A Disturbance-Rejection Problem with Minimal Order Dynamic Compensator for Linear ω-Periodic Discrete-Time Systems

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A Disturbance-Rejection Problem with Minimal Order Dynamic Compensator For Linear ω-Periodic Discrete-Time Systems

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Abstract

In this paper necessary and sufficient conditions for a disturbance-rejection problem with dynamic compensator which was investigated by Grasselli and Longhi to be solvable is given for linear ω-periodic discrete time systems without assuming that the order of dynamic compensator is equal to that of system plant. Further, the minimal order of compensator which is necessary for the solution of the problem is also investigated.

Keywords : Disturbance-Rejection, Dynamic Compensator, ω-periodic Systems

1 Introduction

In the framework of the so-called geometric approach Grasselli and Longhi[1, 2] studied the notion of $(A(\cdot), B(\cdot))$-invariance and $(C(\cdot), A(\cdot))$-invariance for linear ω-periodic discrete time systems, and disturbance-rejection problem with state feedback has been studied. Further, they studied the notion of $(C(\cdot), A(\cdot), B(\cdot))$-pair for linear ω-periodic discrete time systems and by using the pair necessary and sufficient conditions for the disturbance-rejection problem with dynamic compensator (DRPDC) to be solvable were given under the assumption that the order (n) of system plant is equal to the order (μ) of dynamic compensator[3].

The objective of this paper is to give necessary and sufficient conditions for the DRPDC to be solvable without assuming $n = \mu$. Further, the minimal extension order which is necessary for the solution of the DRPDC is given.

The present investigation is organized as follows. Section 2 gives the notions of two invariances which were investigated by Grasselli and Longhi. In Section 3 the notion of $(C(\cdot), A(\cdot), B(\cdot))$-pair and its useful results are given. In Section 4 the main results are given. Finally, we make some concluding remarks in Section 5.

2 Preliminaries

The following notations are used throughout this investigation. $\boldsymbol{N}$:= the set of all natural numbers, $\boldsymbol{Z}$:= the set of all integers, $\boldsymbol{Z}^{\omega}_{k_0} := \{k_0+1, k_0+2, \ldots, k_0+\omega\}$ for $k_0 \in \boldsymbol{Z}$ and $\omega \in \boldsymbol{N}$, $\boldsymbol{R}^s$:= s dimensional Euclidean space and $\boldsymbol{R}^{p\times q}$:= the set of all $p \times q$ real matrices. For a linear map A from a vector space X to a vector space Y ImA := the image of A, Ker$A$:= the kernel of A, $A|_V$:= the restriction map of A on a subspace V of X and $A^{-1}\Omega := \{x \in X \mid Ax \in \Omega\}$ for a subspace Ω of Y. The notations $\oplus, \cong$ and ϕ indicate the direct sum, the isomorphic and the empty set, respectively. And $A(\cdot)$ is said to be ω-periodic for a given $\omega \in \boldsymbol{N}$ if $A(k) = A(k+\omega)$ for all $k \in \boldsymbol{Z}$. For a subspace-valued function $V(\cdot)$ $(V(k) \subset \boldsymbol{R}^s, k \in \boldsymbol{Z})$, $V(\cdot)$ is said to be ω-periodic for a given $\omega \in \boldsymbol{N}$ if $V(k) = V(k+\omega)$ for all $k \in \boldsymbol{Z}$.

Now, consider a linear ω-periodic discrete-time system S given by

$$S : \begin{cases} x(k+1) &= A(k)x(k) + B(k)u(k) \\ y(k) &= C(k)x(k), \ k \in \boldsymbol{Z} \end{cases}$$

where $x(k) \in X := \boldsymbol{R}^n$ is the state, $u(k) \in U := \boldsymbol{R}^m$ is the input, $y(k) \in Y := \boldsymbol{R}^p$ is the measurement output and $A(\cdot)$ $(A(k) \in \boldsymbol{R}^{n\times n})$, $B(\cdot)$ $(B(k) \in \boldsymbol{R}^{n\times m})$ and $C(\cdot)$ $(C(k) \in \boldsymbol{R}^{p\times n})$ are ω-periodic.

Definition 2.1 Let $V(\cdot)$ $(V(k) \subset X)$ be ω-periodic subspace-valued function.

