Generalized Controlled and Conditioned Invariances for Linear ω-Periodic Discrete-Time Systems

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ISE-TR-00-172

June 1, 2000

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Generalized Controlled and Conditioned Invariances for Linear ω-Periodic Discrete-Time Systems

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Abstract

In this paper the three notions of generalized controlled $(A_\alpha(\cdot),B_\beta(\cdot))$-invariance, generalized conditioned $(C_\gamma(\cdot),A_\alpha(\cdot))$-invariance and generalized $(A_\alpha(\cdot),B_\beta(\cdot),C_\gamma(\cdot))$-invariance for uncertain linear ω-periodic discrete-time systems are studied, and then the parameter insensitive disturbance-rejection problems are formulated and their solvability conditions are presented.

Keywords : Generalized invariances, Uncertain Discrete-Time Systems, ω-Periodic Systems, Disturbance-Rejection, Geometric Approach

1 Introduction

The notions of controlled (A, B)-invariant subspaces and conditioned (C, A)-invariant subspaces were studied by Basile and Marro[2] and Wonham[14] and then various disturbance-rejection problems have been studied using those invariant subspaces (e.g., [1], [5], [11], [14]). Further, from the practical viewpoint the parameter insensitive disturbance-rejection problem with state feedback was first considered by Bhattacharyya[3] in the case in which the matrices depend linearly on uncertain parameters using the notion of generalized controlled (A, B)-invariant subspaces. Recently, the present author[9], [10] investigated the notions of generalized conditioned (C, A)-invariant subspaces and generalized (C, A, B)-pairs, and the corresponding problems with static output feedback and / or with dynamic compensator were studied. Further, simultaneous versions of controlled (A, B)-invariant subspaces, conditioned (C, A)-invariant subspaces and (C, A, B)-pairs were investigated by Ghosh[4] and Otsuka et al.[8] and various parameter insensitive disturbance-rejection problems for uncertain linear systems in the sense that system's matrices are represented as convex combinations of given matrices were investigated.

On the other hand, Grasselli and Longhi[6] investigated the ω-periodic versions of controlled (A, B)-invariance and conditioned (C, A)-invariance and Shiomi, Otsuka and Inaba[12] investigated the ω-periodic versions of simultaneous controlled (A, B)-invariance and simultaneous conditioned (C, A)-invariance.

The objective of this paper is to study the notions of generalized controlled $(A_\alpha(\cdot),B_\beta(\cdot))$-invariance, generalized conditioned $(C_\gamma(\cdot),A_\alpha(\cdot))$-invariance and generalized $(A_\alpha(\cdot),B_\beta(\cdot),C_\gamma(\cdot))$-invariance for linear ω-periodic discrete-time systems and to study the parameter insensitive disturbance-rejection problems.

The present investigation is organized as follows. Section 2 gives the notions of some generalized invariances and their properties. In Section 3 the parameter insensitive disturbance-rejection problems are studied. Finally, we make some conclusions in Section 4.

2 Generalized Invariances

First, the following notations are used throughout this investigation. $\boldsymbol{N}$:= the set of all natural numbers, $\boldsymbol{Z}$:= the set of all integers, $\boldsymbol{Z}_{k_0}^{\omega} := \{k_0 + 1, k_0 + 2, \ldots, k_0 + \omega \mid k_0 \in \boldsymbol{Z}\}$ for $\omega \in \boldsymbol{N}$, $\boldsymbol{R}^s$:= s dimensional

Euclidean space and $\boldsymbol{R}^{p\times q}$:= the set of all $p \times q$ real matrices. For a matrix-valued function $A(\cdot)$ ($A(k) \in \boldsymbol{R}^{p\times q}, k \in \boldsymbol{Z}$), Im$A(k)$:= the image of $A(k)$, Ker$A(k)$:= the kernel of $A(k)$ and $A^{-1}(k)\Omega := \{x \in \boldsymbol{R}^q \mid A(k)x \in \Omega\}$ for a subspace Ω of $\boldsymbol{R}^p$. And $A(\cdot)$ is said to be ω-periodic for a given $\omega \in \boldsymbol{N}$ if $A(k) = A(k+\omega)$ for all $k \in \boldsymbol{Z}$. For a subspace-valued function $V(\cdot)$ ($V(k) \subset \boldsymbol{R}^s, k \in \boldsymbol{Z}$), $V(\cdot)$ is said to be ω-periodic for a given $\omega \in \boldsymbol{N}$ if $V(k) = V(k+\omega)$ for all $k \in \boldsymbol{Z}$. Further, $\boldsymbol{0}$ indicates the zero vector of a linear space X and a subspace V of X dim$V$ indicates the dimension of V.

Next, consider a family of linear ω-periodic discrete-time systems $\{S(\alpha,\beta,\gamma)\}$ given by

$$S(\alpha,\beta,\gamma): \begin{cases} x(k+1) = A_\alpha(k)x(k) + B_\beta(k)u(k), \\ y(k) = C_\gamma(k)x(k), \ k \in \boldsymbol{Z} \end{cases}$$

where $x(k) \in X := \boldsymbol{R}^n, u(k) \in U := \boldsymbol{R}^m$, $y(k) \in Y := \boldsymbol{R}^\ell$ are the state, the input and the measurement output, respectively. And matrix-valued functions $A_\alpha(\cdot)$, $B_\beta(\cdot)$ and $C_\gamma(\cdot)$ are all ω-periodic and have unknown parameters in the sense that

$$\begin{aligned} A_\alpha(k) &= A_0(k) + \alpha_1 A_1(k) + \cdots + \alpha_p A_p(k) := A_0(k) + \Delta A_\alpha(k) \in \boldsymbol{R}^{n\times n} \\ B_\beta(k) &= B_0(k) + \beta_1 B_1(k) + \cdots + \beta_q B_q(k) := B_0(k) + \Delta B_\beta(k) \in \boldsymbol{R}^{n\times m}, \\ C_\gamma(k) &= C_0(k) + \gamma_1 C_1(k) + \cdots + \gamma_r C_r(k) := C_0(k) + \Delta C(\gamma)(k) \in \boldsymbol{R}^{\ell\times n}, \end{aligned}$$

where $\alpha := (\alpha_1,\cdots,\alpha_p) \in \boldsymbol{R}^p$, $\beta := (\beta_1,\cdots,\beta_q) \in \boldsymbol{R}^q$ and $\gamma := (\gamma_1,\cdots,\gamma_r) \in \boldsymbol{R}^r$.

