A CHARACTERIZATION OF PARADOXES IN DISTRIBUTED OPTIMIZATION OF PERFORMANCE FOR MULTIPLICATED SYSTEMS

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The present work was done during her visit to the University of Tsukuba.

A Characterization of Paradoxes in Distributed Optimization of Performance for Multiplicated Systems

Hisao Kameda* and Odile Pourtallier†

Abstract

We show the existence and the characterization of paradoxical cases, in load balancing among multiplicated nodes (hosts) of a system, where adding a communication capacity to the system can lead to large degradation of system performance, in an intermediately distributed performance optimization. In these cases such paradoxical degradation of performance occurs neither in the completely centralized optimization nor in the completely distributed optimization. The degradation reduces and finally disappears as the optimization decision becomes more and more distributed. We study the model of a system which consists of parallel identical nodes with identical arrivals of jobs of different classes while the values of the service time, arrival rate, and communication time parameters for jobs of each class are distinct within each node. Preliminary numerical studies suggest that such paradoxes appear most strongly in symmetrical models. We characterize conditions under which such paradoxical behaviors appear. It is notable that we can find paradoxes that may bring unlimitedly large degradation of performance in such a common system.

keywords Distributed decision, Braess paradox, Nash equilibrium, Wardrop equilibrium, performance optimization, parallel queues, load balancing.

1 Introduction

We can consider systems that consist of a finite number of facilities and arriving threads or flows of infinitely many customers to be served by the facilities. For example, distributed computer systems have continuing arrivals of infinitely many jobs to be processed by computers, communication networks have flows of infinitely many packets or calls to pass through communication links, and transportation flow networks have incoming threads of infinitely many vehicles to drive through roads, *etc.* We may have various optimum issues, depending on the degree of the distribution of decisions. Among them, we have three typical optima corresponding to three typical decision schemes:

(A) [Completely centralized decision]: The system is run by a single decision maker that optimizes the total cost over all users or the mean response time of the entire system over all jobs, packets, or vehicles, as a single performance measure. This optimized situation is called the system optimum, overall optimum, cooperative optimum or social optimum.

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All users, jobs, packets, or vehicles are unified into one group which has only one decision maker that seeks a single performance objective. We call it the *overall optimum* here.

(B) [Completely distributed decision]: Each of infinitely many individuals, users, jobs, *etc.*, optimizes its own cost or the expected response time for itself independently of others. In this optimized situation each of infinitely many individuals cannot receive any further benefit by changing its own decision. It is further assumed that the decision of a single individual has a negligible impact on the performance of other individuals. This optimized situation is called the individual optimum, Wardrop equilibrium, or user optimum (by some people). We call it the *individual optimum* or *Wardrop equilibrium* here.

(C) [Intermediately distributed decision]: Infinitely many users, jobs, packets, or vehicles are classified into a finite number ($N(>1)$) of groups, each of which has its own decision maker and is regarded as one player, user, or class. Each decision maker optimizes non-cooperatively its own cost or the expected response time over only the jobs of the class. The decision of a single decision maker of a class has a nonnegligible impact on the performance of other classes. In this optimized situation each of a finite number of users, classes, or players cannot receive any further benefit by changing its decision. This optimized situation is called the class optimum, Nash non-cooperative equilibrium, or user optimum (by some other people). We call it the *class optimum* or *Nash equilibrium* here. We may have different levels in the intermediately distributed optimization.

Note that (C) is reduced to (A) when the number of players reduces to 1 ($N=1$) and approaches (B) when the number of players becomes infinitely many ($N \to \infty$) [6].

Intuitively, we can think that the total processing capacity of a system will increase when the capacity of a part of the system increases, and so we expect improvements in performance objectives accordingly in that case. The famous Braess paradox tells us that this is not always the case; *i.e.*, increased capacity of a part of the system may sometimes lead to the degradation in the benefits of all users in an individual optimization or Wardrop equilibrium [2, 3, 4, 6]. We can expect that, in the class optimum (*i.e.*, Nash equilibrium) a similar type of paradox occurs (with large N), whenever it occurs for the Wardrop equilibrium ($N \to \infty$). Indeed, Korilis et al. found examples wherein the Braess-like paradox appears in a Nash equilibrium where all user classes are identical in the same topology for which the original Braess paradox (for the Wardrop equilibrium) was in fact obtained [14, 15].

As it is known that the Nash equilibrium converges to the Wardrop equilibrium as the number of users becomes large [6], it is natural to expect the same type of paradox in the Nash equilibrium context (for a large number of players), whenever it occurs for the Wardrop equilibrium, although it never occurs in the overall optimum where the total cost is minimized.

Kameda et al. [9] have obtained, however, numerical examples where a paradox similar to Braess's appears in the Nash equilibrium but does not occur in the Wardrop equilibrium in the same environment. These cases look quite strange if we note that such a paradox should never occur in the overall optimum and if we regard the Nash equilibrium as an intermediate between the overall optimum and the Wardrop equilibrium. In particular, the numerical examples show that the increased capacity of a part of a system would degrade the benefits of all classes up to a few 10 percent, in a class optimum (Nash equilibrium) whereas it should not degrade the benefits of all classes at the same time in a Wardrop equilibrium in the same environment. In the background of this work, it has been observed that increased capacity of a part of a system may lead to somewhat awkward behavior in terms of a system-wide measure, in a model of distributed computer system [9, 10, 21]. The methods and algorithms for obtaining the optima and the equilibria are described in

[10, 12, 13, 16, 20].

In this paper, we present an analytic study of a model of static load balancing among identical nodes each of which has an identical arrival. The results look quite counter-intuitive and show that the ratio of the performance degradation in the paradoxical cases can be *unlimitedly large*. We particularly investigate the model of symmetrical nodes here since *in our preliminary numerical investigation on asymmetrical node models such counter-intuitive phenomena appeared most strongly in symmetrical models* [7], which itself looks quite counter-intuitive to us.

In the model studied in this paper, each node (or processor) may have a communication means for forwarding jobs to be processed by other nodes. It is intuitively clear that in the overall optimum, no forwarding of jobs should occur. In the individual optimum, no forwarding of jobs occurs also. In the class optimum, no forwarding of jobs occurs only for some parameter setting.

For some other parameter setting, however, mutual forwarding does occur in the class optimum, and the mean response time for each class can be unlimitedly many times larger than in the case of no mutual forwarding. The ratio of performance degradation decreases and finally disappears as the number of classes increases unlimitedly. These situations look quite paradoxical and surprising to us, although we know the existence of the prisoners' dilemma and although it has been already shown that Nash equilibria of games even with smooth payoff functions are generally Pareto-inefficient [5].

2 The Model and Assumptions

We consider a system with m nodes (host computers or processors) connected with a communication means. The jobs that arrive at each node i, $i = 1, 2, \cdots, m$, are classified into n types k, $k = 1, 2, \cdots, n$. Consequently, we have mn different job classes R_{ik}. Each of class R_{ik} is distinguished by the node i at which its jobs arrive and by the type k of the jobs. We call such a class *local class*, or simply *class*.

We assume that each node has identical arrivals and identical processing capacity. That is, the system has multiplicated nodes that are identical with one another. Jobs of type k arrive at each node with node-independent rate ϕ_k. We denote the total arrival rate to the node by ϕ $(= \sum_k \phi_k)$, and we have the time scale whereby $\phi = 1$ and $\sum_k \phi_k = 1$.

We also consider what we call *global class* J_k that consists in the collection of local class R_{ik}, *i.e.*, $J_k = \bigcup_i R_{ik}$. J_k thus consists of all jobs of type k. Whereas, for local class R_{ik}, all the jobs arrive at the same node i, the arrivals of the jobs of class J_k are equally distributed over all nodes i.

The average processing (service) time (without queueing delays) of a type k job at any node is $1/\mu_k$ and is, in particular, node-independent. We denote ϕ_k/μ_k by ρ_k and $\rho = \sum_k \rho_k$.

Out of type k jobs arriving at node i, the ratio x_{ijk} of jobs is forwarded upon arrival through the communication means to another node j $(\neq i)$ to be processed there. The remaining ratio $x_{iik} = 1 - \sum_{j(\neq i)} x_{ijk}$ is processed at node i. Thus $\sum_j x_{ijk} = 1$. That is, the rate $\phi_k x_{ijk}$ of type k jobs that arrive at node i is forwarded through the communication means to node j, while the rate $\phi_k x_{iik}$ of class R_{ik} jobs is processed at the arrival node i. We have $0 \leq x_{ijk} \leq 1$, for all i, j, k. Within these constraints, a set of values of $\boldsymbol{x}_{ik}$ $(i = 1, 2, \cdots, m, k = 1, 2, \cdots, n)$ are chosen to achieve optimization.