(i) $V(\cdot)$ is said to be $(A(\cdot), B(\cdot))$-invariant if $A(k)V(k) \subset V(k+1) + \mathrm{Im}B(k)$ for all $k \in \boldsymbol{Z}$.

(ii) $V(\cdot)$ is said to be $(C(\cdot), A(\cdot))$-invariant if $A(k)(V(k) \cap \mathrm{Ker}C(k)) \subset V(k+1)$ for all $k \in \boldsymbol{Z}$. □

Lemma 2.1 [1, 2] Let $V(\cdot)$ $(V(k) \subset X)$ be ω-periodic subspace-valued function.

(i) $V(\cdot)$ is $(A(\cdot), B(\cdot))$-invariant if and only if there exists an ω-periodic feedback $F(\cdot)$ $(F(k) \in \boldsymbol{R}^{m\times n})$ such that

$$(A(k) + B(k)F(k))V(k) \subset V(k+1) \ \text{ for all } k \in \boldsymbol{Z}.$$

(ii) $V(\cdot)$ is $(C(\cdot), A(\cdot))$-invariant if and only if there exists an ω-periodic $G(\cdot)$ $(G(k) \in \boldsymbol{R}^{n\times p})$ such that

$$(A(k) + G(k)C(k))V(k) \subset V(k+1) \ \text{ for all } k \in \boldsymbol{Z}. \quad \square$$

For a given ω-periodic subspace-valued function $\Lambda(\cdot)$ $(\Lambda(k) \subset X)$, define the following two classes of ω-periodic subspace-valued functions.

$$\boldsymbol{V}(A(\cdot), B(\cdot); \Lambda(\cdot)) := \Big\{V(\cdot) \mid V(\cdot) \text{ is } (A(\cdot), B(\cdot))\text{-invariant and } V(k) \subset \Lambda(k) \text{ for all } k \in \boldsymbol{Z}\Big\}.$$

$$\boldsymbol{V}(\Lambda(\cdot); C(\cdot), A(\cdot)) := \Big\{V(\cdot) \mid V(\cdot) \text{ is } (C(\cdot), A(\cdot))\text{-invariant and } \Lambda(k) \subset V(k) \text{ for all } k \in \boldsymbol{Z}\Big\}.$$

Further, the following definitions are introduced.

$$\boldsymbol{F}(V(\cdot)) := \{F(\cdot) : \omega\text{-periodic} \mid (A(k) + B(k)F(k))V(k) \subset V(k+1) \text{ for all } k \in \boldsymbol{Z}\}.$$

$$\boldsymbol{G}(V(\cdot)) := \{G(\cdot) : \omega\text{-periodic} \mid (A(k) + G(k)C(k))V(k) \subset V(k+1) \text{ for all } k \in \boldsymbol{Z}\}.$$

Definition 2.2

(i) $V^*(\cdot)$ is said to be a maximal element of $\boldsymbol{V}(A(\cdot), B(\cdot); \Lambda(\cdot))$ if $V^*(\cdot) \in \boldsymbol{V}(A(\cdot), B(\cdot); \Lambda(\cdot))$ and $V(k) \subset V^*(k)$ $(k \in \boldsymbol{Z})$ for all $V(\cdot)$ of $\boldsymbol{V}(A(\cdot), B(\cdot); \Lambda(\cdot))$.

(ii) $V_*(\cdot)$ is said to be a minimal element of $\boldsymbol{V}(\Lambda(\cdot); C(\cdot), A(\cdot))$ if $V_*(\cdot) \in \boldsymbol{V}(\Lambda(\cdot); C(\cdot), A(\cdot))$ and $V_*(k) \subset V(k)$ $(k \in \boldsymbol{Z})$ for all $V(\cdot)$ of $\boldsymbol{V}(\Lambda(\cdot); C(\cdot), A(\cdot))$. □

Then, the following lemma has been given.

