R_{ik} jobs is chosen by the corresponding decision maker (ik). The optimal strategy for decision maker (ik), or equivalently local-class job R_{ik}, is denoted by the m-dimensional vector

$$\tilde{\boldsymbol{x}}_{ik} = (\tilde{x}_{i1k}, \tilde{x}_{i2k}, \cdots, \tilde{x}_{imk}),$$

and an optimal strategy profile for type k jobs, that we will denote by $\tilde{\boldsymbol{x}}_k$, is the collection of strategies $\tilde{\boldsymbol{x}}_{ik}$. We call this optimized situation the *local-class optimum* for type k jobs.

(B-II) In the *global-class optimization* strategy for type k jobs, jobs of local classes R_{ik} for all i are united into one global class J_k that has a single decision maker (k). Each decision maker (k) of global class J_k chooses the amount of job forwarding for the m local classes, $R_{1k}, R_{2k}, \ldots, R_{mk}$. The optimal strategy for decision maker k is consequently an mm-dimensional vector

$$\check{\boldsymbol{x}}_k = (\check{\boldsymbol{x}}_{1k}, \check{\boldsymbol{x}}_{2k}, \cdots, \check{\boldsymbol{x}}_{mk}).$$

We call this optimized situation the *global-class optimum* for type k jobs.

We denote by $\vec{\boldsymbol{x}}$ such a strategy profile that all type k jobs achieve their own performance optimization. We call such an optimized situation a *mixed optimum.*

We define $\boldsymbol{x}_{k-(iik)}$ to be an $m(m-1)$-dimensional vector such that the elements x_{iik}, for all i, k, are excluded from the mm-dimensional vector $\boldsymbol{x}_k$ whereas all its elements are the same as the remaining $m(m-1)$ elements of $\boldsymbol{x}_k$.

We define $(\vec{\boldsymbol{x}}_{-(k)}; \boldsymbol{x}_k)$ to be an mmn-dimensional vector such that the elements $\vec{x}_{ijk}$, for all i, j, are replaced by the mm-dimensional vector $\boldsymbol{x}_k$ whereas all other elements are the same as the remaining $m(m-1)n$ elements of $\vec{\boldsymbol{x}}$.

We define $\vec{\beta}_i = \beta_i(\vec{\boldsymbol{x}})$.

3 Results

We show that the solution for each type k jobs is unique and given as follows.

(A) [Individual optimization] If type k jobs seek the individual optimum (*i.e.*, the Wardrop equilibrium), the solution is given by such $\hat{\boldsymbol{x}}_k$ as satisfies the following for all i,

$$T_{ik}(\vec{\boldsymbol{x}}_{-(k)};\hat{\boldsymbol{x}}_k) = \min_j\{T_{ijk}(\vec{\boldsymbol{x}}_{-(k)};\hat{\boldsymbol{x}}_k)\} \quad \text{and} \quad (\vec{\boldsymbol{x}}_{-(k)};\hat{\boldsymbol{x}}_k) \in \boldsymbol{C}. \tag{12}$$

We define $\hat{g}_{ijk}(\boldsymbol{x})$ as

$$\hat{g}_{ijk}(\boldsymbol{x}) = \phi_k G_{ijk}(\boldsymbol{x}). \tag{13}$$

By Assumption Π3, we have

$$\begin{aligned}
\hat{g}_{ijk}(\boldsymbol{x}) &= \sigma_k \underline{G}(\sigma_k x_{ijk}) \ \text{ for type G-I,} \\
\hat{g}_{ijk}(\boldsymbol{x}) &= \sigma_k \underline{G}(\underline{x}) \ \text{ for type G-II} \\
& \qquad (\underline{x} = \sum_{p,q\neq p} \sigma_k x_{pqk} \ \text{ for type G-II(a),} \\
& \qquad \sum_{p,q(\neq p),k} \sigma_k x_{pqk} \ \text{ for type G-II(b)).}
\end{aligned}$$

Therefore we have the property, for $i \neq j, j'$,

$$\hat{g}_{ijk}(\boldsymbol{x}) \geq \hat{g}_{ij'k}(\boldsymbol{x}) \text{ if } x_{ijk} > x_{ij'k}. \tag{14}$$

We define

$$\hat{t}_{ijk}(\boldsymbol{x}) = \phi_k T_{ijk}(\boldsymbol{x}). \tag{15}$$

The solution $\hat{\boldsymbol{x}}_k$ for (12) is characterized as follows: For all i, j we have

$$\begin{aligned}
\hat{t}_{ijk}(\vec{\boldsymbol{x}}_{-(k)};\hat{\boldsymbol{x}}_k) &= \hat{\alpha}_{ik}, \quad \hat{x}_{ijk} > 0, \\
\hat{t}_{ijk}(\vec{\boldsymbol{x}}_{-(k)};\hat{\boldsymbol{x}}_k) &\geq \hat{\alpha}_{ik}, \quad \hat{x}_{ijk} = 0, \\
& \sum_{j'} \hat{x}_{ij'k} = 1,
\end{aligned} \tag{16}$$

where $\hat{\alpha}_{ik} = \min_{j'}\{\phi_k D(\beta_{j'}(\vec{\boldsymbol{x}}_{-(k)};\hat{\boldsymbol{x}}_k))\}$.

(B-I)[Local-class optimization] If type k jobs seek the local-class optimum, the solution is given by such $\tilde{\boldsymbol{x}}_k$ as satisfies the following for all i,

$$\begin{aligned}
T_{ik}(\vec{\boldsymbol{x}}_{-(k)};\tilde{\boldsymbol{x}}_k) &= \min T_{ik}(\vec{\boldsymbol{x}}_{-(k)};\tilde{\boldsymbol{x}}_{k-(ik)};\boldsymbol{x}_{ik}) \\
& \quad \text{with respect to } \boldsymbol{x}_{ik} \text{ such that } (\vec{\boldsymbol{x}}_{-(k)};\tilde{\boldsymbol{x}}_{k-(ik)};\boldsymbol{x}_{ik}) \in \boldsymbol{C},
\end{aligned} \tag{17}$$

where $(\tilde{\boldsymbol{x}}_{k-(ik)};\boldsymbol{x}_{ik})$ and $(\vec{\boldsymbol{x}}_{-(k)};\tilde{\boldsymbol{x}}_{k-(ik)};\boldsymbol{x}_{ik})$ denote mm and mmn-dimensional vectors in which the elements corresponding to $\tilde{\boldsymbol{x}}_{ik}$ and $\vec{\boldsymbol{x}}_k$ have been replaced by $\boldsymbol{x}_{ik}$ and $(\tilde{\boldsymbol{x}}_{k-(ik)};\boldsymbol{x}_{ik})$ whereas all the other elements are the same as the remaining $m(m-1)$ and $mm(n-1)$ elements of $\tilde{\boldsymbol{x}}_k$ and $\vec{\boldsymbol{x}}$, respectively.

Let us define $\tilde{g}_{ijk}(\cdot)$ as

$$\tilde{g}_{ijk}(\boldsymbol{x}) = \frac{\partial}{\partial x_{ijk}}\{\phi_k \sum_{p\neq i} x_{ipk} G_{ipk}(\boldsymbol{x})\}. \tag{18}$$

By Assumption Π3, we have

$$\begin{aligned}
\tilde{g}_{ijk}(\boldsymbol{x}) &= \sigma_k[\underline{G}(\sigma_k x_{ijk}) + \sigma_k x_{ijk}\underline{G}'(\sigma_k x_{ijk})], \quad \text{for type G-I,} \\
\tilde{g}_{ijk}(\boldsymbol{x}) &= \sigma_k[\underline{G}(\underline{x}) + \sigma_k(1 - x_{iik})\underline{G}'(\underline{x})], \quad \text{for type G-II,} \\
\text{where } \underline{x} &= \sum_{p,q(\neq p)} \sigma_k x_{pqk} \quad \text{for type G-II(a), and} \\
\underline{x} &= \sum_{p,q(\neq p),k} \sigma_k x_{pqk} \quad \text{for type G-II(b).}
\end{aligned}$$

We see that, under the assumption Π3, functions $G_{ijk}(\boldsymbol{x})$ satisfy for all $i, j(\neq i), j'(\neq i)$,

$$\tilde{g}_{ijk}(\boldsymbol{x}) \geq \tilde{g}_{ij'k}(\boldsymbol{x}) \text{ if } x_{ijk} > x_{ij'k}. \tag{19}$$

If Assumption Π3 holds, for $\boldsymbol{x}$ such that $x_{ijk} = x_k$, for all $i, j(\neq i)$, we denote

$$G_k(\boldsymbol{x}) = G_{ijk}(\boldsymbol{x}) \text{ and } \tilde{g}_k(\boldsymbol{x}) = \tilde{g}_{ijk}(\boldsymbol{x}).$$

We define

$$\tilde{t}_{ijk}(\boldsymbol{x}) = \phi_k \frac{\partial}{\partial x_{ijk}} T_{ik}(\boldsymbol{x}). \tag{20}$$

Because T_{ik} are convex functions and $\boldsymbol{C}$ is a convex set, the solution $\tilde{\boldsymbol{x}}_k$ of the problem exists, and the Kuhn-Tucker condition gives the following relations (see, *e.g.*, [22]): There exist $\tilde{\alpha}_{ik}$ such that, for all i, j,

$$\begin{aligned}
\tilde{t}_{ijk}(\vec{\boldsymbol{x}}_{-(k)}; \tilde{\boldsymbol{x}}_k) &= \tilde{\alpha}_{ik}, \quad \tilde{x}_{ijk} > 0, \\
\tilde{t}_{ijk}(\vec{\boldsymbol{x}}_{-(k)}; \tilde{\boldsymbol{x}}_k) &\geq \tilde{\alpha}_{ik}, \quad \tilde{x}_{ijk} = 0, \\
\sum_{j'} \tilde{x}_{ij'k} &= 1,
\end{aligned} \tag{21}$$