We thus define a class R_{ik} strategy as the m vector

$$\boldsymbol{x}_{ik} = (x_{i1k}, \cdots, x_{imk}).$$

We define a global-class J_k strategy as the mm vector

$$\boldsymbol{x}_k = (\boldsymbol{x}_{1k}, \boldsymbol{x}_{2k}, \cdots, \boldsymbol{x}_{mk}).$$

We will also denote $\boldsymbol{x}$ the vector of strategies concerning all job classes, called strategy profile, *i.e.,* the vector of length mmn,

$$\boldsymbol{x} = (\boldsymbol{x}_{11}, \boldsymbol{x}_{12}, \cdots, \boldsymbol{x}_{1n}, \boldsymbol{x}_{21}, \cdots, \boldsymbol{x}_{2n}, \cdots, \boldsymbol{x}_{m1}, \cdots, \boldsymbol{x}_{mn}), \quad \text{or} \quad \boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n).$$

For a strategy profile $\boldsymbol{x}$, the load β_i on node i is

$$\beta_i = \beta_i(\boldsymbol{x}) = \sum_{j,k} \rho_k x_{jik}. \tag{1}$$

The contribution $\beta_i^{(k)}$ on the load of node i by the global class k jobs is

$$\beta_i^{(k)} = \beta_i^{(k)}(\boldsymbol{x}) = \sum_{j} \rho_k x_{jik}, \tag{2}$$

and clearly $\beta_i = \beta_i^{(1)} + \beta_i^{(2)} + \cdots + \beta_i^{(n)}$.

We denote the set of $\boldsymbol{x}$'s that satisfy the constraints (*i.e.,* $\sum_l x_{ilk} = 1, x_{ijk} \geq 0$, for all i, j, k) by $\boldsymbol{C}$. Note that $\boldsymbol{C}$ is a compact set.

We have the following assumptions:

Assumption A1 *We assume that the expected processing (including queueing) time of a type k job that is processed at node i (or the cost function at node i), is a strictly increasing, strictly convex and continuously differentiable function of β_i, denoted by $\mu_k^{-1}D(\beta_i)$ for all i, k.*

Assumption A2 *We assume that the mean communication delay (including queueing delay) or the cost for forwarding type k jobs arriving at node i to node j ($i \neq j$), denoted by $G_{ijk}(\boldsymbol{x})$, is a positive, nondecreasing, convex and continuously differentiable function of $\boldsymbol{x}$. We assume that $G_{iik}(\boldsymbol{x}) = 0$.*

Example 1 We may consider the following simple functions for mean node and communication delays. For the mean node delay:

$$1/\mu_k D(\beta_i) = \frac{1/\mu_k}{1 - \beta_i} \text{ for } \beta_i < 1, \text{ otherwise it is infinite.} \tag{3}$$

For the mean communication delay:

$$G_{ijk}(\boldsymbol{x}) = t. \tag{4}$$

Equation (3) means that we have a simple assumption of the external time-invariant Poisson arrival for each class, and the mean service time (without queueing delays) for each type k jobs is μ_k^{-1} at each node i. The service discipline is processor-sharing or preemptive-resume last-come first-served. When $\mu_k = \mu$ for all k and when no forwarding of jobs occurs, the mean communication delay is, simply, $1/(\mu - 1)$.

By Equation (4) we assume that one communication line is provided separately for sending jobs from one node to another. The line (ij) is used for forwarding a job that arrives at node i to node j ($\neq i$). The expected communication time of a job arriving at node i and being processed at node j ($\neq i$) is expressed simply as t, *i.e.,* independent of the traffic and of the job class, with no queueing delay.

We refer to the length of time between the instant when a job arrives at a node and the instant when it leaves one of the nodes after all processing and communication, if any, are over as *the response time* for the job. The expected response time of a class R_{ik} job that arrives at node i, $T_{ik}(\boldsymbol{x})$, is expressed as,

$$T_{ik}(\boldsymbol{x}) = \sum_j x_{ijk} T_{ijk}(\boldsymbol{x}), \tag{5}$$

where

$$\begin{aligned} T_{iik}(\boldsymbol{x}) &= \mu_k^{-1} D(\beta_i(\boldsymbol{x})), \quad \text{and} && (6) \\ T_{ijk}(\boldsymbol{x}) &= \mu_k^{-1} D(\beta_j(\boldsymbol{x})) + G_{ijk}(\boldsymbol{x}), \text{ for } j \neq i. && (7) \end{aligned}$$

The expected response time of a global-class J_k jobs is

$$T_k(\boldsymbol{x}) = \frac{1}{m} \sum_i T_{ik}(\boldsymbol{x}). \tag{8}$$

The overall expected response time of a job that arrives at the system is

$$\begin{aligned} T(\boldsymbol{x}) &= \sum_k \phi_k T_k(\boldsymbol{x}) = \frac{1}{m} \sum_{i,k} \phi_k T_{ik}(\boldsymbol{x}), \\ &= \frac{1}{m} \{ \sum_i \beta_i(\boldsymbol{x}) D(\beta_i(\boldsymbol{x})) + \sum_{i,j(\neq i),k} \phi_k x_{ijk} G_{ijk}(\boldsymbol{x}) \}. && (9) \end{aligned}$$

Remark 2. 1 *Note that as a consequence of Assumption (A1) and Assumption (A2), the functions $T(.)$, $T_{ik}(.)$ and T_k are strictly convex and differentiable with respect to the strategy profile $\boldsymbol{x}$.*

We analyze several decision schemes.

(A) In the *completely centralized decision* scheme, the forwarding decision over all jobs is taken by a single decision maker. His strategy is therefore the choice of the optimal *mmn* vector $\bar{\boldsymbol{x}}$, with components $\bar{x}_{ijk}$. This optimized situation is the *overall optimum.*

(B) At the opposite, *i.e.*, in the *completely distributed decision* scheme, we consider that each single job chooses the node to be processed. Thus the system has infinitely many decision makers. The resulting optimal ratio of jobs of class R_{ik} that choose the node j to be processed will be $\hat{x}_{ijk}$. This optimized situation is the *individual optimum.* We denote the individually optimal strategy profile by $\hat{\boldsymbol{x}}$.

(C-I) In one intermediately distributed decision scheme between (A) and (B), each class R_{ik} has its own decision maker (ik). The amount of forwarding for class R_{ik} jobs is chosen by the corresponding decision maker (ik). The optimal strategy for decision maker (ik), or equivalently class job R_{ik}, is denoted by the m vector

$$\tilde{\boldsymbol{x}}_{ik} = (\tilde{x}_{i1k}, \tilde{x}_{i2k}, \cdots, \tilde{x}_{imk}),$$

and an optimal strategy profile, that we will denote $\tilde{\boldsymbol{x}}$, is the collection of strategies $\tilde{\boldsymbol{x}}_{ik}$. We call this scheme the *intermediately distributed decision I*, and this optimized situation the *class optimum.*

(C-II) In another possible intermediately distributed decision scheme between (A) and (B), jobs of classes R_{ik} for all i are united into one global class J_k that has a single decision maker (k). Each decision maker (k) of class J_k chooses the amount of job forwarding for the m classes, $R_{1k}, R_{2k}, \ldots, R_{mk}$. The optimal strategy for decision maker k is consequently an mm vector

$$\check{\boldsymbol{x}}_k = (\check{\boldsymbol{x}}_{1k}, \check{\boldsymbol{x}}_{2k}, \cdots, \check{\boldsymbol{x}}_{mk}).$$

An optimal strategy profile is an mmn vector $\check{\boldsymbol{x}} = (\check{\boldsymbol{x}}_1, \check{\boldsymbol{x}}_2, \ldots, \check{\boldsymbol{x}}_n)$. We call this scheme the *intermediately distributed decision II*, and this optimized situation the *global-class optimum.*

3 The Results

(A) [Completely centralized decision: Overall optimization] The overall optimum is given by such $\bar{\boldsymbol{x}}$ as satisfies the following,

$$T(\bar{\boldsymbol{x}}) = \min T(\boldsymbol{x}) \quad \text{with respect to} \quad \boldsymbol{x} \in \boldsymbol{C}. \tag{10}$$

We define $\boldsymbol{x}_{-(iik)}$ to be an $m(m-1)n$ vector such that all elements x_{iik}, for all i, k, are excluded from the mmn vector $\boldsymbol{x}$ whereas all its elements are the same as the remaining $m(m-1)n$ elememts of $\boldsymbol{x}$.

Solution: *The solution* $\bar{\boldsymbol{x}}$ *is unique and given as follows:*

$$\bar{\boldsymbol{x}}_{-(iik)} = \boldsymbol{0}, \quad i.e., \ x_{ijk} = 0, \ \text{and} \ x_{iik} = 1, \ \text{for all} \ i, j (\neq i), k.$$

The mean response time is

$$T_k(\bar{\boldsymbol{x}}) = T_{ik}(\bar{\boldsymbol{x}}) = \mu_k^{-1} D(\rho), \ \text{for all} \ i, k, \quad T(\bar{\boldsymbol{x}}) = \rho D(\rho).$$

Proof: This solution and its uniqueness are clear from the strict convexity assumption on D and the fact that G_{ijk} is positive and nondecreasing. □

(B) [Completely distributed decision: Individual optimization] The individual optimum (*i.e.*, Wardrop equilibrium) is given by such $\hat{\boldsymbol{x}}$ as satisfies the following for all i, k,

$$T_{ik}(\hat{\boldsymbol{x}}) = \min_j \{T_{ijk}(\hat{\boldsymbol{x}})\} \quad \text{and} \quad \hat{\boldsymbol{x}} \in \boldsymbol{C}. \tag{11}$$

Solution: *The solution* $\hat{\boldsymbol{x}}$ *is unique and given as follows:*

$$\hat{\boldsymbol{x}}_{-(iik)} = \boldsymbol{0}, \quad i.e., \ x_{ijk} = 0, \ \text{and} \ x_{iik} = 1, \ \text{for all} \ i, j (\neq i), k.$$