Lemma 2.2 [1, 2]

(i) $\boldsymbol{V}(A(\cdot), B(\cdot); \Lambda(\cdot))$ has a maximal element $V^*(\cdot)$ in the sence of Definition 2.2 and its computational algorithm is given as

$$V^0 := \Lambda(k) \quad (k \in \boldsymbol{Z}),$$

$$V^i(k) := \Lambda(k) \cap A^{-1}(k)(V^{i-1}(k+1) + \mathrm{Im}B(k)) \quad (k \in \boldsymbol{Z}), \quad i = 1, 2, \cdots.$$

(ii) $\boldsymbol{V}(\Lambda(\cdot); C(\cdot), A(\cdot))$ has a minimal element $V_*(\cdot)$ in the sence of Definition 2.2 and its computational algorithm is given as

$$V^0 := \mathcal{A}(k) \quad (k \in \boldsymbol{Z}),$$

$$V^i(k) := \mathcal{A}(k+1) + A(k)(V^{i-1}(k) \cap \mathrm{Ker}C(k)) \quad (k \in \boldsymbol{Z}), \quad i = 1, 2, \cdots. \quad \square$$

3 $(C(\cdot), A(\cdot), B(\cdot))$-pairs

In this section the properties of $(C(\cdot), A(\cdot), B(\cdot))$-pair which will be needed in the Section 4 are investigated. Consider the following linear ω-periodic discrete-time system Σ :

$$\begin{cases} x(k+1) &= A(k)x(k) + B(k)u(k) + M(k)\xi(k), \\ y(k) &= C(k)x(k), \\ z(k) &= D(k)x(k), \ k \in \boldsymbol{Z}, \end{cases}$$

where $x(k) \in X := \boldsymbol{R}^n$, $u(k) \in U := \boldsymbol{R}^m$, $\xi(k) \in Q := \boldsymbol{R}^s$, $y(k) \in Y := \boldsymbol{R}^p$ and $z(k) \in Z := \boldsymbol{R}^q$ are the state, the input, the disturbance, the measurement output and the controlled output, respectively.

Now, introduce a compensator $(K(\cdot), L(\cdot), M(\cdot), N(\cdot))$ defined in $W := \boldsymbol{R}^\mu$ of the following form :

$$\begin{cases} w(k+1) = N(k)w(k) + M(k)y(k), \\ u(k) = L(k)w(k) + K(k)y(k), \end{cases} \tag{1}$$

where $N(k) \in \boldsymbol{R}^{\mu \times \mu}$, $M(k) \in \boldsymbol{R}^{\mu \times p}$, $L(k) \in \boldsymbol{R}^{m \times \mu}$ and $K(k) \in \boldsymbol{R}^{m \times p}$.

If a compensator of the form (1) is applied to system Σ, the resulting extended system Σ^e on the extended state space $X^e := X \oplus W$ is easily seen to be

$$\begin{cases} \begin{bmatrix} x(k+1) \\ w(k+1) \end{bmatrix} = \begin{bmatrix} A(k) + B(k)K(k)C(k) & B(k)L(k) \\ M(k)C(k) & N(k) \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} + \begin{bmatrix} M(k) \\ 0 \end{bmatrix} \xi(k), \\ z(k) = \begin{bmatrix} D(k) & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}. \end{cases}$$

For the system Σ^e, define

$$x^e(k) := \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}, \quad A^e(k) := \begin{bmatrix} A(k) + B(k)K(k)C(k) & B(k)L(k) \\ M(k)C(k) & N(k) \end{bmatrix}, \quad M^e(k) := \begin{bmatrix} M(k) \\ 0 \end{bmatrix},$$

$D^e(k) := \begin{bmatrix} D(k) & 0 \end{bmatrix}$, $\Phi^e(k, k_0) := A^e(k-1)A^e(k-2)\cdots A^e(k_0)$ for $k > k_0$ $(k, k_0 \in \boldsymbol{Z})$ and

$\Phi^e(k, k) := I_{n+\mu}$ $(k \in \boldsymbol{Z})$, where $I_{n+\mu}$ is the identity matrix of dimension $(n+\mu)$.

Now, the difinition of $(C(\cdot), A(\cdot), B(\cdot))$-pair is given.

Definition 3.1 Let $V_1(\cdot), V_2(\cdot)$ $(V_1(k), V_2(k) \subset X)$ be ω-periodic subspace-valued functions. A pair $(V_1(\cdot), V_2(\cdot))$ is said to be a $(C(\cdot), A(\cdot), B(\cdot))$-pair if the following three conditions are satisfied.

(i) $V_1(\cdot)$ is a $(C(\cdot), A(\cdot))$-invariant.

(ii) $V_2(\cdot)$ is an $(A(\cdot), B(\cdot))$-invariant.

(iii) $V_1(k) \subset V_2(k)$ for all $k \in \boldsymbol{Z}$. □

For an extended system Σ^e, we give the following definitions.