($\tilde{\alpha}_{ik}$ are the Lagrange multipliers). From Definitions (1), (7) to (9), (18), and (20), we have

$$\tilde{t}_{iik}(\boldsymbol{x}) = \phi_k \frac{\partial T_{ik}}{\partial x_{iik}}(\boldsymbol{x}) = \rho_k[D(\beta_i(\boldsymbol{x})) + \rho_k x_{iik} D'(\beta_i(\boldsymbol{x}))], \tag{22}$$

$$\tilde{t}_{ijk}(\boldsymbol{x}) = \phi_k \frac{\partial T_{ik}}{\partial x_{ijk}}(\boldsymbol{x}) = \rho_k[D(\beta_j(\boldsymbol{x})) + \rho_k x_{ijk} D'(\beta_j(\boldsymbol{x}))] + \tilde{g}_{ijk}(\boldsymbol{x}) \text{ for } j \neq i. \tag{23}$$

(B-II)[Global-class optimization] If type k jobs seek the global-class optimum the solution is given by such $\breve{\boldsymbol{x}}_k$ as satisfies the following for all i,

$$T_k(\vec{\boldsymbol{x}}_{(-k)}; \breve{\boldsymbol{x}}_k) = \min T_k(\vec{\boldsymbol{x}}_{-(k)}; \boldsymbol{x}_k), \text{ with respect to } \boldsymbol{x}_k \text{ such that } (\vec{\boldsymbol{x}}_{-(k)}; \boldsymbol{x}_k) \in \boldsymbol{C}. \tag{24}$$

where $(\vec{\boldsymbol{x}}_{-(k)}; \boldsymbol{x}_k)$ denotes an mmn-dimensional vector in which the elements corresponding to the coordinates of $\vec{\boldsymbol{x}}_k$ has been replaced by the vector $\boldsymbol{x}_k$. We note that

$$\phi_k m T_k(\boldsymbol{x}) = \sum_i \beta_i^{(k)}(\boldsymbol{x}) D(\beta_i(\boldsymbol{x})) + \sum_{i,j\neq i} \phi_k x_{ijk} G_{ijk}(\boldsymbol{x}). \tag{25}$$

Note that we have the assumption Π3 on the function $G_{ijk}(\boldsymbol{x})$.

We define $\breve{g}_{ijk}(\boldsymbol{x})$ as

$$\breve{g}_{ijk}(\boldsymbol{x}) = \frac{\partial}{\partial x_{ijk}} \{ \sum_{p,q\neq p} \phi_k x_{pqk} G_{pqk}(\boldsymbol{x}) \}. \tag{26}$$

By Assumption Π3, we have

$$\begin{aligned}
\breve{g}_{ijk}(\boldsymbol{x}) &= \sigma_k[\underline{G}(x_{ijk}) + \sigma_k x_{ijk}\underline{G}'(x_{ijk})] \text{ for type G-I,} \\
\breve{g}_{ijk}(\boldsymbol{x}) &= \sigma_k[\underline{G}(\underline{x}) + \sigma_k \sum_p (1 - x_{ppk})\underline{G}'(\underline{x})] \text{ for type G-II} \\
& \quad (\underline{x} = \sum_{i,j\neq i} \sigma_k x_{ijk} \text{ for type G-II(a),} \\
& \quad \sum_{i,j(\neq i),k} \sigma_k x_{ijk} \text{ for type G-II(b)).}
\end{aligned}$$

Therefore we have the property

$$\breve{g}_{ijk}(\boldsymbol{x}) \geq \breve{g}_{ij'k}(\boldsymbol{x}) \text{ if } x_{ijk} > x_{ij'k}. \tag{27}$$

We define

$$\breve{t}_{ijk}(\boldsymbol{x}) = m\phi_k \frac{\partial}{\partial x_{ijk}} T_k(\boldsymbol{x}). \tag{28}$$

Again, because T_k is a convex function and $\boldsymbol{C}$ is compact, the solution $\breve{\boldsymbol{x}}$ of the problem exists (see [21]) and from the Kuhn-Tucker condition it is characterized by the relations (see, *e.g.*, [22]):

$$\begin{aligned}
\breve{t}_{ijk}(\vec{\boldsymbol{x}}_{-(k)}; \breve{\boldsymbol{x}}_k) &= \breve{\alpha}_{ik} \text{ for } \breve{x}_{ijk} \text{ such that } \breve{x}_{ijk} > 0, \\
\breve{t}_{ijk}(\vec{\boldsymbol{x}}_{-(k)}; \breve{\boldsymbol{x}}_k) &\geq \breve{\alpha}_{ik} \text{ for } \breve{x}_{ijk} \text{ such that } \breve{x}_{ijk} = 0. \\
& \sum_j \breve{x}_{ijk} = 1, \text{ for all } i, k
\end{aligned} \tag{29}$$

where $\breve{\alpha}_{ik}$ are the Lagrange multipliers. From the definitions (1) to (10), (26), and (28), we have

$$\breve{t}_{iik}(\boldsymbol{x}) = m\phi_k \frac{\partial T_k}{\partial x_{iik}} = \rho_k[D(\beta_i) + \beta_i^{(k)} D'(\beta_i)], \tag{30}$$

$$\breve{t}_{ijk}(\boldsymbol{x}) = m\phi_k \frac{\partial T_k}{\partial x_{ijk}} = \rho_k[D(\beta_j) + \beta_j^{(k)} D'(\beta_j)] + \breve{g}_{ijk}, \quad \text{for } j \neq i. \tag{31}$$

[Mixed optimization] Now we recall the definition of the mixed optimum given near the end of Section 2. That is, a strategy profile $\vec{\boldsymbol{x}}$ is called a mixed optimum if, for all k, its kth component is an optimum for type k jobs given $\vec{\boldsymbol{x}}_{-(k)}$. In other words, if type k jobs seek individual optimization then $\vec{\boldsymbol{x}}_k = \hat{\boldsymbol{x}}_k$, where $\hat{\boldsymbol{x}}_k$ is given in eq.(12), if type k jobs seek local-class optimization then $\vec{\boldsymbol{x}}_k = \tilde{\boldsymbol{x}}_k$, where $\tilde{\boldsymbol{x}}_k$ is given in eq.(17), and if type k jobs seek global-class optimization then $\vec{\boldsymbol{x}}_k = \breve{\boldsymbol{x}}_k$, where $\breve{\boldsymbol{x}}_k$ is given in eq.(24).

Lemma 3. 1 *Consider a network with several types of jobs, where each type optimizes according to (A), (B-I) or (B-II). If there exists a mixed optimum in the network, then in the mixed optimum, we must have*

$$\vec{\beta}_i = \rho, \quad \textit{for all } i.$$

Proof: We show by contradiction that $\vec{\beta}_j = \vec{\beta}_{j'}$ for every pair of (j, j'), and consequently, $\vec{\beta}_i = \rho$ for all i.

Assume that $\vec{\beta}_j > \vec{\beta}_{j'}$ for some j and j'. Then there must exist k such that $\vec{\beta}_j^{(k)} > \vec{\beta}_{j'}^{(k)}$. The type k seeks either one of the individual, local-class, and global-class optimizations.

(a1) Assume first that the job type k seeks individual optimization. We define

$$\hat{\Xi}_{ijk;i'j'k} = \hat{t}_{ijk}(\boldsymbol{x}) - \hat{t}_{i'j'k}(\boldsymbol{x}). \quad (32)$$

From (15) we have for $i \neq j, j'$

$$\hat{\Xi}_{ijk;ij'k} = \rho_k[D(\beta_j) - D(\beta_{j'})] + \hat{g}_{ijk} - \hat{g}_{ij'k}. \quad (33)$$

(a1-1) Since $\vec{\beta}_j > \vec{\beta}_{j'}$, we have

$$\hat{t}_{j'jk}(\vec{\boldsymbol{x}}) \geq \rho_k D(\vec{\beta}_j) > \rho_k D(\vec{\beta}_{j'}) = \hat{t}_{j'j'k}(\vec{\boldsymbol{x}}).$$

Therefore, from the fact that D is increasing (Assumption Π1) and from Property (14), we have $\hat{x}_{j'jk} = 0$ and consequently $\hat{x}_{j'jk} \leq \hat{x}_{jj'k}$.

(a1-2) Suppose we have $\hat{x}_{ijk} > \hat{x}_{ij'k}$ for some $i(\neq j, j')$, then necessarily $\hat{g}_{ijk} \geq \hat{g}_{ij'k}$ by Property (14). Since, by Assumption Π1, $D(\cdot)$ is increasing, $\hat{\Xi}_{ijk;ij'k}(\vec{\boldsymbol{x}}) > 0$. However, from (16), we have

$$\hat{\Xi}_{ijk;ij'k} \leq 0, \quad (34)$$

which contradicts the above. Thus, we must have

$$\hat{x}_{ijk} \leq \hat{x}_{ij'k} \text{ for all } i \neq j, j'.$$

Therefore, from $\hat{\beta}_j^{(k)} > \hat{\beta}_{j'}^{(k)}$, we must have

$$\hat{x}_{jjk} + \hat{x}_{j'jk} > \hat{x}_{j'j'k} + \hat{x}_{jj'k}.$$

Thus we have from (a1-1)

$$\hat{x}_{jj'k} \geq \hat{x}_{j'jk} \text{ and } \hat{x}_{jjk} > \hat{x}_{j'j'k}.$$

(a1-3) Since, from $\check{x}_{jjk} > 0$,

$$\rho_k D(\vec{\beta}_j) = \hat{\alpha}_{jk},$$
$$\rho_k D(\vec{\beta}_{j'}) \geq \hat{\alpha}_{j'k},$$

we have $\hat{\alpha}_{jk} > \hat{\alpha}_{j'k}$.