In system $S(\alpha,\beta,\gamma)$ $(A_0(k), B_0(k), C_0(k))$ and $(\Delta A_\alpha(k), \Delta B_\beta(k), \Delta C_\gamma(k))$ represent the nominal system model and a specific uncertain perturbation, respectively.

Definition 2.1 Let $V(\cdot)$ ($V(k) \subset X$) be ω-periodic subspace-valued function.

(i) $V(\cdot)$ is said to be a generalized controlled ($A_\alpha(\cdot)$,$B_\beta(\cdot)$)-invariant if there exists an ω-periodic feedback $F(\cdot)$ ($F(k) \in \boldsymbol{R}^{m\times n}$) such that

$$(A_\alpha(k) + B_\beta(k)F(k))V(k) \subset V(k+1)$$

for all $(\alpha,\beta) \in \boldsymbol{R}^p \times \boldsymbol{R}^q$ and $k \in \boldsymbol{Z}$. Further, for a given ω-periodic subspace-valued function $\Lambda(\cdot)$ ($\Lambda(k) \subset X$), define the following class of ω-periodic subspace-valued functions.

$$\boldsymbol{V}(A_\alpha(\cdot), B_\beta(\cdot); \Lambda(\cdot)) := \Big\{V(\cdot) \mid V(\cdot) \text{ is a generalized controlled } (A_\alpha(\cdot), B_\alpha(\cdot))\text{-invariant and } V(k) \subset \Lambda(k) \text{ for all } k \in \boldsymbol{Z}\Big\}.$$

(ii) $V(\cdot)$ is said to be a generalized conditioned ($C_\gamma(\cdot)$,$A_\alpha(\cdot)$)-invariant if there exists an ω-periodic $G(\cdot)$ ($G(k) \in \boldsymbol{R}^{n\times\ell}$) such that

$$(A_\alpha(k) + G(k)C_\gamma(k))V(k) \subset V(k+1)$$

for all $(\alpha,\gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^r$ and $k \in \boldsymbol{Z}$. Further, for a given ω-periodic subspace-valued function $\varepsilon(\cdot)$ ($\varepsilon(k) \subset X$), define the following class of ω-periodic subspace-valued functions.

$$\boldsymbol{V}(\varepsilon(\cdot); C_\gamma(\cdot), A_\alpha(\cdot)) := \Big\{V(\cdot) \mid V(\cdot) \text{ is a generalized conditioned } (C_\gamma(\cdot), A_\alpha(\cdot))\text{-invariant and } \varepsilon(k) \subset V(k) \text{ for all } k \in \boldsymbol{Z}\Big\}.$$

(iii) $V(\cdot)$ is said to be a generalized ($A_\alpha(\cdot)$,$B_\beta(\cdot)$,$C_\gamma(\cdot)$)-invariant if there exists an ω-periodic feedback $H(\cdot)$ ($H(k) \in \boldsymbol{R}^{m\times\ell}$) such that

$$(A_\alpha(k) + B_\beta(k)H(k)C_\gamma(k))V(k) \subset V(k+1)$$

for all $(\alpha,\beta,\gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^q \times \boldsymbol{R}^r$ and $k \in \boldsymbol{Z}$. Further, define the following class of ω-periodic subspace-valued functions.

$$\boldsymbol{V}(A_\alpha(\cdot), B_\beta(\cdot), C_\gamma(\cdot)) := \Big\{V(\cdot) \mid V(\cdot) \text{ is a generalized } (A_\alpha(\cdot), B_\beta(\cdot), C_\gamma(\cdot))\text{-invariant } \Big\}. \ \blacksquare$$

Definition 2.2

(i) $V^*(\cdot)$ is said to be a maximal element of $\boldsymbol{V}(A_\alpha(\cdot), B_\beta(\cdot); \Lambda(\cdot))$ if $V^*(\cdot) \in \boldsymbol{V}(A_\alpha(\cdot), B_\beta(\cdot); \Lambda(\cdot))$ and $V(k) \subset V^*(k)$ $(k \in \boldsymbol{Z})$ for all $V(\cdot) \in \boldsymbol{V}(A_\alpha(\cdot), B_\beta(\cdot); \Lambda(\cdot))$.

(ii) $V_*(\cdot)$ is said to be a minimal element of $\boldsymbol{V}(\varepsilon(\cdot); C_\gamma(\cdot), A_\alpha(\cdot))$ if $V_*(\cdot) \in \boldsymbol{V}(\varepsilon(\cdot); C_\gamma(\cdot), A_\alpha(\cdot))$ and $V_*(k) \subset V(k)$ $(k \in \boldsymbol{Z})$ for all $V(\cdot) \in \boldsymbol{V}(\varepsilon(\cdot); C_\gamma(\cdot), A_\alpha(\cdot))$. ■

The following two theorems are the ω-periodic versions of the results of Bhattacharyya[3].

Theorem 2.3 Let $V(\cdot)$ $(V(k) \subset X)$ be ω-periodic subspace-valued function. Then, the following three statements are equivalent.

(i) $V(\cdot) \in \boldsymbol{V}(A_\alpha(\cdot), B_\beta(\cdot); \Lambda(\cdot))$.

(ii) There exists an ω-periodic feedback $F(\cdot)$ $(F(k) \in \boldsymbol{R}^{m\times n})$ such that

$$(A_0(k) + B_0(k)F(k))V(k) \subset V(k+1) \text{ and } B_i(k)F(k)V(k) \subset V(k+1) \ (i = 1, \cdots, q)$$

for all $k \in \boldsymbol{Z}$, and $A_i(k)V(k) \subset V(k+1) \subset \Lambda(k+1)$ $(i = 1, \cdots, p)$ for all $k \in \boldsymbol{Z}$.