Definition 3.2 Let $V^e(\cdot)$ ($V^e(k) \subset X^e$) be an ω-periodic subspace valued function. $V^e(\cdot)$ is said to be an $A^e(\cdot)$-invariant if $A^e(k)V^e(k) \subset V^e(k+1)$ for all $k \in \boldsymbol{Z}$. □

Definition 3.3 Let $V^e(\cdot)$ ($V^e(k) \subset X^e$) be an ω-periodic subspace valued function. Then, the following two subspace-valued functions $V_s(\cdot)$ and $V_p(\cdot)$ are defined:

$$V_s(k) := \left\{ x \in X \,\middle|\, \begin{bmatrix} x \\ 0 \end{bmatrix} \in V^e(k) \right\} \text{ and}$$

$$\begin{aligned} V_p(k) &:= \left\{ x \in X \,\middle|\, \begin{bmatrix} x \\ w \end{bmatrix} \in V^e(k) \text{ for some } w \in W \right\} \\ &= P_X(V^e(k)), \end{aligned}$$

where P_X is the projection map from X^e onto X along W. □

The following lemma was given by Grasselli and Longhi.

Lemma 3.1 [3] If $V^e(\cdot)$ is an $A^e(\cdot)$-invariant ω-periodic subspace-valued function, then the pair $(V_s(\cdot), V_p(\cdot))$ is a $(C(\cdot), A(\cdot), B(\cdot))$-pair. □

The following two lemmas are used to prove Lemma 3.4.

Lemma 3.2 If a pair $(V_1(\cdot), V_2(\cdot))$ is a $(C(\cdot),\ A(\cdot),\ B(\cdot))$-pair, then there exist $F(\cdot) \in \boldsymbol{F}(V_2(\cdot))$, $G(\cdot) \in \boldsymbol{G}(V_1(\cdot))$, $G_0(k) \in \boldsymbol{R}^{n\times p}$, $F_0(k) \in \boldsymbol{R}^{m\times n}$ and $K(k) \in \boldsymbol{R}^{m\times p}$ such that

$F(k) = K(k)C(k) + F_0(k)$, $G(k) = B(k)K(k) + G_0(k)$, $\mathrm{Ker}F_0(k) \supset V_1(k)$ and $\mathrm{Im}G_0(k) \subset V_2(k+1)$ for all $k \in \boldsymbol{Z}$.

(Proof) Suppose that a pair $(V_1(\cdot), V_2(\cdot))$ is a $(C(\cdot),\ A(\cdot),\ B(\cdot))$-pair.

Claim 1: There exists a $G(\cdot) \in \boldsymbol{G}(V_1(\cdot))$ such that $\mathrm{Im}G(k) \subset V_2(k+1) + \mathrm{Im}B(k)$.
To prove Claim 1, choose an arbitrary element $G_1(\cdot) \in \boldsymbol{G}(V_1(\cdot))$. Define a subspace $\Omega(k)$ such that $Y = C(k)V_1(k) \oplus \Omega(k)$. Further, define an ω-periodic linear map $G(k)$ from Y to X such that

$$G(k) := \begin{cases} G_1(k) & \text{on } C(k)V_1(k), \\ 0 & \text{on } \Omega(k). \end{cases}$$

Then, it can be easily obtained that $G(\cdot) \in \boldsymbol{G}(V_1(\cdot))$ and $\mathrm{Im}G(k) \subset V_2(k+1) + \mathrm{Im}B(k)$.

Claim 2: There exists a $K(k) \in \boldsymbol{R}^{m\times p}$ and $G_0(k) \in \boldsymbol{R}^{n\times p}$ such that $G(k) = B(k)K(k) + G_0(k)$ and $\mathrm{Im}G_0(k) \subset V_2(k+1)$.
To prove Claim 2 let $\{y_1, \cdots, y_p\}$ be a basis of Y. Then, it follows from Claim 1 that there exist an $x_i(k) \in V_2(k+1)$ and a $u_i(k) \in U$ such that $G(k)y_i = x_i(k) + B(k)u_i(k)$. Define linear maps $K(k)$ from Y to U and $G_0(k)$ from Y to X such that $K(k)y_i := u_i(k)$ $(i = 1, \cdots, p)$ and $G_0(k)y_i := x_i(k)$ $(i = 1, \cdots, p)$. Then, we have $G(k) = B(k)K(k) + G_0(k)$ and $\mathrm{Im}G_0(k) \subset V_2(k+1)$ which proves Claim 2.