We next show that $\hat{x}_{jlk} \geq \hat{x}_{j'lk}$ $(l \neq j, j')$ by contradiction. Assume $\hat{x}_{jlk} < \hat{x}_{j'lk}$. Then $\hat{x}_{j'lk} > 0$, and we have from (16),

$$\rho_k D(\vec{\beta}_l) + \hat{g}_{j'lk}(\vec{\boldsymbol{x}}) = \hat{\alpha}_{j'k},$$

$$\rho_k D(\vec{\beta}_l) + \hat{g}_{jlk}(\vec{\boldsymbol{x}}) \geq \hat{\alpha}_{jk} > \hat{\alpha}_{j'k},$$

which contradicts the assumption, as we see by noting that $\hat{g}_{jlk}(\vec{\boldsymbol{x}}) \leq \hat{g}_{j'lk}(\vec{\boldsymbol{x}})$ for both of G-I and G-II. Therefore we must have

$$\hat{x}_{jlk} \geq \hat{x}_{j'lk}.$$

From this and (a1-2),

$$\begin{aligned} \hat{x}_{jjk} &> \hat{x}_{j'j'k}, \\ \hat{x}_{jj'k} &\geq \hat{x}_{j'jk}, \\ \hat{x}_{jlk} &\geq \hat{x}_{j'lk} \text{ for all } l(\neq j, j'). \end{aligned}$$

This implies

$$1 = \sum_l \hat{x}_{jlk} > \sum_l \hat{x}_{j'lk} = 1,$$

which is impossible. That is, the assumption leads to a contradiction and we do not have $\hat{\beta}_j^{(k)} > \hat{\beta}_{j'}^{(k)}$ for the type k with individual optimization.

(b1) Secondly assume that the type k seeks local-class optimization.

We define

$$\Xi_{ijk;i'j'k}(\boldsymbol{x}) = \tilde{t}_{ijk}(\boldsymbol{x}) - \tilde{t}_{i'j'k}(\boldsymbol{x}). \tag{35}$$

(b1-1) Assume $\tilde{x}_{ijk} > \tilde{x}_{ij'k}$ for some $i(\neq j, j')$. Then, we have $\tilde{g}_{ijk}(\vec{\boldsymbol{x}}) \geq \tilde{g}_{ij'k}(\vec{\boldsymbol{x}})$ by (19). From Equation (23) and Definition (35) we have

$$\begin{aligned} \Xi_{ijk;ij'k}(\boldsymbol{x}) &= \rho_k[D(\beta_j(\boldsymbol{x})) - D(\beta_{j'}(\boldsymbol{x}))] \\ &+ \rho_k^2[x_{ijk}D'(\beta_j(\boldsymbol{x})) - x_{ij'k}D'(\beta_{j'}(\boldsymbol{x}))] + \tilde{g}_{ijk}(\boldsymbol{x}) - \tilde{g}_{ij'k}(\boldsymbol{x}). \end{aligned} \tag{36}$$

Together with the fact that $D(\cdot)$ and $D'(\cdot)$ are increasing (Π1), it follows that

$$\Xi_{ijk;ij'k}(\vec{\boldsymbol{x}}) > 0.$$

However, from (21) we must have

$$\Xi_{ijk;ij'k}(\vec{\boldsymbol{x}}) \leq 0, \tag{37}$$

which contradicts the above. Thus, we must have $\tilde{x}_{ijk} \leq \tilde{x}_{ij'k}$ for all $i(\neq j, j')$.

(b1-2) Then, from the assumption $\tilde{\beta}_j^{(k)} > \tilde{\beta}_{j'}^{(k)}$, we have,

$$\tilde{x}_{jjk} + \tilde{x}_{j'jk} > \tilde{x}_{jj'k} + \tilde{x}_{j'j'k}.$$

(b1-2-1) If $\tilde{x}_{j'jk} = 0$, we have

$$\tilde{x}_{jjk} > \tilde{x}_{j'j'k} \quad \text{and} \quad \tilde{x}_{jj'k} \geq \tilde{x}_{j'jk} \quad \textbf{(Condition I)}.$$

(b1-2-2) If $\tilde{x}_{j'jk} > 0$, since $\tilde{g}_{j'j'k}(\boldsymbol{x}) = 0$, we have

$$\begin{aligned} \Xi_{j'jk;j'j'k}(\boldsymbol{x}) &= \rho_k[D(\beta_j(\boldsymbol{x})) - D(\beta_{j'}(\boldsymbol{x}))] \\ &+ \rho_k^2[x_{j'jk}D'(\beta_j(\boldsymbol{x})) - x_{j'j'k}D'(\beta_{j'}(\boldsymbol{x}))] + \tilde{g}_{j'jk}(\boldsymbol{x}) > 0. \end{aligned}$$

Thus, similarly as in (b1-1), we see that if $\tilde{x}_{j'jk} > \tilde{x}_{j'j'k}$, we have $\Xi_{j'jk;j'j'k}(\vec{\boldsymbol{x}}) > 0$, which contradicts (37). Thus we have $\tilde{x}_{j'jk} \leq \tilde{x}_{j'j'k}$. Then $\tilde{x}_{jjk} > \tilde{x}_{jj'k}$ and $\tilde{x}_{jjk} > 0$. Thus from (21), (22), and (23),

$$\tilde{t}_{jjk}(\vec{\boldsymbol{x}}) = \rho_k[D(\vec{\beta}_j) + \rho_k \tilde{x}_{jjk} D'(\vec{\beta}_j)] = \tilde{\alpha}_{jk},$$

$$\tilde{t}_{j'jk}(\vec{\boldsymbol{x}}) = \rho_k[D(\vec{\beta}_j) + \rho_k \tilde{x}_{j'jk} D'(\vec{\beta}_j)] + \tilde{g}_{j'jk}(\vec{\boldsymbol{x}}) = \tilde{\alpha}_{j'k}.$$

Then we have, by adding the last two equations,

$$\rho_k[2D(\vec{\beta}_j) + \rho_k(\tilde{x}_{jjk} + \tilde{x}_{j'jk})D'(\vec{\beta}_j)] + \tilde{g}_{j'jk}(\vec{\boldsymbol{x}}) = \tilde{\alpha}_{jk} + \tilde{\alpha}_{j'k}. \tag{38}$$

Note that we have from (21), (22), and (23),

$$\tilde{t}_{jj'k}(\vec{\boldsymbol{x}}) + \tilde{t}_{j'j'k}(\vec{\boldsymbol{x}}) = \rho_k[2D(\vec{\beta}_{j'}) + \rho_k(\tilde{x}_{jj'k} + \tilde{x}_{j'j'k})D'(\vec{\beta}_{j'})] + \tilde{g}_{jj'k}(\vec{\boldsymbol{x}}) \geq \tilde{\alpha}_{jk} + \tilde{\alpha}_{j'k}. \tag{39}$$

Since D and D' are increasing, $\vec{\beta}_j > \vec{\beta}_{j'}$ and $\tilde{x}_{jjk} + \tilde{x}_{j'jk} > \tilde{x}_{jj'k} + \tilde{x}_{j'j'k}$ by assumption, the only possibility for (38) and (39) not to contradict each other is

$$\tilde{g}_{jj'k}(\tilde{\boldsymbol{x}}) > \tilde{g}_{j'jk}(\tilde{\boldsymbol{x}}). \tag{40}$$

Therefore, in the special case where $\tilde{g}_{jj'k}(\tilde{\boldsymbol{x}}) = \tilde{g}_{j'jk}(\tilde{\boldsymbol{x}})$, these two relations contradict each other. For the other cases, we investigate in the following (b1-2-2-1) and (b1-2-2-2).

(b1-2-2-1) Consider the Type G-I case. From the above (40), the relation (19) on $\tilde{g}$, and $\tilde{x}_{jjk} + \tilde{x}_{j'jk} > \tilde{x}_{jj'k} + \tilde{x}_{j'j'k}$, we have $\tilde{x}_{jj'k} \geq \tilde{x}_{j'jk}$, from which $\tilde{x}_{jjk} > \tilde{x}_{j'j'k}$ follows. We thus have

$$\tilde{x}_{jj'k} \geq \tilde{x}_{j'jk} \text{ and } \tilde{x}_{jjk} > \tilde{x}_{j'j'k} \text{ (Condition I)},$$

which is the same as (b1-2-1).

(b1-2-2-2) Consider the Type G-II case. From the above (40), and the relation (19) on $\tilde{g}$ and $\tilde{x}_{jjk} + \tilde{x}_{j'jk} > \tilde{x}_{jj'k} + \tilde{x}_{j'j'k}$, we have $\tilde{x}_{j'j'k} > \tilde{x}_{jjk}$, from which $\tilde{x}_{j'jk} > \tilde{x}_{jj'k}$ follows. We thus have

$$\tilde{x}_{j'j'k} > \tilde{x}_{jjk} \text{ and } \tilde{x}_{j'jk} > \tilde{x}_{jj'k} \textbf{ (Condition II)}.$$

(b1-3) Now we examine each of Conditions I and II, respectively, in the following (b1-3-1) and (b1-3-2), and will show that both lead to contradictions.