(iii) $A_0(k)V(k) \subset B_0(k)\bigcap_{i=1}^{q} B_i^{-1}(k)V(k+1) + V(k+1)$ and $A_i(k)V(k) \subset V(k+1) \subset \Lambda(k+1)$ $(i = 1, \cdots, p)$ for all $k \in \boldsymbol{Z}$.

Proof. ((i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii)) Since the proofs easily follow, they are omitted.

((iii) $\Rightarrow$ (ii)) Suppose that (iii) holds. Noticing that $V(\cdot)$ is ω-periodic, let $\{v_1(k), \cdots, v_{t_k}(k)\}$ be a basis of $V(k)$ and let $\{v_1(k), \cdots, v_{t_k}(k), v_{t_k+1}, \cdots, V_n(k)\}$ be a basis of X satisfying $v_j(k) = v_j(k+\omega)$ $(j = 1, \cdots, n)$. Then, there exist ω-periodic vector-valued functions $\gamma_j(\cdot)$ and $w_j(\cdot)$ $(j = 1, \cdots, t_k)$ such that

$$A_0(k)v_j(k) = B_0(k)\gamma_j(k) + w_j(k), \text{ where } \gamma_j(k) \in \bigcap_{i=1}^{q} B_i^{-1}(k)V(k+1) \text{ and } w_j(k) \in V(k+1).$$

Define ω-periodic feedback $F(\cdot)$ $(F(k) \in \boldsymbol{R}^{m\times n})$ such that

$$F(k)v_j(k) = \begin{cases} -\gamma_j(k) \ (i = 1, \cdots, t_k) \\ \boldsymbol{0} \ (i = t_k + 1, \cdots, n). \end{cases}$$

Then,

$$(A_0(k) + B_0(k)F(k))v_j(k) = w_j(k) \in V(k+1) \quad (j = 1, \cdots, t_k)$$

and

$$B_i(k)F(k)v_j(k) = B_i(k)(-\gamma_j(k)) \in V(k+1) \quad (i = 1, \cdots, q; j = 1, \cdots, t_k),$$

which proves (ii). ■

The following theorem gives a computational algorithm of the maximal element of $\boldsymbol{V}(A_\alpha(\cdot), B_\beta(\cdot); \Lambda(\cdot))$.

Theorem 2.4 Let $\Lambda(\cdot)$ $(\Lambda(k) \subset X)$ be ω-periodic subspace-valued function. For each $k \in \boldsymbol{Z}$, define the sequence $V^\mu(k)$ according to

$V^0(k) := \Lambda(k)$ and

$$V^{\mu+1}(k) := V^\mu(k) \cap A_0^{-1}(k)(B_0(k)\bigcap_{i=1}^{q} B_i^{-1}(k)V^\mu(k+1) + V^\mu(k+1)) \cap A_1^{-1}(k)V^\mu(k+1) \cap \cdots \cap A_p^{-1}(k)V^\mu(k+1)$$

$(\mu \geq 0)$.

Then, the following statements hold.

(i) $V^{\mu+1}(k) \subset V^{\mu}(k)$ for all $k \in \boldsymbol{Z}$ and $\mu \geq 0$.

(ii) For fixed $k_0 \in \boldsymbol{Z}$ there exists a $j_0 \leq \max\{\dim[\Lambda(k)] \mid k \in \boldsymbol{Z}_{k_0}^{\omega}\}$ such that $V^{j_0}(\cdot)$ is the maximal element of $\boldsymbol{V}(A_\alpha(\cdot), B_\beta(\cdot); \Lambda(\cdot))$.

Proof. Since the proof of (i) follows easily, we prove (ii). To prove (ii) it suffices to show the following two claims.

Claim 1 : There exists a $j_0 \leq \max\{\dim[\Lambda(k)] \mid k \in \boldsymbol{Z}_{k_0}^{\omega}\}$ such that $V^{j_0}(\cdot) \in \boldsymbol{V}(A_\alpha(\cdot), B_\beta(\cdot); \Lambda(\cdot))$.

Claim 2 : $V(k) \subset V^{\mu}(k)$ $(k \in \boldsymbol{Z},\ \mu \geq 0)$ for all element $V(\cdot) \in \boldsymbol{V}(A_\alpha(\cdot), B_\beta(\cdot); \Lambda(\cdot))$.

(Proof of Claim 1) It remarks that ω-periodicity of $V^{\mu}(\cdot)$ $(\mu \geq 0)$ is obvious. Further, it follows from (i) and ω-periodicity of $\Lambda(\cdot)$ that there exists a $j_0 \leq \max\{\dim[\Lambda(k)] \mid k \in \boldsymbol{Z}_{k_0}^{\omega}\}$ such that

$$V^{j_0}(k) := V^{j_0}(k) \cap A_0^{-1}(k)(B_0(k)\bigcap_{i=1}^{q} B_i^{-1}(k)V^{j_0}(k+1) + V^{j_0}(k+1)) \cap A_1^{-1}(k)V^{j_0}(k+1) \cap \cdots \cap A_p^{-1}(k)V^{j_0}(k+1)\ (k \in \boldsymbol{Z}).$$

Hence,

$$V^{j_0}(k) \subset \Lambda(k),\ A_0(k)V^{j_0}(k) \subset B_0(k)\bigcap_{i=1}^{q} B_i^{-1}(k)V^{j_0}(k+1) + V^{j_0}(k+1) \text{ and } A_i(k)V^{j_0}(k) \subset V^{j_0}(k+1)\ (i = 1, \cdots, p).$$

It follows from Theorem 2.3 that $V^{j_0}(\cdot) \in \boldsymbol{V}(A_\alpha(\cdot), B_\beta(\cdot); \Lambda(\cdot))$.