Now, it can be easily obtained that $V_2(\cdot)$ is $((A(\cdot) + G(\cdot)C(\cdot)), B(\cdot))$-invariant.

Claim 3: There exists an $F_0(k)$ from X to U such that $\mathrm{Ker}F_0(k) \supset V_1(k)$ and $(A(k)+G(k)C(k)+B(k)F_0(k))V_2(k) \subset V_2(k+1)$.

To prove Claim 3 define a subspace $\Lambda(k)$ such that $X = \underbrace{V_1(k) \oplus \Omega(k)}_{V_2(k)} \oplus \Lambda(k)$. Since $V_2(\cdot)$ is $((A(\cdot)+G(\cdot)C(\cdot)), B(\cdot))$-invariant, there exists a linear map $F(k)$ from X to U such that $(A(k)+G(k)C(k)+B(k)F(k))V_2(k) \subset V_2(k+1)$. Further, define a linear map $F_0(k)$ from X to U such that

$$F_0(k) := \begin{cases} F(k) & \text{on } \Omega(k) \oplus \Lambda(k), \\ 0 & \text{on } V_1(k). \end{cases}$$

Then, $F_0(k)$ satisfies $\mathrm{Ker} F_0(k) \supset V_1(k)$ and $(A(k) + G(k)C(k) + B(k)F_0(k))V_2(k) \subset V_2(k+1)$ which proves Claim 3. Finally, define $F(k) := K(k)C(k) + F_0(k)$. Then, it can be easily obtained that $F(\cdot) \in \boldsymbol{F}(V_2(\cdot))$. This completes the proof. □

Define a class of pairs of maps as follows.

$\boldsymbol{P}(V_1(\cdot), V_2(\cdot)) := \{(F(\cdot), G(\cdot)) \in \boldsymbol{F}(V_2(\cdot)) \times \boldsymbol{G}(V_1(\cdot)) \mid$ there exists a $K(k) \in \boldsymbol{R}^{m\times p}$ such that $\mathrm{Ker}(F(k) - K(k)C(k)) \supset V_1(k)$ and $\mathrm{Im}(G(k) - B(k)K(k)) \subset V_2(k+1)$ for all $k \in \boldsymbol{Z}.\}$

In fact, if $(V_1(\cdot), V_2(\cdot))$ is a $(C(\cdot), A(\cdot), B(\cdot))$-pair, then $\boldsymbol{P}(V_1(\cdot), V_2(\cdot)) \neq \phi$.

Lemma 3.3 Suppose that V_1 and V_2 are subspaces of X satisfying $V_1 \subset V_2$ and $W \cong \boldsymbol{R}^{\mathrm{dim}V_2 - \mathrm{dim}V_1}$. Then, a linear map R from V_2 to W can be defined such that $\mathrm{Ker}R = V_1$.
(Proof) Since the proof follows easily, it was omitted. □

The following lemma plays an important role to prove the main theorem in the next section.

Lemma 3.4 If a pair $(V_1(\cdot), V_2(\cdot))$ is a $(C(\cdot), A(\cdot), B(\cdot))$-pair, then there exist a compensator $(K(\cdot), L(\cdot), M(\cdot), N(\cdot))$ on W and a $V^e(\cdot)$ such that

$$\mathrm{dim}W = \mathrm{dim}(\sum_{k\in \boldsymbol{Z}^{\omega}_{k_0}} V_2(k)) - \mathrm{dim}(\sum_{k\in \boldsymbol{Z}^{\omega}_{k_0}} V_1(k)),\ V_1(k) = V_s(k),\ V_2(k) = V_p(k) \text{ and } V^e(\cdot) \text{ is } A^e(\cdot)\text{-invariant.}$$

(Proof) Suppose that $(V_1(\cdot), V_2(\cdot))$ is a $(C(\cdot), A(\cdot), B(\cdot))$-pair.
Define $W := \boldsymbol{R}^{\mu}$, where $\mu := \mathrm{dim}(\sum_{k\in \boldsymbol{Z}^{\omega}_{k_0}} V_2(k)) - \mathrm{dim}(\sum_{k\in \boldsymbol{Z}^{\omega}_{k_0}} V_1(k))$. From Lemma 3.3 it can be defined that a linear map R from $(\sum_{k\in \boldsymbol{Z}^{\omega}_{k_0}} V_2(k))$ to W such that

$$\mathrm{Ker}R = (\sum_{k\in \boldsymbol{Z}^{\omega}_{k_0}} V_1(k)).$$

Further, define

$$V^e(k) := \left\{ \begin{bmatrix} x(k) \\ Rx(k) \end{bmatrix} | x(k) \in V_2(k) \right\} \subset X \oplus W.$$