The mean response time is

$$T_k(\hat{\boldsymbol{x}}) = T_{ik}(\hat{\boldsymbol{x}}) = \mu_k^{-1} D(\rho), \ \text{for all} \ i, k, \quad T(\bar{\boldsymbol{x}}) = \rho D(\rho).$$

Proof: This can be easily seen in the following way. The solution $\hat{\boldsymbol{x}}$ for (11) is characterized as follows: For all i, k we have

$$T_{ijk}(\hat{\boldsymbol{x}}) = \hat{\alpha}_{ik}, \quad \hat{x}_{ijk} \geq 0, \tag{12}$$

$$T_{ijk}(\hat{\boldsymbol{x}}) > \hat{\alpha}_{ik}, \quad \hat{x}_{ijk} = 0, \tag{13}$$

$$\sum_j \hat{x}_{ijk} = 1,$$

where $\hat{\alpha}_{ik} = \min_{j'}\{\mu_k^{-1} D(\beta_{j'}(\hat{\boldsymbol{x}}))\}$. We can easily see that these relations are satisfied if and only if $\hat{x}_{ijk} = 0$ for all $i, j (\neq i), k$, by noting that the uniqueness of the solution is given in [1] and by the method presented in [11]. □

We suppose in the following (C-I) scheme and in the subsequent (C-II) scheme that the following assumption holds true:

[Assumption A3] *We assume that $G_{ijk}(\boldsymbol{x})$ is one of the following functions:*

Type G-I

$$G_{ijk}(\boldsymbol{x}) = \omega_k^{-1} \underline{G}(\sigma_k x_{ijk})$$

(one dedicated line for each combination of a pair of origin and destination nodes, and a class: i.e., $m(m-1)n$ lines in total),

Type G-II(a)

$$G_{ijk}(\boldsymbol{x}) = \omega_k^{-1} \underline{G}(\sum_{p,q \neq p} \sigma_k x_{pqk})$$

(one bus line for each global class: i.e., n bus lines in total),

Type G-II(b)

$$G_{ijk}(\boldsymbol{x}) = \omega_k^{-1} \underline{G}(\sum_{p,q(\neq p),k} \sigma_k x_{pqk})$$

(one common bus line for the entire system: i.e., 1 bus line),

where $\sigma_k = \phi_k / \omega_k$ and $\underline{G}(x)$ is a nondecreasing, convex, and differentiable function of x.

Remark 3. 1 *ω_k^{-1} can be regarded as the mean communication time (without queueing delays) for forwarding a type k job from the arrival node to another processing node if $\underline{G}(0) = 1$. $\sigma_k x_{ijk} (j \neq i)$ is the traffic intensity of the communication line for the class R_{ik} jobs being forwardeded to node j.*

Example 2 We use the same definition (3) for the mean node delay as in Example 1. We define $G_{ijk}(\boldsymbol{x})$ for the mean communication delay as follows. We assume $\omega_k = \theta$ for all k and thus $\sigma_k = \phi_k/\theta$, and set

$$G_{ijk}(\boldsymbol{x}) = \frac{1/\theta}{1 - \sum_{p,q(\neq p),k} \sigma_k x_{pqk}} \text{ for } \sum_{p,q(\neq p),k} \sigma_k x_{pqk} < 1, \text{ and otherwise infinite.} \quad (14)$$

This is identical to:

$$G_{ijk}(\boldsymbol{x}) = \frac{1}{\theta - \sum_{p,q(\neq p),k} \phi_k x_{pqk}} \text{ for } \sum_{p,q(\neq p),k} \phi_k x_{pqk} < \theta, \text{ and otherwise infinite.} \quad (15)$$

By this, we assume that one bus-type communication line is provided commonly for all the nodes to be used for forwarding of jobs to other nodes in the same way as in Example 1, whereas the transmission time without queueing delay is exponentially distributed with mean θ^{-1} and the scheduling discipline is First-Come-First-Served. Thus, the expected communication time of a job arriving at node i and being processed at node j ($\neq i$) is expressed as $1/(\theta - \sum_{p,q(\neq p),k} \phi_k x_{pqk})$, *i.e.*, independent of the job class and the origin and destination nodes.

(C-I)[Intermediately distributed decision I: Class optimization] The class optimum (or a Nash equilibrium) is given by such $\tilde{\boldsymbol{x}}$ as satisfies the following for all i, k,

$$T_{ik}(\tilde{\boldsymbol{x}}) = \min T_{ik}(\tilde{\boldsymbol{x}}_{-(ik)}; \boldsymbol{x}_{ik}) \text{ with respect to } \boldsymbol{x}_{ik} \text{ such that } (\tilde{\boldsymbol{x}}_{-(ik)}; \boldsymbol{x}_{ik}) \in \boldsymbol{C}.$$

where $(\tilde{\boldsymbol{x}}_{-(ik)}; \boldsymbol{x}_{ik})$ denotes the mmn vector in which the elements corresponding to $\tilde{\boldsymbol{x}}_{ik}$ has been replaced by $\boldsymbol{x}_{ik}$.

Let us define $\tilde{g}_{ijk}(.)$ as

$$\tilde{g}_{ijk}(\boldsymbol{x}) = \frac{\partial}{\partial x_{ijk}} \{\phi_k \sum_{p \neq i} x_{ipk} G_{ipk}(\boldsymbol{x})\}. \tag{16}$$

By Assumption A3, we have

$$\begin{aligned}
\tilde{g}_{ijk}(\boldsymbol{x}) &= \sigma_k[\underline{G}(\sigma_k x_{ijk}) + \sigma_k x_{ijk} \underline{G}'(\sigma_k x_{ijk})], \quad \text{for type G-I,} \\
\tilde{g}_{ijk}(\boldsymbol{x}) &= \sigma_k[\underline{G}(\underline{x}) + \sigma_k(1 - x_{iik})\underline{G}'(\underline{x})], \quad \text{for type G-II,} \\
\text{where } \underline{x} &= \sum_{p,q(\neq p)} \sigma_k x_{pqk} \quad \text{for type G-II(a), and} \\
\underline{x} &= \sum_{p,q(\neq p),k} \sigma_k x_{pqk} \quad \text{for type G-II(b).}
\end{aligned}$$

We see that under the assumption A3 the class of functions $G_{ijk}(\boldsymbol{x})$ satisfy for all $i, j(\neq i), j'(\neq i), k$,

$$\tilde{g}_{ijk}(\boldsymbol{x}) \geq \tilde{g}_{ij'k}(\boldsymbol{x}) \text{ if } x_{ijk} > x_{ij'k}. \tag{17}$$

If Assumption (A3) holds, for $\boldsymbol{x}$ such that $x_{ijk} = x_k$, for all $i, j(\neq i), k$ we denote

$$G_k(\boldsymbol{x}) = G_{ijk}(\boldsymbol{x}) \quad \text{and} \quad \tilde{g}_k(\boldsymbol{x}) = \tilde{g}_{ijk}(\boldsymbol{x}).$$

In particular, for $\boldsymbol{x}$ such that $x_{ijk} = x$, for all $i, j(\neq i), k$ we denote

$$G(x) = G_k(\boldsymbol{x}) = G_{ijk}(\boldsymbol{x}) \quad \text{and} \quad \tilde{g}(x) = \tilde{g}_k(\boldsymbol{x}) = \tilde{g}_{ijk}(\boldsymbol{x}).$$

Solution: *We denote $\Gamma_k = \rho_k^2 \sigma_k^{-1}$. The solution $\tilde{\boldsymbol{x}}$ is unique and is given as follows:*
For Types G-I and G-II(a)

(a) *For class R_{ik} such that $\rho_k^2 D'(\rho) \leq \tilde{g}_k(0) = \sigma_k \underline{G}(0)$, i.e., $\Gamma_k D'(\rho) \leq \underline{G}(0)$,*

$$x_{ijk} = 0, \ \ \textit{and} \ x_{iik} = 1, \ \ \textit{for all} \ i, j(\neq i).$$

The mean response time is

$$T_k(\tilde{\boldsymbol{x}}) = T_{ik}(\tilde{\boldsymbol{x}}) = \mu_k^{-1} D(\rho), \ \textit{for all} \ i, k, \quad T(\tilde{\boldsymbol{x}}) = \rho D(\rho).$$

(b) *For class R_{ik} such that $\rho_k^2 D'(\rho) > \tilde{g}_k(0) = \sigma_k \underline{G}(0)$, i.e., $\Gamma_k D'(\rho) > \underline{G}(0)$, the solution is given as follows:*

$$\tilde{x}_{ijk} = \tilde{x}_k, \ \textit{for all} \ i, \ j(\neq i), \tag{18}$$

where $\tilde{x}_k$ is the unique solution of

$$\begin{aligned}
&\rho_k^2(1 - m\tilde{x}_k)D'(\rho) = \tilde{g}_k(\tilde{x}_k) \\
&= \sigma_k[\underline{G}(m(m-1)\sigma_k \tilde{x}_k) + \sigma_k(m-1)\tilde{x}_k \underline{G}'(m(m-1)\sigma_k \tilde{x}_k)].
\end{aligned} \tag{19}$$