(Proof of Claim 2) Let $V(\cdot)$ be an arbitrary element of $\boldsymbol{V}(A_\alpha(\cdot), B_\beta(\cdot); \Lambda(\cdot))$. Then, $V(k) \subset \Lambda(k) = V_0(k)$ for all $k \in \boldsymbol{Z}$. Assume that $V(k) \subset V^{\mu}(k)$ for all $k \in \boldsymbol{Z}$. Then, it follows from Theorem 2.3 that

$$\begin{aligned}
V(k) &\subset V^{\mu}(k) \cap A_0^{-1}(k)(B_0(k)\bigcap_{i=1}^{q} B_i^{-1}(k)V(k+1) + V(k+1)) \cap A_1^{-1}(k)V(k+1) \cap \cdots \cap A_p^{-1}(k)V(k+1) \\
&\subset V^{\mu}(k) \cap A_0^{-1}(k)(B_0(k)\bigcap_{i=1}^{q} B_i^{-1}(k)V^{\mu}(k+1) + V^{\mu}(k+1)) \cap A_1^{-1}(k)V^{\mu}(k+1) \cap \cdots \cap A_p^{-1}(k)V^{\mu}(k+1) \\
&= V^{\mu+1}(k)\ (k \in \boldsymbol{Z}),
\end{aligned}$$

which proves $V(k) \subset V^{\mu}(k)$ for all $k \in \boldsymbol{Z}$ and $\mu \geq 0$. This completes the proof of Theorem 2.4. ■

The following two theorems are ω-periodic versions of the results of Otsuka[10] and are dualities of Theorems 2.3 and 2.4, respectively.

Theorem 2.5 Let $V(\cdot)$ $(V(k) \subset X)$ be ω-periodic subspace-valued function. Then, the following three statements are equivalent.

(i) $V(\cdot) \in \boldsymbol{V}(\varepsilon(\cdot); C_\gamma(\cdot), A_\alpha(\cdot))$.

(ii) There exists an ω-periodic $G(\cdot)$ $(G(k) \in \boldsymbol{R}^{n \times \ell})$ such that $(A_0(k) + G(k)C_0(k))V(k) \subset V(k+1)$ and $G(k)C_i(k)V(k) \subset V(k+1)$ $(i = 1, \cdots, r)$ for all $k \in \boldsymbol{Z}$, $A_i(k)V(k) \subset V(k+1)$ $(i = 1, \cdots, p)$ and $\varepsilon(k) \subset V(k)$ for all $k \in \boldsymbol{Z}$.

(iii) $A_0(k)(V(k) \cap C_0^{-1}(k)\sum_{i=1}^{r} C_i(k)V(k)) \subset V(k+1)$, $A_i(k)V(k) \subset V(k+1)$ $(i = 1, \cdots, p)$ and $\varepsilon(k) \subset V(k)$ for all $k \in \boldsymbol{Z}$. ■

Theorem 2.6 Let $\varepsilon(\cdot)$ $(\varepsilon(k) \subset X)$ be ω-periodic subspace-valued function. For each $k \in \boldsymbol{Z}$, define the sequence $V_\mu(k)$ according to

$V_0(k) := \varepsilon(k)$ and

$$V_{\mu+1}(k+1) := V_\mu(k+1) + A_0(k)(V_\mu(k) \cap C_0^{-1}(k)\sum_{i=1}^{r} C_i(k)V_\mu(k)) + A_1(k)V_\mu(k) + \cdots + A_p(k)V_\mu(k) \quad (\mu \geq 0).$$

Then, the following statements hold.

(i) $V_\mu(k) \subset V_{\mu+1}(k)$ for all $k \in \boldsymbol{Z}$ and $\mu \geq 0$.

(ii) For fixed $k_0 \in \boldsymbol{Z}$ there exists an $i_0 \leq \max\{n - \dim\varepsilon(k) \mid k \in \boldsymbol{Z}_{k_0}^{\omega}\}$ such that $V_{i_0}(\cdot)$ is the minimal element of $\boldsymbol{V}(\varepsilon(\cdot); C_\gamma(\cdot), A_\alpha(\cdot))$. ∎

The following two lemmas are used to prove Theorem 2.9.

Lemma 2.7 [7] Let V, W ($\subset X$) be subspaces. Then, there exist subspaces X_0 and X_1 such that $V = X_1 \oplus (V \cap W)$, $X = X_0 \oplus W$ and $X_1 \subset X_0$. ∎

Lemma 2.8 [13] Let $F \in \boldsymbol{R}^{m\times n}$ and $T \in \boldsymbol{R}^{\ell\times n}$. Then, there exists a $K \in \boldsymbol{R}^{m\times \ell}$ such that $F = KT$ if and only if $\mathrm{Ker}T \subset \mathrm{Ker}F$. ∎

Theorem 2.9 Let $V(\cdot)$ ($V(k) \subset X$) be ω-periodic subspace-valued function. Then, the following three statements are equivalent.

(i) $V(\cdot) \in \boldsymbol{V}(A_\alpha(\cdot), B_\beta(\cdot), C_\gamma(\cdot))$.

(ii) There exists an ω-periodic feedback $H(\cdot)$ ($H(k) \in \boldsymbol{R}^{m\times\ell}$) such that $(A_0(k)+B_0(k)H(k)C_0(k))V(k) \subset V(k+1)$ and $B_i(k)H(k)C_j(k)V(k) \subset V(k+1)$ $(i = 0, \cdots, q, j = 0, \cdots, r; (i,j) \neq (0,0))$ for all $k \in \boldsymbol{Z}$, and $A_i(k)V(k) \subset V(k+1)$ $(i = 1, \cdots, p)$ for all $k \in \boldsymbol{Z}$.

(iii) $A_0(k)V(k) \subset B_0(k)\bigcap_{i=1}^{q} B_i^{-1}(k)V(k+1) + V(k+1)$, $A_0(k)(V(k) \cap C_0^{-1}(k)\sum_{i=1}^{r} C_i(k)V(k)) \subset V(k+1)$ and $A_i(k)V(k) \subset V(k+1)$ $(i = 1, \cdots, p)$ for all $k \in \boldsymbol{Z}$.

Proof. ((i) $\Leftrightarrow$ (ii)) The proofs follow easily.
((ii) $\Rightarrow$ (iii)) The proof follows from Theorems 2.3 and 2.5.
((iii) $\Rightarrow$ (ii)) Suppose that (iii) holds. Then, from Lemma 2.7 there exist subspaces $\Phi(k)$ and $X_0(k)$ such that $V(k) = \Phi(k) \oplus (V(k) \cap C_0^{-1}(k)\sum_{i=1}^{r} C_i(k)V(k))$, $X = X_0(k) \oplus C_0^{-1}(k)\sum_{i=1}^{r} C_i(k)V(k)$ and $\Phi(k) \subset X_0(k)$.