(b1-3-1) Consider the case where Condition I holds. Since

$$\tilde{t}_{jjk}(\tilde{\boldsymbol{x}}) = \rho_k[D(\vec{\beta}_j) + \rho_k \tilde{x}_{jjk} D'(\vec{\beta}_j)] = \tilde{\alpha}_{jk},$$

$$\tilde{t}_{j'j'k}(\tilde{\boldsymbol{x}}) = \rho_k[D(\vec{\beta}_{j'}) + \rho_k \tilde{x}_{j'j'k} D'(\vec{\beta}_{j'})] \geq \tilde{\alpha}_{j'k},$$

we have $\tilde{\alpha}_{jk} > \tilde{\alpha}_{j'k}$, because D and D' are increasing and $\vec{\beta}_j > \vec{\beta}_{j'}$ by assumption.

We next show that $\tilde{x}_{jlk} \geq \tilde{x}_{j'lk}$ by contradiction. Assume $\tilde{x}_{jlk} < \tilde{x}_{j'lk}$. Then $\tilde{x}_{j'lk} > 0$, and we have from (21) and (23),

$$\tilde{t}_{j'lk}(\tilde{\boldsymbol{x}}) = \rho_k[D(\vec{\beta}_l) + \rho_k \tilde{x}_{j'lk} D'(\vec{\beta}_l)] + \tilde{g}_{j'lk}(\tilde{\boldsymbol{x}}) = \tilde{\alpha}_{j'k},$$

$$\tilde{t}_{jlk}(\tilde{\boldsymbol{x}}) = \rho_k[D(\vec{\beta}_l) + \rho_k \tilde{x}_{jlk} D'(\vec{\beta}_l)] + \tilde{g}_{jlk}(\tilde{\boldsymbol{x}}) \geq \tilde{\alpha}_{jk} > \tilde{\alpha}_{j'k},$$

which contradicts the assumption, as we see by noting that here for G-I

$$\tilde{g}_{jlk}(\tilde{\boldsymbol{x}}) \leq \tilde{g}_{j'lk}(\tilde{\boldsymbol{x}})$$

due to the fact that $\tilde{x}_{jlk} < \tilde{x}_{jl'k}$. Therefore we must have

$$\tilde{x}_{jlk} \geq \tilde{x}_{j'lk}.$$

From this and Condition I, it follows

$$\begin{aligned} \tilde{x}_{jjk} &> \tilde{x}_{j'j'k}, \\ \tilde{x}_{jj'k} &\geq \tilde{x}_{j'jk}, \\ \tilde{x}_{jlk} &\geq \tilde{x}_{j'lk} \text{ for all } l(\neq j, j'). \end{aligned}$$

This implies

$$1 = \sum_l \tilde{x}_{jlk} > \sum_l \tilde{x}_{j'lk} = 1,$$

which is impossible. That is, the assumption leads to a contradiction.

(b1-3-2) Consider the case where Condition II holds. This implies $\tilde{x}_{j'jk} > 0$ and we have

$$\tilde{t}_{j'jk}(\vec{\boldsymbol{x}}) = \rho_k[D(\vec{\beta}_j) + \rho_k\tilde{x}_{j'jk}D'(\vec{\beta}_j)] + \tilde{g}_{j'jk}(\vec{\boldsymbol{x}}) = \tilde{\alpha}_{j'k},$$

$$\tilde{t}_{jj'k}(\vec{\boldsymbol{x}}) = \rho_k[D(\vec{\beta}_{j'}) + \rho_k\tilde{x}_{jj'k}D'(\vec{\beta}_{j'})] + \tilde{g}_{jj'k}(\vec{\boldsymbol{x}}) \geq \tilde{\alpha}_{jk}.$$

Since D and D' are increasing, $\beta_j > \beta_{j'}$ and $\tilde{x}_{j'jk} > \tilde{x}_{jj'k}$, we have

$$\tilde{\alpha}_{j'k} - \tilde{g}_{j'jk}(\vec{\boldsymbol{x}}) > \tilde{\alpha}_{jk} - \tilde{g}_{jj'k}(\vec{\boldsymbol{x}}).$$

By noting that for type G-II, we have $\tilde{g}_{jlk}(\vec{\boldsymbol{x}}) = \tilde{g}_{jj'k}(\vec{\boldsymbol{x}})$ and $\tilde{g}_{j'lk}(\vec{\boldsymbol{x}}) = \tilde{g}_{j'jk}(\vec{\boldsymbol{x}})$ for any $l(\neq j, j')$, and from the Kuhn-Tucker condition, we have

$$\begin{aligned}
\rho_k[D(\vec{\beta}_l) + \rho_k\tilde{x}_{j'lk}D'(\vec{\beta}_l)] &= \tilde{\alpha}_{j'k} - \tilde{g}_{j'lk}(\vec{\boldsymbol{x}}) = \tilde{\alpha}_{j'k} - \tilde{g}_{j'jk}(\vec{\boldsymbol{x}}),\ \tilde{x}_{j'lk} > 0,\\
\rho_k[D(\vec{\beta}_l) + \rho_k\tilde{x}_{j'lk}D'(\vec{\beta}_l)] &\geq \tilde{\alpha}_{j'k} - \tilde{g}_{j'lk}(\vec{\boldsymbol{x}}) = \tilde{\alpha}_{j'k} - \tilde{g}_{j'jk}(\vec{\boldsymbol{x}}),\ \tilde{x}_{j'lk} = 0,\\
\rho_k[D(\vec{\beta}_l) + \rho_k\tilde{x}_{jlk}D'(\vec{\beta}_l)] &= \tilde{\alpha}_{jk} - \tilde{g}_{jlk}(\vec{\boldsymbol{x}}) = \tilde{\alpha}_{jk} - \tilde{g}_{jj'k}(\vec{\boldsymbol{x}}),\ \tilde{x}_{jlk} > 0,\\
\rho_k[D(\vec{\beta}_l) + \rho_k\tilde{x}_{jlk}D'(\vec{\beta}_l)] &\geq \tilde{\alpha}_{jk} - \tilde{g}_{jlk}(\vec{\boldsymbol{x}}) = \tilde{\alpha}_{jk} - \tilde{g}_{jj'k}(\vec{\boldsymbol{x}}),\ \tilde{x}_{jlk} = 0,
\end{aligned}$$

which can hold only when $\tilde{x}_{jlk} \leq \tilde{x}_{j'lk}$ for all $l(\neq j, j')$.

From this and Condition II,

$$\begin{aligned}
\tilde{x}_{jjk} &< \tilde{x}_{j'j'k},\\
\tilde{x}_{jj'k} &< \tilde{x}_{j'jk},\\
\tilde{x}_{jlk} &\leq \tilde{x}_{j'lk} \text{ for all } l(\neq j, j').
\end{aligned}$$

This implies

$$1 = \sum_l \tilde{x}_{jlk} < \sum_l \tilde{x}_{j'lk} = 1,$$

which is impossible. That is, the assumption leads to a contradiction.

Thus we see that the assumption $\vec{\beta}_j > \vec{\beta}_{j'}$ leads to either Condition I [(b1-2-1) and (b1-3-1)] or Condition II [(b1-3-2)], both of which lead to contradictions.

Therefore, we cannot have $\tilde{\beta}_j^{(k)} > \tilde{\beta}_{j'}^{(k)}$ for the type k with the local-class optimization.

(c1) Thirdly assume that the type k seeks global-class optimization. We define

$$\check{\Xi}_{ijk;i'j'k} = \check{t}_{ijk}(\boldsymbol{x}) - \check{t}_{i'j'k}(\boldsymbol{x}). \tag{41}$$

From (31) we have, for $j, j'(\neq i)$

$$\check{\Xi}_{ijk;ij'k} = \rho_k[D(\beta_j) - D(\beta_{j'})] + \rho_k[\beta_j^{(k)}D'(\beta_j) - \beta_{j'}^{(k)}D'(\beta_{j'})] + \check{g}_{ijk} - \check{g}_{ij'k}. \tag{42}$$

(c1-1) Since $\check{\beta}_j^{(k)} > \check{\beta}_{j'}^{(k)}$, we have from (29), (30), and (31)

$$\begin{aligned}
\check{t}_{j'j'k}(\vec{\boldsymbol{x}}) = \rho_k[D(\vec{\beta}_{j'}) + \check{\beta}_{j'}^{(k)}D'(\vec{\beta}_{j'})] &= \check{\alpha}_{j'k},\ \ \check{x}_{j'j'k} > 0,\\
\check{t}_{j'j'k}(\vec{\boldsymbol{x}}) = \rho_k[D(\vec{\beta}_{j'}) + \check{\beta}_{j'}^{(k)}D'(\vec{\beta}_{j'})] &\geq \check{\alpha}_{j'k},\ \ \check{x}_{j'j'k} = 0,\\
\check{t}_{j'jk}(\vec{\boldsymbol{x}}) = \rho_k[D(\vec{\beta}_j) + \check{\beta}_j^{(k)}D'(\vec{\beta}_j)] + \check{g}_{j'jk}(\vec{\boldsymbol{x}}) &= \check{\alpha}_{j'k},\ \ \check{x}_{j'jk} > 0,\\
\check{t}_{j'jk}(\vec{\boldsymbol{x}}) = \rho_k[D(\vec{\beta}_j) + \check{\beta}_j^{(k)}D'(\vec{\beta}_j)] + \check{g}_{j'jk}(\vec{\boldsymbol{x}}) &\geq \check{\alpha}_{j'k},\ \ \check{x}_{j'jk} = 0.
\end{aligned}$$

Therefore, from the fact that D and D' are increasing functions (Assumption Π1) and from Property (27), we have $\check{x}_{j'jk} = 0$ and consequently $\check{x}_{j'jk} \le \check{x}_{jj'k}$.