The mean response time is

$$T_k(\tilde{\boldsymbol{x}}) = T_{ik}(\tilde{\boldsymbol{x}}) = \mu_k^{-1} D(\rho) + (m-1)\tilde{x}_k G_k(\tilde{\boldsymbol{x}}), \quad \textit{for all} \ \ i. \tag{20}$$

For Type G-II(b)

The solution is given as in the following. We first change the numbering of k such that $\Gamma_1 \geq \Gamma_2 \geq \cdots \geq \Gamma_k \geq \cdots \geq \Gamma_n$. The following three situations can occur:

$$\begin{aligned} \text{We can find } K \text{ such that } \Gamma_K D'(\rho) &> \underline{G}(0) \text{ and } \Gamma_{K+1}D'(\rho) \leq \underline{G}(0), & (21)\\ \text{or } \Gamma_n D'(\rho) &> \underline{G}(0) \quad (i.e., K = n), & (22)\\ \text{or } \Gamma_1 D'(\rho) &\leq \underline{G}(0). & (23) \end{aligned}$$

When (23) holds, we have a unique solution of $\tilde{x}_k = 0$ for all k. When (21) or (22) holds, we can find a unique solution as follows. Let us define the function $F_k(X)$ as

$$F_k(X) = \{\sum_{l=1}^{k} \frac{\sigma_l[\Gamma_l D'(\rho) - \underline{G}(X)]}{m\Gamma_k D'(\rho) + (m-1)\sigma_l \underline{G}'(X)}\} - \frac{X}{m(m-1)}. \tag{24}$$

We obtain the largest $k = \tilde{k} \leq K$ and $X = \tilde{X}_{\tilde{k}} (> 0)$ that satisfies $F_{\tilde{k}}(\tilde{X}_{\tilde{k}}) = 0$ and $\sigma_{\tilde{k}}[\Gamma_{\tilde{k}} D'(\rho) - \underline{G}(\tilde{X}_{\tilde{k}})] > 0$. Then by using

$$\sigma_k[\Gamma_k D'(\rho) - \underline{G}(\tilde{X}_{\tilde{k}})] = \sigma_k \tilde{x}_k [m\Gamma_k D'(\rho) + (m-1)\sigma_k \underline{G}'(\tilde{X}_{\tilde{k}})], \tag{25}$$

for $k = 1, 2, \cdots, \tilde{k}$, we can obtain the unique set of values such that $\tilde{x}_1 \geq \tilde{x}_2 \geq \cdots \geq \tilde{x}_{\tilde{k}} > 0$ and that $\tilde{x}_{\tilde{k}+1} = \tilde{x}_{\tilde{k}+2} = \cdots = \tilde{x}_n = 0$ that satisfies the above relation, which is a unique solution. The mean response time is

$$T_k(\tilde{\boldsymbol{x}}) = T_{ik}(\tilde{\boldsymbol{x}}) = \mu_k^{-1} D(\rho) + (m-1)\tilde{x}_k G_k(\tilde{\boldsymbol{x}}), \quad \text{for all } i. \tag{26}$$

In particular, for a special case where $\phi_k = 1/n$, $\mu_k = \mu$, $\omega_k = \omega$, and therefore $\rho_k = \rho$ and $\sigma_k = \sigma$ for all k, we have the following simpler form:

(a) *If $\mu^{-2} D'(\rho) \leq n^2 \tilde{g}(0) = n^2 \sigma \underline{G}(0)$, the solution $\tilde{\boldsymbol{x}}$ is unique and is given as follows:*

$$\tilde{\boldsymbol{x}}_{-(iik)} = \mathbf{0}, \quad i.e., \ x_{ijk} = 0, \text{ and } x_{iik} = 1, \text{ for all } i, j (\neq i), k.$$

The mean response time is

$$T(\tilde{\boldsymbol{x}}) = T_k(\tilde{\boldsymbol{x}}) = T_{ik}(\tilde{\boldsymbol{x}}) = \mu^{-1} D(\rho), \text{ for all } i, k.$$

(b) *If $\mu^{-2} D'(\rho) > n^2 \tilde{g}(0) = n^2 \sigma \underline{G}(0)$, the solution $\tilde{\boldsymbol{x}}$ is given as follows:*

$$\begin{aligned} &\tilde{\boldsymbol{x}}_{-(iik)} = (\tilde{x}, \tilde{x}, \cdots, \tilde{x}),\\ &i.e., \ \tilde{x}_{ijk} = \tilde{x} \text{ and } \tilde{x}_{iik} = 1 - (m-1)\tilde{x}, \text{ for all } i, \ j(\neq i) \ k, \end{aligned} \tag{27}$$

where $\tilde{x}$ is the unique solution of

$$\frac{1}{n\mu^2}(1 - m\tilde{x})D'(\rho) = \tilde{g}(\tilde{x}). \tag{28}$$

The mean response time is

$$T(\tilde{\boldsymbol{x}}) = T_k(\tilde{\boldsymbol{x}}) = T_{ik}(\tilde{\boldsymbol{x}}) = \mu^{-1} D(\rho) + (m-1)\tilde{x} G(\tilde{x}), \quad \text{for all } i, k. \tag{29}$$

Remark 3. 2 *From the above we see that the chances of the Braess-like paradoxical performance degradation vary from class to class on the basis of the values of* Γ_k $(= \rho_k^2/\sigma_k = \phi_k\omega_k/\mu_k^2)$. *That is, the performance for the classes that have the larger arrival rate* (ϕ_k)*, the larger processing time requirement* (μ_k^{-1})*, and the smaller communication time requirment* (ω_k^{-1})*, has more chances to be degraded. The performance for all the classes has more chances to be degraded with a larger value of* ρ $(= \sum_k \rho_k)$.

Proof: We define

$$\tilde{t}_{ijk}(\boldsymbol{x}) = \phi_k \frac{\partial}{\partial x_{ijk}} T_{ik}(\boldsymbol{x}). \tag{30}$$

Because T_{ik} are convex functions and C is a convex set, the solution $\tilde{\boldsymbol{x}}$ of the problem exists (see, [18]), and from the Kuhn-Tucker condition it is characterized by the relations (see, *e.g.*, [19]):

$$\begin{aligned} \tilde{t}_{ijk}(\tilde{\boldsymbol{x}}) &= \tilde{\alpha}_{ik}, \quad \tilde{x}_{ijk} > 0, \\ &\geq \tilde{\alpha}_{ik}, \quad \tilde{x}_{ijk} = 0, \\ \sum_j \tilde{x}_{ijk} &= 1, \text{ for all } i, k, \end{aligned} \tag{31}$$

where $\tilde{\alpha}_{ik}$ are the Lagrange multipliers. From Definitions (1), (5) to (7), (16), and (30), we have

$$\tilde{t}_{iik}(\boldsymbol{x}) = \phi_k \frac{\partial T_{ik}}{\partial x_{iik}}(\boldsymbol{x}) = \rho_k[D(\beta_j(\boldsymbol{x})) + \rho_k x_{iik} D'(\beta_j(\boldsymbol{x}))], \tag{32}$$

$$\tilde{t}_{ijk}(\boldsymbol{x}) = \phi_k \frac{\partial T_{ik}}{\partial x_{ijk}}(\boldsymbol{x}) = \rho_k[D(\beta_j(\boldsymbol{x})) + \rho_k x_{ijk} D'(\beta_j(\boldsymbol{x}))] + \tilde{g}_{ijk}(\boldsymbol{x}) \text{ for } j \neq i. \tag{33}$$

Let $\tilde{\boldsymbol{x}}$ be any class optimum strategy profile, and define $\tilde{\beta}_i = \beta_i(\tilde{\boldsymbol{x}})$.

(1) First, we show by contradiction that $\tilde{\beta}_j = \tilde{\beta}_{j'}$ for every pair of (j, j'), and consequently, $\tilde{\beta}_i = \rho$ for all i.

We define

$$\Xi_{ijk;i'j'k}(\boldsymbol{x}) = \tilde{t}_{ijk}(\boldsymbol{x}) - \tilde{t}_{i'j'k}(\boldsymbol{x}). \tag{34}$$

Assume that $\tilde{\beta}_j > \tilde{\beta}_{j'}$ for some j and j'.

(1-1) Assume $\tilde{x}_{ijk} > \tilde{x}_{ij'k}$ for some $i (\neq j, j'), k$. Then, we have $\tilde{g}_{ijk}(\tilde{\boldsymbol{x}}) \geq \tilde{g}_{ij'k}(\tilde{\boldsymbol{x}})$ by (17). From Equation (33) and Definition (34) we have

$$\begin{aligned} \Xi_{ijk;ij'k}(\boldsymbol{x}) &= \rho_k[D(\beta_j(\boldsymbol{x})) - D(\beta_{j'}(\boldsymbol{x}))] \\ &+ \rho_k^2[x_{ijk} D'(\beta_j(\boldsymbol{x})) - x_{ij'k} D'(\beta_{j'}(\boldsymbol{x}))] + \tilde{g}_{ijk}(\boldsymbol{x}) - \tilde{g}_{ij'k}(\boldsymbol{x}). \end{aligned} \tag{35}$$

Together with the fact that $D(.)$ and $D'(.)$ are increasing (A1), it comes that $\Xi_{ijk;ij'k}(\tilde{\boldsymbol{x}}) > 0$. However, from (31) we must have

$$\begin{aligned} \Xi_{ijk;ij'k} &= 0 \text{ and } \tilde{x}_{ijk} > \tilde{x}_{ij'k} > 0, \text{ or} \\ &\leq 0 \text{ and } \tilde{x}_{ijk} > \tilde{x}_{ij'k} = 0, \end{aligned} \tag{36}$$

which contradicts the above. Thus, we must have $\tilde{x}_{ijk} \leq \tilde{x}_{ij'k}$ for all $i (\neq j, j'), k$.