Now, let $L(k) : \boldsymbol{R}^n \to \boldsymbol{R}^n$, be a projection map onto $X_0(k)$ along $C_0^{-1}(k)\sum_{i=1}^{r} C_i(k)V(k)$. Further, let $Q_{V(k+1)} : \boldsymbol{R}^m \to \boldsymbol{R}^m$, be a projection map onto $\bigcap_{i=1}^{q} B_i^{-1}(k)V(k+1)$ along $(\bigcap_{i=1}^{q} B_i^{-1}(k)V(k+1))^{\perp}$.
Then, since $A_0(k)V(k) \subset \mathrm{Im}B_0(k)Q_{V(k+1)} + V(k+1)$, there exists an $F_0(k) \in \boldsymbol{R}^{m\times n}$ such that $(A_0(k) + B_0(k)Q_{V(k+1)}F_0(k))V(k) \subset V(k+1)$.
Define $\tilde{F}(k) := F_0(k)L(k)$. Further, let $P_{V(k)} : \boldsymbol{R}^\ell \to \boldsymbol{R}^\ell$, be a projection map onto $(\sum_{i=1}^{r} C_i(k)V(k))^{\perp}$ along $\sum_{i=1}^{r} C_i(k)V(k)$. Then, it can be easily obtained that $\mathrm{Ker}P_{V(k)}C_0(k) = C_0^{-1}(k)\sum_{i=1}^{r} C_i(k)V(k)$. Thus, we have $\mathrm{Ker}(P_{V(k)}C_0(k)) \subset \mathrm{Ker}\tilde{F}(k)$.

Hence, it follows from Lemmma 2.8 that there exists a $K(k) \in \boldsymbol{R}^{m\times\ell}$ such that $\tilde{F}(k) = K(k)P_{V(k)}C_0(k)$. Now, define $H(k) := Q_{V(k+1)}K(k)P_{V(k)}$.

Then, we have

$$(A_0(k) + B_0(k)H(k)C_0(k))V(k)$$
$$= (A_0(k) + B_0(k)Q_{V(k+1)}\tilde{F}(k))\Phi(k) + (A_0(k) + B_0(k)Q_{V(k+1)}\tilde{F}(k))(V(k) \cap C_0^{-1}(k)\sum_{i=1}^{r} C_i(k)V(k))$$

$$= (A_0(k)+B_0(k)Q_{V(k+1)}F_0(k)L(k))\Phi(k)+(A_0(k)+B_0(k)Q_{V(k+1)}K(k)P_{V(k)}C_0(k))(V(k)\cap C_0^{-1}(k)\sum_{i=1}^{r} C_i(k)V(k))$$
$$= (A_0(k) + B_0(k)Q_{V(k+1)}F_0(k))\Phi(k) + A_0(k)(V(k) \cap C_0^{-1}(k)\sum_{i=1}^{r} C_i(k)V(k))$$
$$\subset V(k+1).$$

On the other hand

$$\begin{aligned}
B_i(k)H(k)C_j(k)V(k) &= B_i(k)Q_{V(k+1)}K(k)P_{V(k)}C_j(k)V(k) \\
&\subset B_i(k)Q_{V(k+1)}K(k)P_{V(k)}\sum_{i=1}^{r} C_i(k)V(k) \\
&= \{\mathbf{0}\} \\
&\subset V(k) \quad (i = 0, \cdots, q, j = 1, \cdots, r).
\end{aligned}$$

Further,

$$\begin{aligned}
B_i(k)H(k)C_0(k)V(k) &= B_i(k)Q_{V(k+1)}K(k)P_{V(k)}C_0(k)V(k) \\
&\subset B_i(k)\mathrm{Im}Q_{V(k+1)} \\
&= B_i(k)\bigcap_{i=1}^{q} B_i^{-1}(k)V(k+1) \\
&= V(k+1) \quad (i = 1, \cdots, q).
\end{aligned}$$

This completes the proof. ∎

Concerning the three generalized invariances, the following corollary holds.

Corollary 2.10 $V(\cdot) \in \boldsymbol{V}(A_\alpha(\cdot), B_\beta(\cdot), C_\gamma(\cdot))$ if and only if $V(\cdot) \in \boldsymbol{V}(A_\alpha(\cdot), B_\beta(\cdot); X(\cdot)) \cap \boldsymbol{V}(\mathbf{0}(\cdot); C_\gamma(\cdot), A_\alpha(\cdot))$, where $X(k) := X$ and $\mathbf{0}(k) := \mathbf{0}$ for all $k \in \boldsymbol{Z}$. ∎

3 Parameter Insensitive Disturbance-Rejection

Consider the following linear ω-periodic discrete-time systems $S(\alpha, \beta, \gamma, \delta, \sigma)$.

$$S(\alpha, \beta, \gamma, \delta, \sigma) : \begin{cases} x(k+1) &= A_\alpha(k)x(k) + B_\beta(k)u(k) + M_\sigma(k)\xi(k) \\ y(k) &= C_\gamma(k)x(k), \\ z(k) &= D_\delta(k)x(k), \ k \in \boldsymbol{Z} \end{cases}$$

where $x(k) \in X := \boldsymbol{R}^n$, $u(k) \in U := \boldsymbol{R}^m$, $\xi(k) \in Q := \boldsymbol{R}^\eta$, $y(k) \in Y := \boldsymbol{R}^\ell$ and $z(k) \in Z := \boldsymbol{R}^\mu$ are the state, the input, the disturbance, the measurement output and the controlled output, respectively, and $A_\alpha(\cdot), B_\beta(\cdot), C_\gamma(\cdot), D_\delta(\cdot)$ and $M_\sigma(\cdot)$ are all ω-periodic and have uncertainties which are assumed to have the following unknown parameters, respectively.