(c1-2) Suppose we have $\check{x}_{ijk} > \check{x}_{ij'k}$ for some $i(\neq j, j')$, then necessarily $\check{g}_{ijk} \ge \check{g}_{ij'k}$ by Property (27). Since, by Assumption Π1, $D(\cdot)$ and $D'(\cdot)$ are increasing, $\check{\Xi}_{ijk;ij'k}(\boldsymbol{\vec{x}}) > 0$. However, from (29), we have

$$\check{\Xi}_{ijk;ij'k}(\boldsymbol{\vec{x}}) \le 0, \tag{43}$$

which contradicts the above. Thus, we must have

$$\check{x}_{ijk} \le \check{x}_{ij'k} \text{ for all } i.$$

Therefore, from $\check{\beta}_j^{(k)} > \check{\beta}_{j'}^{(k)}$, we must have

$$\check{x}_{jjk} + \check{x}_{j'jk} > \check{x}_{j'j'k} + \check{x}_{jj'k}.$$

Thus we have from (c1-1)

$$\check{x}_{jj'k} \ge \check{x}_{j'jk} \text{ and } \check{x}_{jjk} > \check{x}_{j'j'k}.$$

(c1-3) Since

$$\rho_k[D(\vec{\beta}_j) + \check{\beta}_j^{(k)} D'(\vec{\beta}_j)] = \check{\alpha}_{jk},$$

$$\rho_k[D(\vec{\beta}_{j'}) + \check{\beta}_{j'}^{(k)} D'(\vec{\beta}_{j'})] \ge \check{\alpha}_{j'k},$$

we have $\check{\alpha}_{jk} > \check{\alpha}_{j'k}$.

We next show that $\check{x}_{jlk} \ge \check{x}_{j'lk}$ by contradiction. Assume $\check{x}_{jlk} < \check{x}_{j'lk}$. Then $\check{x}_{j'lk} > 0$, and we have from (29), (30), and (31),

$$\rho_k[D(\vec{\beta}_l) + \check{\beta}_l^{(k)} D'(\vec{\beta}_l)] + \check{g}_{j'lk}(\boldsymbol{\vec{x}}) = \check{\alpha}_{j'k},$$

$$\rho_k[D(\vec{\beta}_l) + \check{\beta}_l^{(k)} D'(\vec{\beta}_l)] + \check{g}_{jlk}(\boldsymbol{\vec{x}}) \ge \check{\alpha}_{jk} > \check{\alpha}_{j'k},$$

which contradicts the assumption, as we see by noting that $\check{g}_{jlk}(\boldsymbol{\vec{x}}) \le \check{g}_{j'lk}(\boldsymbol{\vec{x}})$ for both of G-I and G-II due to the fact that $\tilde{x}_{jlk} < \tilde{x}_{jl'k}$ for G-I and $\check{g}_{jlk}(\boldsymbol{\vec{x}}) = \check{g}_{j'lk}(\boldsymbol{\vec{x}})$ for G-II.

Therefore we must have

$$\check{x}_{jlk} \ge \check{x}_{j'lk}.$$

From this and (c1-2),

$$\begin{aligned} \check{x}_{jjk} &> \check{x}_{j'j'k}, \\ \check{x}_{jj'k} &\ge \check{x}_{j'jk}, \\ \check{x}_{jlk} &\ge \check{x}_{j'lk} \text{ for all } l(\neq j, j'). \end{aligned}$$

This implies

$$1 = \sum_l \check{x}_{jlk} > \sum_l \check{x}_{j'lk} = 1,$$

which is impossible. That is, the assumption leads to a contradiction and we do not have $\check{\beta}_j^{(k)} > \check{\beta}_{j'}^{(k)}$ for the type k with global-class optimization.

Thus we see that the assumption $\vec{\beta}_j > \vec{\beta}_{j'}$ leads to a contradiction. Therefore necessarily $\vec{\beta}_j = \vec{\beta}'_j$, and consequently $\vec{\beta}_i = \rho$ for all i. □

Lemma 3. 2 *If there exists a mixed optimum for the network, then in the mixed optimum, the solution* $\hat{\boldsymbol{x}}_k$ *for type k with individual optimization is unique and given as follows:*

$$\hat{\boldsymbol{x}}_{k-(iik)} = \mathbf{0}, \quad i.e., \ \hat{x}_{ijk} = 0, \ and \ \hat{x}_{iik} = 1, \ for\ all\ i, j(\neq i),$$

The mean response time is

$$T_k(\hat{\boldsymbol{x}}) = T_{ik}(\hat{\boldsymbol{x}}) = \mu_k^{-1} D(\rho), \ for\ all\ i.$$

Proof: From lemma 3. 1 this can be easily seen in the following way. We can easily see that the set of relations (16) is satisfied if and only if $\hat{x}_{ijk} = 0$ for all $i, j(\neq i)$. □

Lemma 3. 3 *We denote* $\Gamma_k = \rho_k^2 \sigma_k^{-1}$. *If there exists a mixed optimum for the network, then in the mixed optimum, the solution* $\tilde{\boldsymbol{x}}_k$ *for type k jobs with local-class optimization is unique and is given as follows:*
For Types G-I and G-II(a)

(*i*) *For local class* R_{ik} *such that* $\rho_k^2 D'(\rho) \leq \tilde{g}_k(0) = \sigma_k$, *i.e.,* $\Gamma_k D'(\rho) \leq 1$,

$$\tilde{x}_{ijk} = 0, \ and \ \tilde{x}_{iik} = 1, \ for\ all\ i, j(\neq i).$$

The mean response time is

$$T_k(\tilde{\boldsymbol{x}}) = T_{ik}(\tilde{\boldsymbol{x}}) = \mu_k^{-1} D(\rho), \ for\ all\ i.$$

(*ii*) *For local class* R_{ik} *such that* $\rho_k^2 D'(\rho) > \tilde{g}_k(0) = \sigma_k$, *i.e.,* $\Gamma_k D'(\rho) > 1$, *the solution is given as follows:*

$$\tilde{x}_{ijk} = \tilde{x}_k, \ for\ all\ i, \ j(\neq i), \tag{44}$$

where $\tilde{x}_k$ *is the unique solution of*

$$\begin{aligned} &\rho_k^2(1 - m\tilde{x}_k)D'(\rho) = \tilde{g}_k(\tilde{x}_k) \\ &= \sigma_k[\underline{G}(m(m-1)\sigma_k\tilde{x}_k) + \sigma_k(m-1)\tilde{x}_k\underline{G}'(m(m-1)\sigma_k\tilde{x}_k)]. \end{aligned} \tag{45}$$

The mean response time is

$$T_k(\tilde{\boldsymbol{x}}) = T_{ik}(\tilde{\boldsymbol{x}}) = \mu_k^{-1} D(\rho) + (m-1)\tilde{x}_k G_k(\tilde{\boldsymbol{x}}), \quad for\ all\ \ i. \tag{46}$$

For Type G-II(b)

The solution is given as in the following. We first change the numbering of k such that $\Gamma_1 \geq \Gamma_2 \geq \cdots \geq \Gamma_k \geq \cdots \geq \Gamma_{n'}$, *where* n' *is the number of job types that seek the local-class optimization. The following three situations can occur:*

$$We\ can\ find\ K\ such\ that\ \Gamma_K D'(\rho) > 1 \ and \ \Gamma_{K+1} D'(\rho) \leq 1, \tag{47}$$

$$or\ \ \Gamma_{n'} D'(\rho) > 1 \ \ (i.e., K = n'), \tag{48}$$

$$or\ \ \Gamma_1 D'(\rho) \leq 1. \tag{49}$$

When (49) holds, we have a unique solution of $\tilde{x}_k = 0$ *for all* $k \leq n'$. *When (47) or (48) holds, we can find a unique solution as follows. Let us define the function* $F_k(X)$ *as*

$$F_k(X) = \{\sum_{l=1}^{k} \frac{\sigma_l[\Gamma_l D'(\rho) - \underline{G}(X)]}{m\Gamma_k D'(\rho) + (m-1)\sigma_l \underline{G}'(X)}\} - \frac{X}{m(m-1)}. \tag{50}$$

We obtain the largest $k = \tilde{k} \leq K$ *and* $X = \tilde{X}_{\tilde{k}} (> 0)$ *that satisfies* $F_{\tilde{k}}(\tilde{X}_{\tilde{k}}) = 0$ *and* $\sigma_{\tilde{k}}[\Gamma_{\tilde{k}} D'(\rho) - \underline{G}(\tilde{X}_{\tilde{k}})] > 0$. *Then by using*

$$\sigma_k[\Gamma_k D'(\rho) - \underline{G}(\tilde{X}_{\tilde{k}})] = \sigma_k \tilde{x}_k [m\Gamma_k D'(\rho) + (m-1)\sigma_k \underline{G}'(\tilde{X}_{\tilde{k}})], \quad (51)$$

for $k = 1, 2, \cdots, \tilde{k}$, *we can obtain the unique set of values, such that* $\tilde{x}_k > 0, k = 1, 2, \cdots, \tilde{k}$, *and* $\tilde{x}_{\tilde{k}+1} = \tilde{x}_{\tilde{k}+2} = \cdots = \tilde{x}_{n'} = 0$, *that satisfies the above relation, which is a unique solution. The mean response time is*

$$T_k(\tilde{\boldsymbol{x}}) = T_{ik}(\tilde{\boldsymbol{x}}) = \mu_k^{-1} D(\rho) + (m-1)\tilde{x}_k G_k(\tilde{\boldsymbol{x}}), \quad \textit{for all } i. \quad (52)$$

Proof: (b1) By Lemma 3. 1 , we have shown that $\vec{\beta}_j = \vec{\beta}_{j'}$ for every pair of (j, j'), and consequently, $\vec{\beta}_i = \rho$ for all i.