(1-2) Then, from the assumption $\tilde{\beta}_j > \tilde{\beta}_{j'}$, we have at least for some k,

$$\tilde{x}_{jjk} + \tilde{x}_{j'jk} > \tilde{x}_{jj'k} + \tilde{x}_{j'j'k}.$$

(1-2-1) If $\tilde{x}_{j'jk} = 0$, we have

$$\tilde{x}_{jjk} > \tilde{x}_{j'j'k} \text{ and } \tilde{x}_{jj'k} \geq \tilde{x}_{j'jk} \quad (\textbf{Condition I}).$$

(1-2-2) If $\tilde{x}_{j'jk} > 0$, similarly as in (1-1), we see that if $\tilde{x}_{j'jk} > \tilde{x}_{j'j'k}$, we have $\Xi_{j'jk;j'j'k}(\tilde{\boldsymbol{x}}) > 0$, which contradicts (36). Thus we have $\tilde{x}_{j'jk} \leq \tilde{x}_{j'j'k}$. Then $\tilde{x}_{jjk} > \tilde{x}_{jj'k}$ and $\tilde{x}_{jjk} > 0$. Thus from (31), (32), and (33),

$$\tilde{t}_{jjk}(\boldsymbol{x}) = \rho_k[D(\tilde{\beta}_j) + \rho_k \tilde{x}_{jjk} D'(\tilde{\beta}_j)] = \tilde{\alpha}_{jk},$$

$$\tilde{t}_{j'jk}(\boldsymbol{x}) = \rho_k[D(\tilde{\beta}_j) + \rho_k \tilde{x}_{j'jk} D'(\tilde{\beta}_j)] + \tilde{g}_{j'jk}(\tilde{\boldsymbol{x}}) = \tilde{\alpha}_{j'k}.$$

Then we have, by adding the last two equations,

$$\rho_k[2D(\tilde{\beta}_j) + \rho_k(\tilde{x}_{jjk} + \tilde{x}_{j'jk})D'(\tilde{\beta}_j)] + \tilde{g}_{j'jk}(\tilde{\boldsymbol{x}}) = \tilde{\alpha}_{jk} + \tilde{\alpha}_{j'k}. \tag{37}$$

Note that we have from (31), (32), and (33),

$$\tilde{t}_{jj'k}(\tilde{\boldsymbol{x}}) + \tilde{t}_{j'j'k}(\tilde{\boldsymbol{x}}) = \rho_k[2D(\tilde{\beta}_{j'}) + \rho_k(\tilde{x}_{jj'k} + \tilde{x}_{j'j'k})D'(\tilde{\beta}_{j'})] + \tilde{g}_{jj'k}(\tilde{\boldsymbol{x}}) \geq \tilde{\alpha}_{jk} + \tilde{\alpha}_{j'k}. \tag{38}$$

Since D and D' are increasing, $\tilde{\beta}_j > \tilde{\beta}_{j'}$ and $\tilde{x}_{jjk} + \tilde{x}_{j'jk} > \tilde{x}_{jj'k} + \tilde{x}_{j'j'k}$ by assumption, the only possibility for (37) and (38) not to contradict each other is

$$\tilde{g}_{j'jk}(\tilde{\boldsymbol{x}}) > \tilde{g}_{jj'k}(\tilde{\boldsymbol{x}}). \tag{39}$$

Therefore, in the special case where $\tilde{g}_{j'jk}(\tilde{\boldsymbol{x}}) = \tilde{g}_{jj'k}(\tilde{\boldsymbol{x}})$, these two relations contradict each other. For the other cases, we investigate in the following (1-2-2-1) and (1-2-2-2).

(1-2-2-1) Consider the Type G-I case. From the above (39), the relation (17) on $\tilde{g}$, and $\tilde{x}_{jjk} + \tilde{x}_{j'jk} > \tilde{x}_{jj'k} + \tilde{x}_{j'j'k}$, we have $\tilde{x}_{jj'k} \geq \tilde{x}_{j'jk}$, from which $\tilde{x}_{jjk} > \tilde{x}_{j'j'k}$ follows. We thus have

$$\tilde{x}_{jj'k} \geq \tilde{x}_{j'jk} \text{ and } \tilde{x}_{jjk} > \tilde{x}_{j'j'k} \quad (\text{Condition I}),$$

which is the same as (1-2-1).

(1-2-2-2) Consider the Type G-II case. From the above (39), and the relation (17) on $\tilde{g}$ and $\tilde{x}_{jjk} + \tilde{x}_{j'jk} > \tilde{x}_{jj'k} + \tilde{x}_{j'j'k}$, we have $\tilde{x}_{j'j'k} > \tilde{x}_{jjk}$, from which $\tilde{x}_{j'jk} > \tilde{x}_{jj'k}$ follows. We thus have

$$\tilde{x}_{j'j'k} > \tilde{x}_{jjk} \text{ and } \tilde{x}_{j'jk} > \tilde{x}_{jj'k} \quad (\textbf{Condition II}).$$

(1-3) Now we examine each of Conditions I and II, respectively, in the following (1-3-1) and (1-3-2), and will show that both lead to contradictions.

(1-3-1) Consider the case where Condition I holds. Since

$$\tilde{t}_{jjk}(\tilde{\boldsymbol{x}}) = \rho_k[D(\tilde{\beta}_j) + \rho_k \tilde{x}_{jjk} D'(\tilde{\beta}_j)] = \tilde{\alpha}_{jk},$$

$$\tilde{t}_{j'j'k}(\tilde{\boldsymbol{x}}) = \rho_k[D(\tilde{\beta}_{j'}) + \rho_k \tilde{x}_{j'j'k} D'(\tilde{\beta}_{j'})] \geq \tilde{\alpha}_{j'k},$$

we have $\tilde{\alpha}_{jk} > \tilde{\alpha}_{j'k}$, because D and D' are increasing and $\tilde{\beta}_j > \tilde{\beta}_{j'}$ by assumption.

We next show that $\tilde{x}_{jlk} \geq \tilde{x}_{j'lk}$ by contradiction. Assume $\tilde{x}_{jlk} < \tilde{x}_{j'lk}$. Then $\tilde{x}_{j'lk} > 0$, and we have from (31) and (33),

$$\tilde{t}_{j'lk}(\tilde{\boldsymbol{x}}) = \rho_k[D(\tilde{\beta}_l) + \rho_k \tilde{x}_{j'lk} D'(\tilde{\beta}_l)] + \tilde{g}_{j'lk}(\tilde{\boldsymbol{x}}) = \tilde{\alpha}_{j'k},$$

$$\tilde{t}_{jlk}(\tilde{\boldsymbol{x}}) = \rho_k[D(\tilde{\beta}_l) + \rho_k \tilde{x}_{jlk} D'(\tilde{\beta}_l)] + \tilde{g}_{jlk}(\tilde{\boldsymbol{x}}) \geq \tilde{\alpha}_{jk} > \tilde{\alpha}_{j'k},$$

which contradicts the assumption, as we see by noting that here for both G-I and G-II

$$\tilde{g}_{jlk}(\tilde{\boldsymbol{x}}) \leq \tilde{g}_{j'lk}(\tilde{\boldsymbol{x}}).$$

Therefore we must have

$$\tilde{x}_{jlk} \geq \tilde{x}_{j'lk}.$$

From this and Condition I, it comes

$$\begin{aligned} \tilde{x}_{jjk} &> \tilde{x}_{j'j'k}, \\ \tilde{x}_{jj'k} &> \tilde{x}_{j'jk}, \\ \tilde{x}_{jlk} &\geq \tilde{x}_{j'lk} \text{ for all } l(\neq j, j'). \end{aligned}$$

This implies

$$1 = \sum_l \tilde{x}_{jlk} > \sum_l \tilde{x}_{j'lk} = 1,$$

which is impossible. That is, the assumption leads to a contradiction.