Then, it can be easily obtained $V_s(k) = V_1(k)$ and $V_p(k) = V_2(k)$ for all $k \in \boldsymbol{Z}$. Now, it follows from Lemma 3.2 that there exist $F(\cdot) \in \boldsymbol{F}(V_2(\cdot))$, $G(\cdot) \in \boldsymbol{G}(V_1(\cdot))$, $G_0(k) \in \boldsymbol{R}^{n\times p}$, $F_0(k) \in \boldsymbol{R}^{m\times n}$ and $K(k) \in \boldsymbol{R}^{m\times p}$ such that

$$F(k) = K(k)C(k) + F_0(k),\ G(k) = B(k)K(k) + G_0(k),\ \mathrm{Ker}F_0(k) \supset V_1(k) \text{ and } \mathrm{Im}G_0(k) \subset V_2(k) \text{ for all } k \in \boldsymbol{Z}.$$

Then, there exists a linear map $L(k)$ from W to X satisfying

$$L(k)R = F_0(k)|_{(\sum_{k\in \boldsymbol{Z}^{\omega}_{k_0}} V_2(k))}.$$

Further, there exists a linear map $N(k)$ from W to W satisfying

$$N(k)R|_{V_2(k)} = R(A(k) + B(k)F(k) + G_0(k)C(k))|_{V_2(k)}.$$

Define $M(k) := -RG_0(k)$. Then, it can be proved that $V^e(\cdot)$ is $A^e(\cdot)$-invariant. This completes the proof. □

4 Disturbance-Rejection

In this section a disturbance-rejection problem with dynamic compensator is formulated and then necessary and sufficient conditions for the problem to be solvable are given without assuming that the order of dynamic compensator is equal to that of system plant.

Consider an extended system Σ^e in the Section 3. The disturbance-rejection problem can be stated as follows. Given ω-periodic matrix-valued functions $A(\cdot)$, $B(\cdot)$, $M(\cdot)$, $C(\cdot)$ and $D(\cdot)$ of the system Σ, find (if possible) a compensator ($K(\cdot)$, $L(\cdot)$, $M(\cdot)$, $N(\cdot)$) such that

$$D^e(k) \sum_{h=k_0}^{k-1} \Phi^e(k, h+1) M^e(h) \xi(h) = 0$$

for all $\xi(\cdot)$ and all $k \in \boldsymbol{Z}$, where k_0 is an initial time.

Noticing that a subspace generated by the disturbances $\xi(\cdot)$ is $\sum_{h=k_0}^{k-1} \Phi^e(k, h+1)\mathrm{Im}M^e(h)$, disturbance rejection problem with dynamic compensator can be formulated as follows.

Disturbance-Rejection Problem with Dynamic Compensator (DRPDC) Given ω-periodic matrix-valued functions $A(\cdot)$, $B(\cdot)$, $M(\cdot)$, $C(\cdot)$ and $D(\cdot)$ of the system Σ, find (if possible) a compensator ($K(\cdot)$, $L(\cdot)$, $M(\cdot)$, $N(\cdot)$) such that

$$\sum_{h=k_0}^{k-1} \Phi^e(k, h+1)\mathrm{Im}M^e(h) \subset \mathrm{Ker}D^e(k)$$

for all $k \in \boldsymbol{Z}$. □

The following theorem is the main result.