(b2) Hence for all $i, j(\neq i), j'(\neq i)$,

$$\Xi_{ijk;ij'k}(\tilde{\boldsymbol{x}}) = \rho_k^2(\tilde{x}_{ijk} - \tilde{x}_{ij'k})D'(\rho) + \tilde{g}_{ijk}(\tilde{\boldsymbol{x}}) - \tilde{g}_{ij'k}(\tilde{\boldsymbol{x}}). \quad (53)$$

Thus, if $\tilde{x}_{ijk} > \tilde{x}_{ij'k}$ for some $i, j(\neq i), j'(\neq i)$, we have $\Xi_{ijk;ij'k} > 0$ since $D'(\rho) > 0$, which contradicts (37). Therefore, we must have

$$\tilde{x}_{ijk} = \tilde{x}_k \text{ for all } \ i, j(\neq i). \quad (54)$$

(b3) We note that since $\sum_j x_{ijk} = 1$, from the assumption on the arrival ratio of each local-class job, $\tilde{x}_k$ has to belong to the interval $[0, 1/(m-1)]$. We discuss the case for Types G-I and G-II(a) and that for Type G-II(b), separately.

The case for Types G-I and G-II(a)

We have

$$\Xi_{ijk;iik}(\tilde{\boldsymbol{x}}) = -\rho_k^2(1 - m\tilde{x}_k)D'(\rho) + \tilde{g}_k(\tilde{x}_k).$$

where

$$\begin{aligned} \tilde{g}_k(\tilde{x}_k) &= \sigma_k[\underline{G}(\sigma_k\tilde{x}_k) + \sigma_k\tilde{x}_k\underline{G}'(\sigma_k\tilde{x}_k)] \quad \text{(Type G-I) or} \\ \tilde{g}_k(\tilde{x}_k) &= \sigma_k[\underline{G}(m(m-1)\sigma_k\tilde{x}_k) + (m-1)\sigma_k\tilde{x}_k\underline{G}'(m(m-1)\sigma_k\tilde{x}_k)] \quad \text{(Type G-II(a))}. \end{aligned}$$

Let us define the function H_k as

$$H_k(x) = -\rho_k^2(1 - mx)D'(\rho) + \tilde{g}_k(x). \quad (55)$$

Clearly, H_k is continuous and monotonically increasing.

(i) For local class R_{ik} such that $\rho_k^2 D'(\rho) \leq \tilde{g}_k(0) = \sigma_k$, we have $H_k(x) > H_k(0) \geq 0$ for any $x > 0$, which proves that $x = \tilde{x}_k = 0$ is the unique optimal solution.

Therefore, for local class R_{ik} such that $\rho_k^2 D'(\rho) \leq \tilde{g}_k(0) = \sigma_k$,

$$\tilde{x}_{ijk} = 0, \text{ and } \tilde{x}_{iik} = 1, \text{ for all } i, j(\neq i).$$

The mean response time is

$$T_k(\tilde{\boldsymbol{x}}) = T_{ik}(\tilde{\boldsymbol{x}}) = \mu_k^{-1} D(\rho), \text{ for all } i.$$

(*ii*) For local class R_{ik} such that $\rho_k^2 D'(\rho) > \tilde{g}_k(0) = \sigma_k$, the optimal solution is uniquely given as follows:

$$\tilde{x}_{ijk} = \tilde{x}_k, \text{ for all } i,\ j(\neq i), \tag{56}$$

where $\tilde{x}_k$ is the unique solution of

$$\rho_k^2(1 - m\tilde{x}_k)D'(\rho) = \tilde{g}_k(\tilde{x}_k).$$

Therefore, the mean response time is

$$T_{ik}(\tilde{\boldsymbol{x}}) = \mu_k^{-1}D(\rho) + (m-1)\tilde{x}_k G_k(\tilde{\boldsymbol{x}}), \quad \text{for all } \ i.$$

Therefore, we have a unique local-class optimum solution for type k.

The case for Type G-II(b) We have

$$\Xi_{ijk;iik}(\tilde{\boldsymbol{x}}) = -\rho_k^2(1 - m\tilde{x}_k)D'(\rho) + \tilde{g}_k(\tilde{\boldsymbol{x}}).$$

We can find the set of $\tilde{x}_k$, $k = 1, 2, \cdots, n$, as the unique solution of the following system of relations:

$$\begin{aligned} \rho_k^2(1 - m\tilde{x}_k)D'(\rho) &= \tilde{g}_k(\tilde{\boldsymbol{x}}) \text{ and } \tilde{x}_k \geq 0, \\ \rho_k^2 D'(\rho) &< \tilde{g}_k(\tilde{\boldsymbol{x}}) \text{ and } \tilde{x}_k = 0, \\ 0 \leq \ \tilde{x}_k & \leq 1/(m-1), \end{aligned} \tag{57}$$

where $\tilde{g}_k(\tilde{\boldsymbol{x}}) = \sigma_k[\underline{G}(m(m-1)\sum_k \sigma_k\tilde{x}_k) + \sigma_k(m-1)\tilde{x}_k\underline{G}'(m(m-1)\sum_k \sigma_k\tilde{x}_k)]$.

The relations (57) are equivalent to the following:

$$\begin{aligned} \sigma_k[\Gamma_k D'(\rho) - \underline{G}(\tilde{X})] &= \sigma_k\tilde{x}_k[m\Gamma_k D'(\rho) + (m-1)\sigma_k\underline{G}'(\tilde{X})] \text{ and } \tilde{x}_k \geq 0, \\ \sigma_k[\Gamma_k D'(\rho) - \underline{G}(\tilde{X})] &< 0 \text{ and } \tilde{x}_k = 0, \\ 0 \leq \ \tilde{x}_k & \leq 1/(m-1), \end{aligned} \tag{58}$$

where we recall $\Gamma_k = \rho_k^2\sigma_k^{-1}$ and we denote $\tilde{X} = m(m-1)\sum_k \sigma_k\tilde{x}_k$. Thus if denote by $\tilde{\boldsymbol{k}}$ the set of k such that $\tilde{x}_k > 0$, then we have

$$\sum_{k\in\tilde{\boldsymbol{k}}} \frac{\sigma_k[\Gamma_k D'(\rho) - \underline{G}(\tilde{X})]}{m\Gamma_k D'(\rho) + (m-1)\sigma_k\underline{G}'(\tilde{X})} = \frac{\sigma_k\tilde{X}}{m(m-1)}. \tag{59}$$

We easily see that we can change the numbering of k such that $\Gamma_1 \geq \Gamma_2 \geq \cdots \geq \Gamma_k \geq \cdots \geq \Gamma_{n'}$. The following three situations can occur:

$$\begin{aligned} \text{We can find } K \text{ such that } \Gamma_K D'(\rho) &> 1 \text{ and } \Gamma_{K+1}D'(\rho) \leq 1, \quad \text{(rel. (47))} \\ \text{or } \Gamma_{n'}D'(\rho) &> 1 \ \ (\textit{i.e.}, K = n'), \quad \text{(rel. (48))} \\ \text{or } \Gamma_1 D'(\rho) &\leq 1 \ \ \text{(rel. (49))}. \end{aligned}$$

When (49) holds, we can find a unique solution of $\tilde{x}_k = 0$ for all $k \leq n'$, where we recall that n' is the number of job types that seek local-class optimization.

When (47) or (48) holds, we can find a unique solution as follows. Recall the definition (50) of the function $F_k(X)$ as

$$F_k(X) = \{\sum_{l=1}^{k} \frac{\sigma_l[\Gamma_l D'(\rho) - \underline{G}(X)]}{m\Gamma_k D'(\rho) + (m-1)\sigma_l \underline{G}'(X)}\} - \frac{X}{m(m-1)}.$$

Clearly, for $k \leq K$, $F_k(0) > 0$ and $F_k(X)$ is continuous and monotonically decreasing with increase in X (≥ 0). Thus for each k ($\leq K$) there exists $X = \tilde{X}_k (> 0)$ that satisfies $F_k(\tilde{X}_k) = 0$. Since (47) or (48) holds, we can find the largest $k = \tilde{k}$ such that $\tilde{x}_{\tilde{k}} > 0$ (*i.e.*, $\Gamma_{\tilde{k}} D'(\rho) - \underline{G}(\tilde{X}_{\tilde{k}}) > 0$). Then given $\tilde{X}_{\tilde{k}}$, from the first equation of (58) we can obtain a unique set of values for $\tilde{x}_k, 1 \leq k \leq \tilde{k}$. We easily see that this equation assures that $0 < \tilde{x}_k \leq 1/m < 1/(m-1)$, $1 \leq k \leq k'$. The set of values, $\tilde{x}_k$, $k = 1, 2, \cdots, \tilde{k}$, as obtained above and $\tilde{x}_k = 0$, $k = \tilde{k}+1, \tilde{k}+2, \cdots, n'$, is the unique solution.

We can see it as follows: From definition (50) we have

$$F_{\tilde{k}}(\tilde{X}_{\tilde{k}}) = F_{\tilde{k}-1}(\tilde{X}_{\tilde{k}}) + \frac{\sigma_{\tilde{k}}[\Gamma_{\tilde{k}} D'(\rho) - \underline{G}(\tilde{X}_{\tilde{k}})]}{m\Gamma_k D'(\rho) + (m-1)\sigma_{\tilde{k}} \underline{G}'(\tilde{X}_{\tilde{k}})}. \quad (60)$$

Thus we have $F_{\tilde{k}-1}(\tilde{X}_{\tilde{k}}) < 0$ since $F_{\tilde{k}}(\tilde{X}_{\tilde{k}}) = 0$ and the second term of the right hand side of (60) must be positive.