(1-3-2) Consider the case where Condition II holds. This implies $\tilde{x}_{j'jk} > 0$ and we have

$$\tilde{t}_{j'jk}(\tilde{\boldsymbol{x}}) = \rho_k[D(\tilde{\beta}_j) + \rho_k \tilde{x}_{j'jk} D'(\tilde{\beta}_j)] + \tilde{g}_{j'jk}(\tilde{\boldsymbol{x}}) = \tilde{\alpha}_{j'k},$$

$$\tilde{t}_{jj'k}(\tilde{\boldsymbol{x}}) = \rho_k[D(\tilde{\beta}_{j'}) + \rho_k \tilde{x}_{jj'k} D'(\tilde{\beta}_{j'})] + \tilde{g}_{jj'k}(\tilde{\boldsymbol{x}}) \geq \tilde{\alpha}_{jk}.$$

Since D and D' are increasing, $\beta_j > \beta_{j'}$ and $\tilde{x}_{j'jk} > \tilde{x}_{jj'k}$, we have

$$\tilde{\alpha}_{j'k} - \tilde{g}_{j'jk}(\tilde{\boldsymbol{x}}) > \tilde{\alpha}_{jk} - \tilde{g}_{jj'k}(\tilde{\boldsymbol{x}}).$$

By noting that for type G-II, we have $\tilde{g}_{jlk}(\tilde{\boldsymbol{x}}) = \tilde{g}_{jj'k}(\tilde{\boldsymbol{x}})$ and $\tilde{g}_{j'lk}(\tilde{\boldsymbol{x}}) = \tilde{g}_{j'jk}(\tilde{\boldsymbol{x}})$ for any $l(\neq j, j')$, and from the Kuhn-Tucker condition, we have

$$\begin{aligned} \rho_k[D(\tilde{\beta}_l) + \rho_k \tilde{x}_{j'lk} D'(\tilde{\beta}_l)] &= \tilde{\alpha}_{j'k} - \tilde{g}_{j'lk}(\tilde{\boldsymbol{x}}) = \tilde{\alpha}_{j'k} - \tilde{g}_{j'jk}(\tilde{\boldsymbol{x}}),\ \tilde{x}_{j'lk} > 0, \\ \rho_k[D(\tilde{\beta}_l) + \rho_k \tilde{x}_{j'lk} D'(\tilde{\beta}_l)] &\geq \tilde{\alpha}_{j'k} - \tilde{g}_{j'lk}(\tilde{\boldsymbol{x}}) = \tilde{\alpha}_{j'k} - \tilde{g}_{j'jk}(\tilde{\boldsymbol{x}}),\ \tilde{x}_{j'lk} = 0, \\ \rho_k[D(\tilde{\beta}_l) + \rho_k \tilde{x}_{jlk} D'(\tilde{\beta}_l)] &= \tilde{\alpha}_{jk} - \tilde{g}_{jlk}(\tilde{\boldsymbol{x}}) = \tilde{\alpha}_{jk} - \tilde{g}_{jj'k}(\tilde{\boldsymbol{x}}),\ \tilde{x}_{jlk} > 0, \\ \rho_k[D(\tilde{\beta}_l) + \rho_k \tilde{x}_{jlk} D'(\tilde{\beta}_l)] &\geq \tilde{\alpha}_{jk} - \tilde{g}_{jlk}(\tilde{\boldsymbol{x}}) = \tilde{\alpha}_{jk} - \tilde{g}_{jj'k}(\tilde{\boldsymbol{x}}),\ \tilde{x}_{jlk} = 0, \end{aligned}$$

which can hold only when $\tilde{x}_{jlk} \leq \tilde{x}_{j'lk}$ for all $l(\neq j, j')$.

From this and Condition II,

$$\begin{aligned} \tilde{x}_{jjk} &< \tilde{x}_{j'j'k}, \\ \tilde{x}_{jj'k} &< \tilde{x}_{j'jk}, \\ \tilde{x}_{jlk} &\leq \tilde{x}_{j'lk} \text{ for all } l(\neq j, j'). \end{aligned}$$

This implies

$$1 = \sum_l \tilde{x}_{jlk} < \sum_l \tilde{x}_{j'lk} = 1,$$

which is impossible. That is, the assumption leads to a contradiction.

Thus we see that the assumption $\tilde{\beta}_j > \tilde{\beta}_{j'}$ leads to either Condition I [(1-2-1) and (1-3-1)] or Condition II [(1-3-2)], both of which lead to contradictions.

Therefore, we must have $\tilde{\beta}_j = \tilde{\beta}_{j'}$, and consequently $\tilde{\beta}_i = \rho$ for all i.

(2) Hence for all $i, j(\neq i), j'(\neq i), k$,

$$\Xi_{ijk;ij'k}(\tilde{\boldsymbol{x}}) = \rho_k^2(\tilde{x}_{ijk} - \tilde{x}_{ij'k})D'(\rho) + \tilde{g}_{ijk}(\tilde{\boldsymbol{x}}) - \tilde{g}_{ij'k}(\tilde{\boldsymbol{x}}). \tag{40}$$

Thus, if $\tilde{x}_{ijk} > \tilde{x}_{ij'k}$ for some $i, j(\neq i), j'(\neq i), k$, we have $\Xi_{ijk;ij'k} > 0$ since $D'(\rho) > 0$, which contradicts (36). Therefore, we must have

$$\tilde{x}_{ijk} = \tilde{x}_k \text{ for all } \; i, j(\neq i), k. \tag{41}$$

(3) We note that since $\sum_j x_{ijk} = 1$, from the assumption on the arrival ratio of each class job, $\tilde{x}_k$ has to belong to the interval $[0, 1/(m-1)]$. We discuss the case for Types G-I and G-II(a) and that for Type G-II(b), separately.

The case for Types G-I and G-II(a)

We have

$$\Xi_{ijk;iik}(\tilde{\boldsymbol{x}}) = -\rho_k^2(1 - m\tilde{x}_k)D'(\rho) + \tilde{g}_k(\tilde{x}_k).$$

where

$$\begin{aligned} \tilde{g}_k(\tilde{x}_k) &= \sigma_k[\underline{G}(\sigma_k\tilde{x}_k) + \sigma_k\tilde{x}_k\underline{G}'(\sigma_k\tilde{x}_k)] \quad \text{(Type G-I) or} \\ \tilde{g}_k(\tilde{x}_k) &= \sigma_k[\underline{G}(m(m-1)\sigma_k\tilde{x}_k) + (m-1)\sigma_k\tilde{x}_k\underline{G}'(m(m-1)\sigma_k\tilde{x}_k)] \quad \text{(Type G-II(a))}. \end{aligned}$$

Let us define the function F_k as

$$F_k(x) = -\rho_k^2(1 - mx)D'(\rho) + \tilde{g}_k(x). \tag{42}$$

Clearly, F_k is continuous and monotonically increasing.

(*a*) For class R_{ik} such that $\rho_k^2 D'(\rho) \leq \tilde{g}_k(0) = \sigma_k\underline{G}(0)$, we have $F_k(x) > F_k(0) \geq 0$ for any $x > 0$, which proves that $x = \tilde{x}_k = 0$ is the unique optimal solution.

Therefore, for class R_{ik} such that $\rho_k^2 D'(\rho) \leq \tilde{g}_k(0) = \sigma_k\underline{G}(0)$,

$$x_{ijk} = 0, \text{ and } x_{iik} = 1, \text{ for all } i, j(\neq i).$$

The mean response time is

$$T_k(\tilde{\boldsymbol{x}}) = T_{ik}(\tilde{\boldsymbol{x}}) = \mu_k^{-1}D(\rho), \text{ for all } i, k, \quad T(\tilde{\boldsymbol{x}}) = \rho D(\rho).$$

(*b*) For class R_{ik} such that $\rho_k^2 D'(\rho) > \tilde{g}_k(0) = \sigma_k\underline{G}(0)$, the optimal solution is uniquely given as follows:

$$\tilde{x}_{ijk} = \tilde{x}_k, \text{ for all } i, \; j(\neq i), \tag{43}$$

where $\tilde{x}_k$ is the unique solution of

$$\rho_k^2(1 - m\tilde{x}_k)D'(\rho) = \tilde{g}_k(\tilde{x}_k).$$

Therefore, the mean response time is

$$T_{ik}(\tilde{\boldsymbol{x}}) = \mu_k^{-1}D(\rho) + (m-1)\tilde{x}_k G_k(\tilde{\boldsymbol{x}}), \quad \text{for all } \; i.$$

Therefore, we have a unique Nash equilibrium or class optimum.

The case for Type G-II(b)
We have

$$\Xi_{ijk;iik}(\tilde{\boldsymbol{x}}) = -\rho_k^2(1 - m\tilde{x}_k)D'(\rho) + \tilde{g}_k(\tilde{\boldsymbol{x}}).$$

We can find the set of $\tilde{x}_k$, $k = 1, 2, \cdots, n$, as the unique solution of the following system of relations:

$$\begin{array}{rcl} \rho_k^2(1 - m\tilde{x}_k)D'(\rho) & = & \tilde{g}_k(\tilde{\boldsymbol{x}}) \text{ and } \tilde{x}_k \geq 0, \\ \rho_k^2 D'(\rho) & < & \tilde{g}_k(\tilde{\boldsymbol{x}}) \text{ and } \tilde{x}_k = 0, \\ 0 \leq & \tilde{x}_k & \leq 1/(m-1), \end{array} \tag{44}$$

where $\tilde{g}_k(\tilde{\boldsymbol{x}}) = \sigma_k[\underline{G}(m(m-1)\sum_k \sigma_k\tilde{x}_k) + \sigma_k(m-1)\tilde{x}_k\underline{G}'(m(m-1)\sum_k \sigma_k\tilde{x}_k)]$.

The relations (44) are equivalent to the following:

$$\begin{array}{rcl} \sigma_k[\Gamma_k D'(\rho) - \underline{G}(\tilde{X})] & = & \sigma_k\tilde{x}_k[m\Gamma_k D'(\rho) + (m-1)\sigma_k\underline{G}'(\tilde{X})] \text{ and } \tilde{x}_k \geq 0, \\ \sigma_k[\Gamma_k D'(\rho) - \underline{G}(\tilde{X})] & < & 0 \text{ and } \tilde{x}_k = 0, \\ 0 \leq & \tilde{x}_k & \leq 1/(m-1), \end{array} \tag{45}$$

where we recall $\Gamma_k = \rho_k^2\sigma_k^{-1}$ and we denote $\tilde{X} = m(m-1)\sum_k \sigma_k\tilde{x}_k$.