Theorem 4.1 The DRPDC is solvable if and only if there exists a $(C(\cdot), A(\cdot), B(\cdot))$-pair $(V_1(\cdot), V_2(\cdot))$ such that

$$\mathrm{Im}M(k-1) \subset V_1(k) \subset V_2(k) \subset \mathrm{Ker}D(k) \quad \text{for all } k \in \boldsymbol{Z}.$$

In this case, the minimal extension order which is necessary for the solution of the DRPDC is given by

$$\min\{\dim(\sum_{k\in \boldsymbol{Z}^{\omega}_{k_0}} V_2(k)) - \dim(\sum_{k\in \boldsymbol{Z}^{\omega}_{k_0}} V_1(k)) \mid (V_1(\cdot), V_2(\cdot)) \text{ is a } (C(\cdot), A(\cdot), B(\cdot)) - \text{pair and}$$

$$\mathrm{Im}M(k-1) \subset V_1(k) \subset V_2(k) \subset \mathrm{Ker}D(k) \text{ for all } k \in \boldsymbol{Z}\}.$$

(Proof) (Necessity) Suppose that the DRPDC is solvable. Then, there exists a compensator ($K(\cdot)$, $L(\cdot)$, $M(\cdot)$, $N(\cdot)$) such that

$$\sum_{h=k_0}^{k-1} \Phi^e(k, h+1)\mathrm{Im}M^e(h) \subset \mathrm{Ker}D^e(k) \quad \text{for all } k \in \boldsymbol{Z}.$$

Define, a subspace

$$V^e(k) := \sum_{h=k_0}^{k-1} \Phi^e(k,h+1)\mathrm{Im}M^e(h).$$

Then, since $V^e(\cdot)$ is ω-periodic and $A^e(\cdot)$-invariant, it follows from Lemma 3.1 that $(V_s(\cdot), V_p(\cdot))$ is a $(C(\cdot), A(\cdot), B(\cdot))$-pair. Futher, it can be easily obtained that

$$\mathrm{Im}M(k-1) \subset V_s(k) \text{ and } V_p(k) \subset \mathrm{Ker}D(k).$$

Thus, we have

$$\mathrm{Im}M(k-1) \subset V_s(k) \subset V_p(k) \subset \mathrm{Ker}D(k) \text{ for all } k \in \boldsymbol{Z}.$$

(Sufficiency) Suppose that there exists a $(C(\cdot), A(\cdot), B(\cdot))$-pair $(V_1(\cdot), V_2(\cdot))$ such that

$$\mathrm{Im}M(k-1) \subset V_1(k) \subset V_2(k) \subset \mathrm{Ker}D(k) \quad \text{for all } k \in \boldsymbol{Z}.$$

Then, it follows from Lemma 3.4 that there exist a compensator $(K(\cdot), L(\cdot), M(\cdot), N(\cdot))$ on W and a $V^e(\cdot)$ such that

$$\dim W = \dim(\sum_{k \in \boldsymbol{Z}^{\omega}_{k_0}} V_2(k)) - \dim(\sum_{k \in \boldsymbol{Z}^{\omega}_{k_0}} V_1(k)), \ V_1(k) = V_s(k), \ V_2(k) = V_p(k) \text{ and}$$

$$V^e(\cdot) \text{ is } A^e(\cdot) - \text{invariant, where } V^e(k) := \left\{ \begin{bmatrix} x(k) \\ Rx(k) \end{bmatrix} \mid x(k) \in V_2(k) \right\}.$$

Then, we have

$$\mathrm{Im}M^e(k-1) \subset V^e(k) \subset \mathrm{Ker}D^e(k).$$

Thus,

$$\sum_{h=k_0}^{k-1} \Phi^e(k,h+1)\mathrm{Im}M^e(h) \subset \sum_{h=k_0}^{k-1} V^e(k) = V^e(k) \subset \mathrm{Ker}D^e(k) \quad \text{for all } k \in \boldsymbol{Z}.$$

which implies the DRPDC is solvable.

The minimal extension order of compensator which is necessary for the solution of the problem follows from the proof of sufficiency. □

Corollary 4.1 The DRPDC is solvable if and only if $V_*(k) \subset V^*(k)$ for all $k \in \boldsymbol{Z}$, where $V^*(\cdot) := \max \boldsymbol{V}(A(\cdot), B(\cdot); \mathrm{Ker}D(\cdot))$ and $V_*(\cdot) := \min \boldsymbol{V}(\mathrm{Im}M(\cdot - 1); C(\cdot), A(\cdot))$.

(Proof) The proof follows from Theorem 4.1. □

5 Concluding Remarks

In this paper necessary and sufficient conditions for a disturbance-rejection problem with dynamic compensator to be solvable were given for linear ω-periodic discrete-time systems without assuming that the order of dynamic compensator is equal to that of system plant. Further, the minimal extension order of compensator which is necessary for the solution of the problem was given. The results in this paper are extensions of the results of Schumacher[4] to ω-periodic version.

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