Assume that we have another feasible solution for $k' = \tilde{k} - 1$. Then we have $X_{\tilde{k}-1} > 0$ such that $F_{\tilde{k}-1}(X_{\tilde{k}-1}) = 0$ and $\tilde{x}_{\tilde{k}} = 0$. Therefore we have

$$F_{\tilde{k}-1}(\tilde{X}_{\tilde{k}}) < 0 = F_{\tilde{k}-1}(X_{\tilde{k}-1}).$$

Therefore, since $F_{\tilde{k}-1}(\cdot)$ is monotonically decreasing, we must have $\tilde{X}_{\tilde{k}} > X_{\tilde{k}-1}$ and thus $\underline{G}(\tilde{X}_{\tilde{k}}) > \underline{G}(X_{\tilde{k}-1})$. Consequently, since $\Gamma_{\tilde{k}} D'(\rho) - \underline{G}(\tilde{X}_{\tilde{k}}) > 0$, we have $\Gamma_{\tilde{k}} D'(\rho) - \underline{G}(X_{\tilde{k}-1}) > 0$. Therefore, from (51), *i.e.*, the first relation of (58), we must have $\tilde{x}_{\tilde{k}} > 0$, which is a contradiction.

In a similar way, for k and k' such that $\Gamma_k = \Gamma_{k'}$, we can show that either $\tilde{x}_k = \tilde{x}_{k'} = 0$ or $\tilde{x}_k, \tilde{x}_{k'} > 0$.

Therefore, we see that we have the unique solution. That is, we can obtain the unique set of values such that $\tilde{x}_k > 0, k = 1, 2, \cdots, \tilde{k}$, and $\tilde{x}_{\tilde{k}+1} = \tilde{x}_{\tilde{k}+2} = \cdots = \tilde{x}_{n'} = 0$, which satisfies the above relation.

The mean response time (52) is obtained by noting the definitions (1), (7), (8), (9), and (11). □

Lemma 3. 4 *If there exists a mixed optimum for the network, then in the mixed optimum, the solution $\check{\mathbf{x}}_k$ for type k jobs with global-class optimization is unique and is given as follows:*

$$\check{\mathbf{x}}_{k-(iik)} = \mathbf{0}, \quad i.e., \ \check{x}_{ijk} = 0, \ and \ \check{x}_{iik} = 1, \ for\ all\ i, j (\neq i).$$

The mean response time is

$$T_k(\check{\mathbf{x}}) = T_{ik}(\check{\mathbf{x}}) = \mu_k^{-1} D(\rho), \ for\ all\ i.$$

Proof:

(c1) By Lemma 3. 1 , we have shown that $\vec{\beta}_j = \vec{\beta}_{j'}$ for every pair of (j, j'), and consequently, $\vec{\beta}_i = \rho$ for all i.

(c2) We next show by contradiction $\check{\beta}_j^{(k)} = \check{\beta}_{j'}^{(k)}$ for every pair of (j, j'), which implies that $\check{\beta}_i^{(k)} = \rho_k$ for all i.

From (42) we have for all $i, j(\neq i), j'(\neq i)$,

$$\check{\Xi}_{ijk;ij'k}(\check{\boldsymbol{x}}) \quad = \quad \rho_k(\beta_j^{(k)} - \beta_{j'}^{(k)})D'(\beta_1) + \check{g}_{ijk}(\check{\boldsymbol{x}}) - \check{g}_{ij'k}(\check{\boldsymbol{x}}). \tag{61}$$

Assume $\check{\beta}_j^{(k)} > \check{\beta}_{j'}^{(k)}$ for some j and j'. We can follow the same line of logic as (c1-1), (c1-2), and (c1-3) in the proof of Lemma 3. 1 , even though $\beta_i = \rho$ for all i, and we see that the above assumption leads to a contradiction. Therefore necessarily $\check{\beta}_j^{(k)} = \check{\beta}_{j'}^{(k)}$, and consequently $\check{\beta}_i^{(k)} = \rho_k$ for all i.

(c3) Now from (42) we have for all $i, j(\neq i), j'(\neq i)$,

$$\check{\Xi}_{ijk;ij'k}(\check{\boldsymbol{x}}) \quad = \quad \check{g}_{ijk}(\check{\boldsymbol{x}}) - \check{g}_{ij'k}(\check{\boldsymbol{x}}). \tag{62}$$

Thus, if $\check{x}_{ijk} > \check{x}_{ij'k}$ for some $i, j(\neq i), j'(\neq i), k$, we have $\check{\Xi}_{ijk;ij'k} \geq 0$, which contradicts relation (43). Therefore, we must have

$$\check{x}_{ijk} = \check{x}_k \text{ for all } i, j(\neq i),$$

and from (31) and (42) we have for all $i, j(\neq i)$,

$$\check{\Xi}_{ijk;iik}(\check{\boldsymbol{x}}) = \check{g}(\check{\boldsymbol{x}}) > 0,$$

and consequently from (29) we have $\check{x}_k = 0$. □

Now we have the main result of this paper.

Theorem 3. 1 *Consider a network of homogeneous computers with several types of jobs where each type optimizes according to (A), (B-I) or (B-II). Then for the network there exists a unique mixed optimum, which is given as follows.*
(A) The solution $\hat{\boldsymbol{x}}_k$ for type k with individual optimization is unique and given as follows:

$$\hat{\boldsymbol{x}}_{k-(iik)} = \mathbf{0}, \quad i.e., \ \hat{x}_{ijk} = 0, \ \text{and } \hat{x}_{iik} = 1, \ \text{for all } i, j(\neq i),$$

The mean response time is

$$T_k(\hat{\boldsymbol{x}}) = T_{ik}(\hat{\boldsymbol{x}}) = \mu_k^{-1} D(\rho), \ \text{for all } i.$$

(B-I) The solution $\tilde{\boldsymbol{x}}_k$ for type k jobs with local-class optimization is unique and is given as follows:
For Types G-I and G-II(a)

(i) For local class R_{ik} such that $\rho_k^2 D'(\rho) \leq \tilde{g}_k(0) = \sigma_k$, i.e., $\Gamma_k D'(\rho) \leq 1$,

$$\tilde{x}_{ijk} = 0, \ \text{and } \tilde{x}_{iik} = 1, \ \text{for all } i, j(\neq i).$$

The mean response time is

$$T_k(\tilde{\boldsymbol{x}}) = T_{ik}(\tilde{\boldsymbol{x}}) = \mu_k^{-1} D(\rho), \ \text{for all } i.$$

(*ii*) *For local class* R_{ik} *such that* $\rho_k^2 D'(\rho) > \tilde{g}_k(0) = \sigma_k$, *i.e.,* $\Gamma_k D'(\rho) > 1$, *the solution is given as follows:*

$$\tilde{x}_{ijk} = \tilde{x}_k, \text{ for all } i,\ j(\neq i), \tag{63}$$

where $\tilde{x}_k$ *is the unique solution of*

$$\begin{aligned} &\rho_k^2(1 - m\tilde{x}_k)D'(\rho) = \tilde{g}_k(\tilde{x}_k) \\ &= \sigma_k[\underline{G}(m(m-1)\sigma_k\tilde{x}_k) + \sigma_k(m-1)\tilde{x}_k\underline{G}'(m(m-1)\sigma_k\tilde{x}_k)]. \end{aligned} \tag{64}$$

The mean response time is

$$T_k(\tilde{\mathbf{x}}) = T_{ik}(\tilde{\mathbf{x}}) = \mu_k^{-1}D(\rho) + (m-1)\tilde{x}_k G_k(\tilde{\mathbf{x}}), \quad \text{for all } i. \tag{46}$$

For Type G-II(b)

The solution is given as in the following. We first change the numbering of k such that $\Gamma_1 \geq \Gamma_2 \geq \cdots \geq \Gamma_k \geq \cdots \geq \Gamma_{n'}$, *where* n' *is the number of job types that seek the local-class optimization. The following three situations can occur:*

$$\begin{aligned} \text{We can find } K \text{ such that } \Gamma_K D'(\rho) &> 1 \text{ and } \Gamma_{K+1}D'(\rho) \leq 1, && (47) \\ \text{or } \Gamma_{n'}D'(\rho) &> 1 \ \ (i.e., K = n'), && (48) \\ \text{or } \Gamma_1 D'(\rho) &\leq 1. && (49) \end{aligned}$$

When (49) *holds, we have a unique solution of* $\tilde{x}_k = 0$ *for all* $k \leq n'$. *When* (47) *or* (48) *holds, we can find a unique solution as follows. Let us define the function* $F_k(X)$ *as*

$$F_k(X) = \{\sum_{l=1}^{k} \frac{\sigma_l[\Gamma_l D'(\rho) - \underline{G}(X)]}{m\Gamma_k D'(\rho) + (m-1)\sigma_l \underline{G}'(X)}\} - \frac{X}{m(m-1)}. \tag{50}$$