We easily see that we can change the numbering of k such that $\Gamma_1 \geq \Gamma_2 \geq \cdots \geq \Gamma_k \geq \cdots \geq \Gamma_n$. The following three situations can occur:

$$\begin{array}{rcl} \text{We can find } K \text{ such that } \Gamma_K D'(\rho) & > & \underline{G}(0) \text{ and } \Gamma_{K+1}D'(\rho) \leq \underline{G}(0), \quad \text{(rel. (21))} \\ \text{or } \Gamma_n D'(\rho) & > & \underline{G}(0) \quad (i.e., K = n), \quad \text{(rel. (22))} \\ \text{or } \Gamma_1 D'(\rho) & \leq & \underline{G}(0) \quad \text{(rel. (23))}. \end{array}$$

When (23) holds, we can find a unique solution of $\tilde{x}_k = 0$ for all k. When (21) or (22) holds, we can find a unique solution as follows.

Recall the definition (24) of the function $F_k(X)$ as

$$F_k(X) = \{\sum_{l=1}^{k} \frac{\sigma_l[\Gamma_l D'(\rho) - \underline{G}(X)]}{m\Gamma_k D'(\rho) + (m-1)\sigma_l\underline{G}'(X)}\} - \frac{X}{m(m-1)}.$$

Clearly, $F_k(X)$ is continuous and monotonically decreasing with the increase in X (≥ 0). Thus for each $k = k'$ ($\leq K$) there exists $X = \tilde{X}_{k'}(> 0)$ that satisfies $F_{k'}(\tilde{X}_{k'}) = 0$. Then given $\tilde{X}_{k'}$, from (45) we can obtain a unique set of values for $\tilde{x}_k, 1 \leq k \leq k'$. Since (21) or (22) holds, we can find the largest $k' = \tilde{k}$ such that $\tilde{x}_{\tilde{k}} > 0$ (*i.e.*, $\Gamma_{\tilde{k}}D'(\rho) - \underline{G}(\tilde{X}_{\tilde{k}}) > 0$, that is the unique solution.

We can see it as follows: From (24) we have

$$F_{\tilde{k}}(\tilde{X}_{\tilde{k}}) = F_{\tilde{k}-1}(\tilde{X}_{\tilde{k}}) + \frac{\sigma_{\tilde{k}}[\Gamma_{\tilde{k}}D'(\rho) - \underline{G}(\tilde{X}_{\tilde{k}})]}{m\Gamma_k D'(\rho) + (m-1)\sigma_{\tilde{k}}\underline{G}'(\tilde{X}_{\tilde{k}})} = 0.$$

Thus we have $F_{\tilde{k}-1}(\tilde{X}_{\tilde{k}}) < 0$.

Assume that we have another feasible solution for $k' = \tilde{k} - 1$. Then we have $X_{\tilde{k}-1} > 0$ such that $F_{\tilde{k}-1}(X_{\tilde{k}-1}) = 0$ and $\tilde{x}_{\tilde{k}} = 0$. Therefore we have

$$F_{\tilde{k}-1}(\tilde{X}_{\tilde{k}}) < F_{\tilde{k}-1}(X_{\tilde{k}-1}).$$

Thus, since $F_k(.)$ is monotonically decreasing, we must have $\tilde{X}_{\tilde{k}} > X_{\tilde{k}-1}$ and $\underline{G}(\tilde{X}_{\tilde{k}}) > \underline{G}(X_{\tilde{k}-1})$. Consequently, since $\Gamma_{\tilde{k}} D'(\rho) - \underline{G}(\tilde{X}_{\tilde{k}}) > 0$, we have $\Gamma_{\tilde{k}} D'(\rho) - \underline{G}(X_{\tilde{k}-1}) > 0$. Therefore, from (25), *i.e.,* (45), we must have $\tilde{x}_{\tilde{k}} > 0$, which is a contradiction.

In a similar way, for k and k' such that $\Gamma_k = \Gamma_{k'}$, we can show that either $\tilde{x}_k = \tilde{x}_{k'} = 0$ or $\tilde{x}_k = \tilde{x}_{k'} > 0$.

Therefore, we see that we have the unique solution. That is, we can obtain the unique set of values such that $\tilde{x}_1 \geq \tilde{x}_2 \geq \cdots \geq \tilde{x}_{\tilde{k}} > 0$ and $\tilde{x}_{\tilde{k}+1} = \tilde{x}_{\tilde{k}+2} = \cdots = \tilde{x}_n = 0$, which satisfies the above relation.

The mean response time (26) is obtained by noting the definitions (1), (5), (6), (7), and (9).

In particular, for a special case where $\phi_k = 1/n$ for all k, we can obtain the solution similarly as the cases for Types G-I and G-II(a). □

(C-II)[Intermediately distributed decision: Global-class optimization] The global-class optimum (Nash equilibrium for another set of decision makers) is given by such $\check{\boldsymbol{x}}$ as satisfies the following for all i, k,

$$T_k(\check{\boldsymbol{x}}) = \min T_k(\check{\boldsymbol{x}}_{-(k)}; \boldsymbol{x}_k), \text{ with respect to } \boldsymbol{x}_k \text{ such that } (\check{\boldsymbol{x}}_{-(k)}; \boldsymbol{x}_k) \in \boldsymbol{C}. \tag{46}$$

where $(\check{\boldsymbol{x}}_{-(k)}; \boldsymbol{x}_k)$ denotes the mmn vector in which the elements corresponding to the coordinates of $\check{\boldsymbol{x}}_k$ has been replaced by the vector $\boldsymbol{x}_k$. We note that

$$\phi_k m T_k(\boldsymbol{x}) = \sum_i \beta_i^{(k)}(\boldsymbol{x}) D(\beta_i(\boldsymbol{x})) + \sum_{i,j \neq i} \phi_k x_{ijk} G_{ijk}(\boldsymbol{x}). \tag{47}$$

Note that we have the assumption A3 on the function $G_{ijk}(\boldsymbol{x})$.

We define $\check{g}_{ijk}(\boldsymbol{x})$ as

$$\check{g}_{ijk}(\boldsymbol{x}) = \frac{\partial}{\partial x_{ijk}} \{ \sum_{p,q \neq p} \phi_k x_{pqk} G_{pqk}(\boldsymbol{x}) \}. \tag{48}$$

By Assumption A3, we have

$$\begin{aligned} \check{g}_{ijk}(\boldsymbol{x}) &= \sigma_k [\underline{G}(x_{ijk}) + \sigma_k x_{ijk} \underline{G}'(x_{ijk})] \text{ for type G-I}, \\ \check{g}_{ijk}(\boldsymbol{x}) &= \sigma_k [\underline{G}(\underline{x}) + \sigma_k \sum_p (1 - x_{ppk}) \underline{G}'(\underline{x})] \text{ for type G-II} \end{aligned}$$

$$(\underline{x} = \sum_{i,j \neq i} \sigma_k x_{ijk} \text{ for type G-II(a)}, \quad \sum_{i,j(\neq i),k} \sigma_k x_{ijk} \text{ for type G-II(b)}).$$

Therefore we have the property

$$\check{g}_{ijk}(\boldsymbol{x}) \geq \check{g}_{ij'k}(\boldsymbol{x}) \text{ if } x_{ijk} > x_{ij'k}. \tag{49}$$

Solution:

$$\check{\boldsymbol{x}}_{-(iik)} = \boldsymbol{0}, \quad \textit{i.e., } x_{ijk} = 0, \textit{ and } x_{iik} = 1, \textit{ for all } i, j (\neq i), k.$$

The mean response time is

$$T_k(\check{\boldsymbol{x}}) = T_{ik}(\check{\boldsymbol{x}}) = \mu_k^{-1} D(\rho), \textit{ for all } i, k, \quad T(\check{\boldsymbol{x}}) = \rho D(\rho).$$

Proof: We define

$$\check{t}_{ijk}(\boldsymbol{x}) = m\phi_k \frac{\partial}{\partial x_{ijk}} T_k(\boldsymbol{x}). \tag{50}$$

Again, because T_k is a convex function and $\boldsymbol{C}$ is compact, the solution $\check{\boldsymbol{x}}$ of the problem exists (see [18]) and from the Kuhn-Tucker condition it is characterized by the relations (see, *e.g.*, [19]):

$$\begin{aligned} \check{t}_{ijk}(\check{\boldsymbol{x}}) &= \check{\alpha}_{ik} \text{ for } \check{x}_{ijk} \text{ such that } \check{x}_{ijk} > 0, \\ &\geq \check{\alpha}_{ik} \text{ for } \check{x}_{ijk} \text{ such that } \check{x}_{ijk} = 0. \\ & \sum_j \check{x}_{ijk} = 1, \text{ for all } i, k \end{aligned} \tag{51}$$

where $\check{\alpha}_{ik}$ are the Lagrange multipliers. From the definitions (1) to (8), (48), and (50), we have

$$\check{t}_{iik}(\boldsymbol{x}) = m\phi_k \frac{\partial T_k}{\partial x_{iik}} = \rho_k[D(\beta_i) + \beta_i^{(k)} D'(\beta_i)], \tag{52}$$

$$\check{t}_{ijk}(\boldsymbol{x}) = m\phi_k \frac{\partial T_k}{\partial x_{ijk}} = \rho_k[D(\beta_j) + \beta_j^{(k)} D'(\beta_j)] + \check{g}_{ijk}, \quad \text{for } j \neq i. \tag{53}$$

We define

$$\check{\Xi}_{ijk;i'j'k} = \check{t}_{ijk}(\boldsymbol{x}) - \check{t}_{i'j'k}(\boldsymbol{x}). \tag{54}$$

From (53) we have

$$\check{\Xi}_{ijk;ij'k} = \rho_k[D(\beta_j) - D(\beta_{j'})] + \rho_k[\beta_j^{(k)} D'(\beta_j) - \beta_{j'}^{(k)} D'(\beta_{j'})] + \check{g}_{ijk} - \check{g}_{ij'k}. \tag{55}$$

Let $\check{\boldsymbol{x}}$ be any global-class optimum. Denote $\check{\beta}_i = \beta_i(\check{\boldsymbol{x}})$.