$$\begin{aligned}
A_\alpha(k) &= A_0(k) + \alpha_1 A_1(k) + \cdots + \alpha_p A_p(k) := A_0(k) + \Delta A_\alpha(k) \in \boldsymbol{R}^{n\times n}, \\
B_\beta(k) &= B_0(k) + \beta_1 B_1(k) + \cdots + \beta_q B_q(k) := B_0(k) + \Delta B_\beta(k) \in \boldsymbol{R}^{n\times m}, \\
C_\gamma(k) &= C_0(k) + \gamma_1 C_1(k) + \cdots + \gamma_r C_r(k) := C_0(k) + \Delta C_\gamma(k) \in \boldsymbol{R}^{\ell\times n}, \\
D_\delta(k) &= D_0(k) + \delta_1 D_1(k) + \cdots + \delta_s D_s(k) := D_0(k) + \Delta D_\delta(k) \in \boldsymbol{R}^{q\times n}, \\
M_\sigma(k) &= M_0(k) + \sigma_1 M_1(k) + \cdots + \sigma_t M_t(k) := M_0(k) + \Delta M_\sigma(k) \in \boldsymbol{R}^{n\times\eta},
\end{aligned}$$

where $\alpha := (\alpha_1, \cdots, \alpha_p) \in \boldsymbol{R}^p$, $\beta := (\beta_1, \cdots, \beta_q) \in \boldsymbol{R}^q$, $\gamma := (\gamma_1, \cdots, \gamma_r) \in \boldsymbol{R}^r$, $\delta := (\delta_1, \cdots, \delta_s) \in \boldsymbol{R}^s$, $\sigma := (\sigma_1, \cdots, \sigma_t) \in \boldsymbol{R}^t$.

In system $S(\alpha, \beta, \gamma, \delta, \sigma)$ $(A_0(k), B_0(k), C_0(k), D_0(k), M_0(k))$ and $(\Delta A_\alpha(k), \Delta B_\beta(k), \Delta C_\gamma(k), \Delta D_\delta(k), \Delta M_\sigma(k))$ represent the nominal system model and a specific uncertain perturbation, respectively.

Now, we apply to system $S(\alpha, \beta, \gamma, \delta, \sigma)$ a static output feedback of the form $u(k) = H(k)y(k)$, where $H(\cdot)$ $(H(k) \in \boldsymbol{R}^{m \times \ell})$ is an ω-periodic. Then, we obtain the following closed loop system $S_{cl}(\alpha, \beta, \gamma, \delta, \sigma)$.

$$S_{cl}(\alpha, \beta, \gamma, \delta, \sigma) : \left\{ \begin{array}{rcl} x(k+1) & = & (A_\alpha(k) + B_\beta(k)H(k)C_\gamma(k))x(k) + M_\sigma(k)\xi(k) \\ z(k) & = & D_\delta(k)x(k), \ k \in \boldsymbol{Z}. \end{array} \right.$$

For the system $S_{cl}(\alpha, \beta, \gamma, \delta, \sigma)$ we use the notations $A^H_{\alpha\beta\gamma}(k) := A_\alpha(k) + B_\beta(k)H(k)C_\gamma(k)$, $\Phi^H_{\alpha\beta\gamma}(k, k_0) := A^H_{\alpha\beta\gamma}(k-1)A^H_{\alpha\beta\gamma}(k-2)\cdots A^H_{\alpha\beta\gamma}(k_0)$ for $k > k_0$ and $\Phi^H_{\alpha\beta\gamma}(k,k) := I_n$, where I_n is an $(n \times n)$ identity matrix.

Our parameter insensitive disturbance-rejection problem with static output feedback (PIDRPSOF) can be stated as follows. Given ω-periodic matrix-valued functions $A_i(\cdot)$, $B_i(\cdot)$, $C_i(\cdot)$, $D_i(\cdot)$ and $M_i(\cdot)$ of the system $S(\alpha, \beta, \gamma, \delta, \sigma)$, find (if possible) a static output feedback $H(\cdot)$ $(H(k) \in \boldsymbol{R}^{m \times \ell})$ which is ω-periodic such that

$$D_\delta(k) \sum_{h=k-n\omega}^{k-1} \Phi^H_{\alpha\beta\gamma}(k, h+1)M_\sigma(h)\xi(h) = \boldsymbol{0}$$

for all parameters $(\alpha, \beta, \gamma, \delta, \sigma) \in \boldsymbol{R}^p \times \boldsymbol{R}^q \times \boldsymbol{R}^r \times \boldsymbol{R}^s \times \boldsymbol{R}^t$, $\xi(\cdot)$ and $k \in \boldsymbol{Z}$ (cf. see e.g., [12]).

This problem can be rephrased as follows.

Problem 3.1 (PIDRPSOF) Given ω-periodic matrix-valued functions $A_i(\cdot)$, $B_i(\cdot)$, $C_i(\cdot)$, $D_i(\cdot)$ and $M_i(\cdot)$ for system $S(\alpha, \beta, \gamma, \delta, \sigma)$, find (if possible) a static output feedback $H(\cdot)$ $(H(k) \in \boldsymbol{R}^{m \times \ell})$ which is ω-periodic such that

$$\sum_{h=k-n\omega}^{k-1} \Phi^H_{\alpha\beta\gamma}(k, h+1)\mathrm{Im}M_\sigma(h) \subset \mathrm{Ker}D_\delta(k)$$

for all parameters $(\alpha, \beta, \gamma, \delta, \sigma) \in \boldsymbol{R}^p \times \boldsymbol{R}^q \times \boldsymbol{R}^r \times \boldsymbol{R}^s \times \boldsymbol{R}^t$ and $k \in \boldsymbol{Z}$. ∎

Remark 3.2 If $C_\gamma(k) = I_n$ for all $k \in \boldsymbol{Z}$, then the Problem 3.1 reduces to the parameter insensitive disturbance-rejection problem with state feedback (PIDRPSF). ∎

First, necessary and sufficient conditions for the Problem 3.1 (PIDRPSOF) to be solvable are given.