We obtain the largest $k = \tilde{k} \leq K$ *and* $X = \tilde{X}_{\tilde{k}} (> 0)$ *that satisfies* $F_{\tilde{k}}(\tilde{X}_{\tilde{k}}) = 0$ *and* $\sigma_{\tilde{k}}[\Gamma_{\tilde{k}}D'(\rho) - \underline{G}(\tilde{X}_{\tilde{k}})] > 0$. *Then by using*

$$\sigma_k[\Gamma_k D'(\rho) - \underline{G}(\tilde{X}_{\tilde{k}})] = \sigma_k\tilde{x}_k[m\Gamma_k D'(\rho) + (m-1)\sigma_k\underline{G}'(\tilde{X}_{\tilde{k}})], \tag{51}$$

for all k. for $k = 1, 2, \cdots, \tilde{k}$, *we can obtain the unique set of values such that* $\tilde{x}_k > 0, k = 1, 2, \cdots, \tilde{k}$, *and that* $\tilde{x}_{\tilde{k}+1} = \tilde{x}_{\tilde{k}+2} = \cdots = \tilde{x}_{n'} = 0$ *that satisfies the above relation, which is a unique solution. The mean response time is*

$$T_k(\tilde{\mathbf{x}}) = T_{ik}(\tilde{\mathbf{x}}) = \mu_k^{-1}D(\rho) + (m-1)\tilde{x}_k G_k(\tilde{\mathbf{x}}), \quad \text{for all } i. \tag{52}$$

(*B-II*) *The solution* $\check{\mathbf{x}}_k$ *for type k jobs with global-class optimization is unique and is given as follows:*

$$\check{\mathbf{x}}_{k-(iik)} = \mathbf{0}, \quad i.e., \ \check{x}_{ijk} = 0, \text{ and } \check{x}_{iik} = 1, \text{ for all } i, j(\neq i).$$

The mean response time is

$$T_k(\check{\mathbf{x}}) = T_{ik}(\check{\mathbf{x}}) = \mu_k^{-1}D(\rho), \text{ for all } i.$$

Proof: We have shown that, under the assumption that there exists a mixed optimum, a unique solution is given as shown in the above lemmata 3. 1 , 3. 2 , 3. 3 , and 3. 4 . On the other hand, we can see that solution given by these lemmata and shown in this theorem satisfies the definition of mixed optimum. We therefore see that a mixed optimum exists and is uniquely given as above. □

Remark 3. 1 *From the above we see that Braess-like paradoxical performance degradation occurs only for the type of jobs seeking local-class optimization and that among such types the chances of it vary on the basis of the values of* Γ_k $(= \rho_k^2/\sigma_k = \phi_k\omega_k/\mu_k^2)$. *That is, the performance for the types that have larger arrival rate* (ϕ_k), *larger processing time requirement* (μ_k^{-1}), *and smaller communication time requirement* (ω_k^{-1}), *has more chances to be degraded. The performance for all the types has more chances to be degraded with a larger value of* ρ $(= \sum_k \rho_k)$. *The chances of paradoxes are independent of the number of nodes,* m.

The types of jobs seeking individual or global-class optimization are not influenced by such performance degradation.

4 Numerical Examples

We consider here Examples 1 introduced in Section 2. We assume that $\mu_k = \mu$ and that $\phi_k = 1/n$ for all k. We have $\rho = 1/\mu$, $\rho_k = 1/(n\mu)$, $D(\rho) = 1/(1-\rho)$, and $\omega_k^{-1} = t$.

Consider that type k jobs seek local-class optimization. We therefore note that

$$\Gamma_k D'(\rho) - 1 = \frac{\phi_k\omega_k}{\mu_k^2} D'(\rho) - 1 = \frac{1}{nt(\mu-1)^2} - 1.$$

(i) If $t > 1/\{n(\mu-1)^2\}$, then $\tilde{\boldsymbol{x}}_k$ is unique and given by

$$\tilde{\boldsymbol{x}}_{-(iik)} = \mathbf{0}, \text{ i.e., } \tilde{x}_{ijk} = 0, \quad \tilde{x}_{iik} = 1, \quad \text{for all } i, j (\neq i).$$

The mean response time is

$$T_k(\tilde{\boldsymbol{x}}) = T_{ik}(\tilde{\boldsymbol{x}}) = \mu_k^{-1} D(\rho) = \frac{1}{\mu - 1}, \quad i = 1, 2, \cdots, m.$$

It is the same for other types of jobs that seek individual, and global-class optima.

(ii) If $t \leq 1/\{n(\mu-1)^2\}$, $\tilde{\boldsymbol{x}}_k$ is given by

$$\tilde{x}_{ijk} = \frac{1}{m}\{1 - nt(\mu-1)^2\}, \quad \tilde{x}_{iik} = \frac{1}{m}\{1 + (m-1)nt(\mu-1)^2\}, \qquad \text{for all } i, j (\neq i). \quad (65)$$

The mean response time is

$$\begin{aligned} T_k(\tilde{\boldsymbol{x}}) &= T_{ik}(\tilde{\boldsymbol{x}}) \\ &= \frac{1}{\mu - 1} + \frac{m-1}{m} t\{1 - nt(\mu-1)^2\}, \quad \text{for all } i. \end{aligned} \quad (66)$$

For some parameters (μ, m, n), $\tilde{T}_k = T_k(\tilde{\boldsymbol{x}})$ attains its maximum in t (*i.e.*, the worst performance), that we denote $\tilde{T}_{k,\max}(\mu, m, n)$, for

$$t = t_{k,\max} = \frac{1}{2n(\mu-1)^2}. \quad (67)$$

We have

$$\tilde{T}_{k,\max}(\mu, m, n) = \frac{1}{\mu - 1}\{1 + \frac{m-1}{4mn(\mu-1)}\}. \quad (68)$$

Thus if we add the communication lines with delay $t = t_{k,\max} = 1/\{2n(\mu-1)^2\}$ to the system that has had no communication means, the mean response time $T_{ik}(\tilde{\boldsymbol{x}})$ for each class increases in the amount of $\frac{m-1}{4mn(\mu-1)^2}$ (*i.e.*, the performance degrades). This is a Braess-like paradox. We define the *worst ratio of the performance degradation* $\Delta_k(\mu, m, n)$ in the paradox for type k jobs, given μ, n, to be

$$\Delta_k(\mu, m, n) = \frac{\tilde{T}_{k,\max}(\mu, m, n) - T_{k,0}(\mu)}{T_{k,0}(\mu)}, \tag{69}$$

where $T_{k,0}(\mu) = 1/(\mu - 1)$ denotes the mean response time of type k jobs for given μ when the system has no communication means. We have

$$\Delta_k(\mu, m, n) = \frac{m-1}{4mn(\mu-1)}. \tag{70}$$

We examine Example 1 with $m = 5$, *i.e.*, the system with five nodes, and consider the case: $\mu = 1.01$. We consider the case where $n = 4$, *i.e.*, the total number of classes R_{ik} is 20. Types 1, 2, 3, and 4 jobs seek individual, individual, local-class, and global-class optimization, respectively.

The mean response time is $T_{k,0}(\mu) = 1/(\mu - 1) = 100$ in the individual optimum (types 1 and 2, *i.e.*, $k = 1, 2$), in the global-class optimum (type 4, *i.e.*, $k = 4$). That is, $T_{1,0} = T_{2,0} = T_{4,0} = 100$ independently of the value of the communication time parameter t. On the other hand, $T_3 = T_{i3}$ depends on t and takes its maximum value

$$\tilde{T}_3(1.01, 5, 4) = 600 \text{ (see (68))},$$

and the worst ratio of the performance degradation $\Delta_3(\mu, m, n)$ in the paradox is

$$\Delta_3(1.01, 5, 4) = 5 \text{ (}i.e.\text{, } 500\% \text{ degradation)} \quad \text{(see (69))},$$

when $t = 1/\{2(\mu - 1)^2\} = 1250$ (see (67)). Then, from (65)

$$\begin{aligned} \tilde{x}_{ij3} &= (1/5)\{1 - 4t(\mu-1)^2\} = 1/10, \text{ for all } i, j(\neq i), \\ \tilde{x}_{ii3} &= (1/5)\{1 + 4 \times 4t(\mu-1)^2\} = 3/5, \text{ for all } i. \end{aligned}$$

In this case, $\tilde{x}_{ij3}$ $(= \tilde{x})$, $i \neq j$, decrease from 1/5 down to 0 as t increases from 0 to 2500 $(= 1/4(\mu - 1)^2)$, and for $t > 2500$, no forwarding of jobs occurs.

It is amazing that only local classes R_{i3}, $i = 1, 2, \cdots, 5$, forward a part of their jobs equally to the other nodes even though the communication delay for forwarding is much greater than the processing delay at the node at which their jobs arrive.

Furthermore we consider other values of μ while the other situations are kept the same as above.

For $\mu = 1.0001$, $\Delta_3(1.0001, 5, 4) = 500$ (*i.e.*, 50000% degradation),

and for $\mu = 1.0000001$, $\Delta_3(1.0000001, 5, 4) = 500000$ (*i.e.*, 50000000% degradation), *etc.*

In this way, we see that the worst ratio of the performance degradation $\Delta_3(\mu, m, n)$ in the paradox becomes unlimitedly large as μ approaches 1.

5 Concluding Remarks

In this paper, we examined the model consisting of symmetrical nodes with identical arrivals to all nodes where forwarding of jobs to the other nodes through communication means with nonzero delays may clearly lead to performance degradation. We considered the mixed optimization where each job type seeks distinct level of distributed optimization. We computed explicitly the equilibrium. We observed a paradoxical behavior in which in equilibrium there is mutual forwarding among nodes. We saw, however, that such a paradoxical behavior may occur only with the job types seeking local-class optimization and does never occur for the job types seeking the individual (Wardrop) and global-class optima, in this symmetrical node model.

We established the uniqueness of a mixed-equilibrium in which different classes may have different types of distributed optimization (local-class, global-class and individual optimization). This was done under different possible assumptions on the communication means (*i.e.*, dedicated lines and bus-type connections). It has been quite hard to extend the proofs to more general assumptions. It is not certain whether in some cases of the communication means the optima may still be unique. It has been also difficult for us to analyze asymmetrical models. These are open future problems.

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