Theorem 3.3 The problem 3.1 (PIDRPSOF) is solvable if and only if there exists an $(A_\alpha(\cdot), B_\beta(\cdot), C_\gamma(\cdot))$-invariant (or equivalently $(A_\alpha(\cdot), B_\beta(\cdot))$ controlled invariant and $(C_\gamma(\cdot), A_\alpha(\cdot))$ conditioned invariant) subspace-valued function $V_{\alpha\beta\gamma}(\cdot)$ such that

(i) $\displaystyle\sum_{i=0}^{t} \mathrm{Im}M_i(k-1) \subset V_{\alpha\beta\gamma}(k) \subset \bigcap_{i=0}^{s} \mathrm{Ker}D_i(k)$ for all $(\alpha, \beta, \gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^q \times \boldsymbol{R}^r$ and $k \in \boldsymbol{Z}$,

(ii) $\displaystyle\bigcap_{\alpha,\beta,\gamma} \boldsymbol{H}(A_\alpha(\cdot), B_\beta(\cdot), C_\gamma(\cdot); V_{\alpha\beta\gamma}(\cdot)) \neq \emptyset$,

where $\boldsymbol{H}(A_\alpha(\cdot), B_\beta(\cdot), C_\gamma(\cdot); V_{\alpha\beta\gamma}(\cdot)) := \{\ H_{\alpha\beta\gamma}(\cdot)\ (H_{\alpha\beta\gamma}(k) \in \boldsymbol{R}^{m \times \ell})\ :\ \omega\text{-periodic}\ |$

$$(A_\alpha(k) + B_\beta(k)H_{\alpha\beta\gamma}(k)C_\gamma(k))V_{\alpha\beta\gamma}(k) \subset V_{\alpha\beta\gamma}(k+1)\}.$$

Proof. (Necessity) Suppose that the Problem 3.1 is solvable. Then, there exists a measurment output feedback $H(\cdot)$ $(H(k) \in \boldsymbol{R}^{m\times \ell})$ which is ω-periodic such that

$$\sum_{h=k-n\omega}^{k-1} \Phi^H_{\alpha\beta\gamma}(k,h+1)\mathrm{Im}M_\sigma(h) \subset \mathrm{Ker}D_\delta(k)$$

for all parameters $(\alpha,\beta,\gamma,\delta,\sigma) \in \boldsymbol{R}^p \times \boldsymbol{R}^q \times \boldsymbol{R}^r \times \boldsymbol{R}^s \times \boldsymbol{R}^t$ and $k \in \boldsymbol{Z}$.
Define a subspace-valued function

$$V_{\alpha\beta\gamma}(k) := \sum_{h=k-n\omega}^{k-1} \Phi^H_{\alpha\beta\gamma}(k,h+1)\sum_{i=0}^{t}\mathrm{Im}M_i(h).$$

Then, $V_{\alpha\beta\gamma}(k)$ is ω-periodic and satisfies

$$\sum_{i=0}^{t}\mathrm{Im}M_i(k-1) \subset V_{\alpha\beta\gamma}(k) \subset \bigcap_{i=0}^{s}\mathrm{Ker}D_i(k)$$

for all α,β,γ which proves (i). Further, $A^H_{\alpha\beta\gamma}(k)V_{\alpha\beta\gamma}(k) \subset V_{\alpha\beta\gamma}(k+1)$ which proves (ii).

(Sufficiency) Suppose that there exists an $(A_\alpha(\cdot),B_\beta(\cdot),C_\gamma(\cdot))$-invariant (or equivalently $(A_\alpha(\cdot),B_\beta(\cdot))$-controlled invariant and $(C_\gamma(\cdot),A_\alpha(\cdot))$-conditioned invariant) subspace-valued function $V_{\alpha\beta\gamma}(\cdot)$ satisfying the stated above conditions (i) and (ii). Now, it can be easily obtained

$$\mathrm{Im}M_\sigma(k-1) \subset V_{\alpha\beta\gamma}(k) \subset \mathrm{Ker}D_\delta(k) \text{ for all } \alpha,\beta,\gamma,\delta,\sigma.$$

Then, we have

$$\begin{aligned}\sum_{h=k-n\omega}^{k-1} \Phi^H_{\alpha\beta\gamma}(k,h+1)\mathrm{Im}M_\sigma(h) &\subset \sum_{h=k-n\omega}^{k-1} \Phi^H_{\alpha\beta\gamma}(k,h+1)V_{\alpha\beta\gamma}(h+1)\\ &\subset \sum_{h=k-n\omega}^{k-1} V_{\alpha\beta\gamma}(k)\\ &= V_{\alpha\beta\gamma}(k)\\ &\subset \mathrm{Ker}D_\delta(k)\end{aligned}$$

for all parameters $(\alpha,\beta,\gamma,\delta,\sigma)$ which proves that the Problem 3.1 is solvable. ∎

It remarks that the solvability conditions for Theorem 3.3 depend on uncertain parameters α,β and γ. Further, since it is not easy to find a subspace-valued function $V_{\alpha\beta\gamma}(\cdot)$ satisfying the conditions (i) and (ii) in Theorem 3.3, we consider a generalized $(A_\alpha(\cdot),B_\beta(\cdot),C_\gamma(\cdot))$-invariant subspace-valued function as one of $V_{\alpha\beta\gamma}(\cdot)$ satisfying those conditions.

Theorem 3.4 If there exists a $V(\cdot) \in \boldsymbol{V}(A_\alpha(\cdot),B_\beta(\cdot),C_\gamma(\cdot))$ such that

$$\sum_{i=0}^{t}\mathrm{Im}M_i(k-1) \subset V(k) \subset \bigcap_{i=0}^{s}\mathrm{Ker}D_i(k) \quad \text{for } k \in \boldsymbol{Z},$$

then the Problem 3.1 (PIDRPSOF) is solvable.

Proof. Suppose that there exists a $V(\cdot) \in \boldsymbol{V}(A_\alpha(\cdot),B_\beta(\cdot),C_\gamma(\cdot))$ such that the stated above conditions are satisfied. Then, there exists an ω-periodic feedback $H(\cdot)$ $(H(k) \in \boldsymbol{R}^{m\times\ell})$ such that

$$A^H_{\alpha\beta\gamma}(k)V(k) \subset V(k+1) \text{ for all } (\alpha,\beta,\gamma) \in \boldsymbol{R}^p \times \boldsymbol{R}^q \times \boldsymbol{R}^r \text{ and } k \in \boldsymbol{Z}.$$

Further,

$$\mathrm{Im}M_\sigma(k-1) \subset \sum_{i=0}^{t} \mathrm{Im}M_i(k-1) \quad \text{and} \quad \bigcap_{i=0}^{s} \mathrm{Ker}D_i(k) \subset \mathrm{Ker}D_\delta(k) \quad \text{for all } (\sigma, \delta) \in \boldsymbol{R}^t \times \boldsymbol{R}^u \text{ and } k \in \boldsymbol{Z}.$$

Then,

$$\begin{aligned} \Phi^H_{\alpha\beta\gamma}(k, h+1)V(h+1) &= A^H_{\alpha\beta\gamma}(k-1)A^H_{\alpha\beta\gamma}(k-2)\cdots A^H_{\alpha\beta\gamma}(h+1)V(h+1) \\ &\subset A^H_{\alpha\beta\gamma}(k-1)A^H_{\alpha\beta\gamma}(k-2)\cdots A^H_{\alpha\beta\gamma}(h+2)V(h+2) \\ &\subset V(k) \ \text{ for all } k, h+1 \in \boldsymbol{Z} \ (k \geq h+1). \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{h=k-n\omega}^{k-1} \Phi^H_{\alpha\beta\gamma}(k, h+1)\mathrm{Im}M_\sigma(h) &\subset \sum_{h=k-n\omega}^{k-1} \Phi^H_{\alpha\beta\gamma}(k, h+1)V(h+1) \\ &\subset \sum_{h=k-n\omega}^{k-1} V(k) \\ &= V(k) \\ &= \mathrm{Ker}D_\delta(k) \ \text{ for all } (\alpha, \beta, \gamma, \delta, \sigma) \in \boldsymbol{R}^p \times \boldsymbol{R}^q \times \boldsymbol{R}^r \times \boldsymbol{R}^s \times \boldsymbol{R}^t, \end{aligned}$$

which imply that the Problem 3.1 is solvable. ∎

Corollary 3.5 Suppose that $V^*(\cdot)$ is a maximal element of $\boldsymbol{V}(A_\alpha(\cdot), B_\beta(\cdot); \bigcap_{i=0}^{s}\mathrm{Ker}D_i(\cdot))$ and $V_*(\cdot)$ is a minimal element of $\boldsymbol{V}(\sum_{i=0}^{t}\mathrm{Im}M_i(\cdot-1); C_\gamma(\cdot), A_\alpha(\cdot))$. If $V^*(\cdot) = V_*(\cdot)$, then the Problem 3.1 (PIDRPSOF) is solvable.

Proof. The proof follows from Corollary 2.10 and Theorem 3.4. ∎

The following three results are the state feedback versions of Theorem 3.3, Theorem 3.4 and Corollary 3.5.

Theorem 3.6 The PIDRPSF is solvable if and only if there exists an $(A_\alpha(\cdot), B_\beta(\cdot))$-controlled invariant subspace-valued function $V_{\alpha\beta}(\cdot)$ such that

(i) $\sum_{i=0}^{t}\mathrm{Im}M_i(k-1) \subset V_{\alpha\beta}(k) \subset \bigcap_{i=0}^{s}\mathrm{Ker}D_i(k)$ for all $(\alpha, \beta) \in \boldsymbol{R}^p \times \boldsymbol{R}^q$ and $k \in \boldsymbol{Z}$,

(ii) $\bigcap_{\alpha,\beta} \boldsymbol{F}(A_\alpha(\cdot), B_\beta(\cdot); V_{\alpha\beta}(\cdot)) \neq \emptyset$,

where $\boldsymbol{F}(A_\alpha(\cdot), B_\beta(\cdot); V_{\alpha\beta}(\cdot)) := \{\, F_{\alpha\beta}(\cdot)\ (F_{\alpha\beta}(k) \in \boldsymbol{R}^{m\times n}) \ : \omega\text{-periodic} \mid$

$$(A_\alpha(k) + B_\beta(k)F_{\alpha\beta}(k)C_\gamma(k))V_{\alpha\beta}(k) \subset V_{\alpha\beta}(k+1).\} \ ∎$$

Theorem 3.7 Assume that $C_\gamma(k) = I_n$ $(k \in \boldsymbol{Z})$. If there exists a subspace valued function $V(\cdot) \in \boldsymbol{V}(A_\alpha(\cdot), B_\beta(\cdot); X(\cdot))$, where $X(k) := X$ for all $k \in \boldsymbol{Z}$ such that

$$\sum_{i=0}^{t}\mathrm{Im}M_i(k-1) \subset V(k) \subset \bigcap_{i=0}^{s}\mathrm{Ker}D_i(k) \quad \text{for all } k \in \boldsymbol{Z},$$

then the PIDRPSF is solvable. ∎

Corollary 3.8 Assume that $C_\gamma(k) = I_n$ $(k \in \boldsymbol{Z})$ and suppose that $V^*(\cdot)$ is a maximal element of $\boldsymbol{V}(A_\alpha(\cdot), B_\beta(\cdot); \bigcap_{i=0}^{s} \mathrm{Ker} D_i(\cdot))$. If

$$\sum_{i=0}^{t} \mathrm{Im} M_i(k-1) \subset V^*(k) \text{ for all } k \in \boldsymbol{Z},$$

then the PIDRPSF is solvable. ■

4 Conclusions

In this paper the notions of generalized controlled $(A_\alpha(\cdot),B_\beta(\cdot))$-invariance, generalized conditioned $(C_\gamma(\cdot),A_\alpha(\cdot))$-invariance and generalized $(A_\alpha(\cdot),B_\beta(\cdot),C_\gamma(\cdot))$-invariance were introduced for linear ω-periodic discrete-time systems and then their properties were studied. Further, the parameter insensitive disturbance-rejection problems with static output feedback and / or with state feedback for uncertain linear ω-periodic discrete-time systems were formulated, and then their solvability conditions were presented.

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