(1)We first show by contradiction that $\check{\beta}_j = \check{\beta}_{j'}$ for every pair of (j, j'), which implies that $\check{\beta}_i = \rho$ for all i.

(1-1) Suppose that $\check{\beta}_j > \check{\beta}_{j'}$ for some j and j'. Then there must exist k such that $\check{\beta}_j^{(k)} > \check{\beta}_{j'}^{(k)}$. From (51), (52), and (53)

$$\begin{aligned} \check{t}_{j'j'k}(\boldsymbol{x}) = \rho_k[D(\check{\beta}_{j'}) + \check{\beta}_{j'}^{(k)} D'(\check{\beta}_{j'})] &= \check{\alpha}_{j'k}, \quad \check{x}_{j'j'k} > 0, \\ \check{t}_{j'j'k}(\boldsymbol{x}) = \rho_k[D(\check{\beta}_{j'}) + \check{\beta}_{j'}^{(k)} D'(\check{\beta}_{j'})] &\geq \check{\alpha}_{j'k}, \quad \check{x}_{j'j'k} = 0, \\ \check{t}_{j'jk}(\boldsymbol{x}) = \rho_k[D(\check{\beta}_j) + \check{\beta}_j^{(k)} D'(\check{\beta}_j)] + \check{g}_{j'jk}(\check{\boldsymbol{x}}) &= \check{\alpha}_{j'k}, \quad \check{x}_{j'jk} > 0, \\ \check{t}_{j'jk}(\boldsymbol{x}) = \rho_k[D(\check{\beta}_j) + \check{\beta}_j^{(k)} D'(\check{\beta}_j)] + \check{g}_{j'jk}(\check{\boldsymbol{x}}) &\geq \check{\alpha}_{j'k}, \quad \check{x}_{j'jk} = 0. \end{aligned}$$

Therefore, from the fact that D and D' are increasing functions (Assumption A1) and from Property (49), we have $\check{x}_{j'jk} = 0$ and consequently $\check{x}_{j'jk} \leq \check{x}_{jj'k}$.

(1-2) Suppose we have $\check{x}_{ijk} > \check{x}_{ij'k}$ for some $i(\neq j, j')$, then necessarily $\check{g}_{ijk} \geq \check{g}_{ij'k}$ by Property (49). Since, by (A1), $D(.)$ and $D'(.)$ are increasing, $\check{\Xi}_{ijk;ij'k}(\check{\boldsymbol{x}}) > 0$. However, from (51), since $\check{\boldsymbol{x}}$ is a global-class optimum we have

$$\begin{aligned} \check{\Xi}_{ijk;ij'k} &= 0 \text{ and } \check{x}_{ijk} > \check{x}_{ij'k} > 0, \text{ or} \\ &\leq 0 \text{ and } \check{x}_{ijk} > \check{x}_{ij'k} = 0, \end{aligned} \tag{56}$$

which contradicts the above. Thus, we must have

$$\check{x}_{ijk} \leq \check{x}_{ij'k} \text{ for all } i.$$

Therefore, from $\check{\beta}_j^{(k)} > \check{\beta}_{j'}^{(k)}$, we must have

$$\check{x}_{jjk} + \check{x}_{j'jk} > \check{x}_{j'j'k} + \check{x}_{jj'k}.$$

Thus we have from (1-1)

$$\check{x}_{jj'k} \geq \check{x}_{j'jk} \text{ and } \check{x}_{jjk} > \check{x}_{j'j'k}.$$

(1-3) Since

$$\rho_k[D(\check{\beta}_j) + \check{\beta}_j^{(k)} D'(\check{\beta}_j)] = \check{\alpha}_{jk},$$

$$\rho_k[D(\check{\beta}_{j'}) + \check{\beta}_{j'}^{(k)} D'(\check{\beta}_{j'})] \geq \check{\alpha}_{j'k},$$

we have $\check{\alpha}_{jk} > \check{\alpha}_{j'k}$.

We next show that $\check{x}_{jlk} \geq \check{x}_{j'lk}$ by contradiction. Assume $\check{x}_{jlk} < \check{x}_{j'lk}$. Then $\check{x}_{j'lk} > 0$, and we have from (51), (52), and (53),

$$\rho_k[D(\check{\beta}_l) + \check{\beta}_l^{(k)} D'(\check{\beta}_l)] + \check{g}_{j'lk}(\check{\boldsymbol{x}}) = \check{\alpha}_{j'k},$$

$$\rho_k[D(\check{\beta}_l) + \check{\beta}_l^{(k)} D'(\check{\beta}_l)] + \check{g}_{jlk}(\check{\boldsymbol{x}}) \geq \check{\alpha}_{jk} > \check{\alpha}_{j'k},$$

which contradicts the assumption, as we see by noting that $\check{g}_{jlk}(\check{\boldsymbol{x}}) \leq \check{g}_{j'lk}(\check{\boldsymbol{x}})$ for both of G-I and G-II. Therefore we must have

$$\check{x}_{jlk} \geq \check{x}_{j'lk}.$$

From this and (1-2),

$$\begin{aligned} \check{x}_{jjk} &> \check{x}_{j'j'k}, \\ \check{x}_{jj'k} &> \check{x}_{j'jk}, \\ \check{x}_{jlk} &\geq \check{x}_{j'lk} \quad \text{for all } l(\neq j, j'). \end{aligned}$$

This implies

$$1 = \sum_l \check{x}_{jlk} > \sum_l \check{x}_{j'lk} = 1,$$

which is impossible. That is, the assumption leads to a contradiction. Thus we see that the assumption $\check{\beta}_j > \check{\beta}_{j'}$ leads to a contradiction. Therefore necessarily $\check{\beta}_j = \check{\beta}'_j$, and consequently $\check{\beta}_i = \rho$ for all i.

(2) We next show by contradiction $\check{\beta}_j^{(k)} = \check{\beta}_{j'}^{(k)}$ for every pair of (j, j'), which implies that $\check{\beta}_i^{(k)} = \rho_k$ for all i, k.

From (55) we have for all $i, j(\neq i), j'(\neq i), k$,

$$\check{\Xi}_{ijk;ij'k}(\check{\boldsymbol{x}}) \quad = \quad \rho_k(\beta_j^{(k)} - \beta_{j'}^{(k)}) D'(\beta_1) + \check{g}_{ijk}(\check{\boldsymbol{x}}) - \check{g}_{ij'k}(\check{\boldsymbol{x}}). \tag{57}$$

Assume $\check{\beta}_j^{(k)} > \check{\beta}_{j'}^{(k)}$ for some j and j'. We can follow the same line of logic as (1-1), (1-2), and (1-3) above, even though $\beta_i = \rho$ for all i, and we see that the above assumption leads to a contradiction. Therefore necessarily $\check{\beta}_j^{(k)} = \check{\beta}_{j'}^{(k)}$, and consequently $\check{\beta}_i^{(k)} = \rho_k$ for all i, k.

(3) Now from (55) we have for all $i, j(\neq i), j'(\neq i), k$,

$$\check{\Xi}_{ijk;ij'k}(\check{\boldsymbol{x}}) = \check{g}_{ijk}(\check{\boldsymbol{x}}) - \check{g}_{ij'k}(\check{\boldsymbol{x}}). \tag{58}$$

Thus, if $\check{x}_{ijk} > \check{x}_{ij'k}$ for some $i, j(\neq i), j'(\neq i), k$, we have $\check{\Xi}_{ijk;ij'k} \geq 0$, which contradicts relation (56). Therefore, we must have

$$\check{x}_{ijk} = \check{x}_k \text{ for all } i, j(\neq i), k,$$

and from (53) and (55) we have for all $i, j(\neq i), k$,

$$\check{\Xi}_{ijk;iik}(\check{\boldsymbol{x}}) = \check{g}(\check{\boldsymbol{x}}) > 0,$$

and consequently from (51) we have $\check{x}_k = 0$ for all k. □

4 Examples

We consider here Examples 1 and 2 introduced in Sections 2 and 3, respectively. We restrict ourselves to the class optimization (C-I) where $\mu_k = \mu$ $\phi_k = 1/n$ for all k. We have $\rho = 1/\mu$.

Example 1 In this case we have

$$\check{g}(x) = t/n.$$

We note that

$$\mu^{-2}D'(\rho) - n^2\check{g}(0) = 1/(\mu-1)^2 - nt.$$

(a) If $t > 1/\{n(\mu-1)^2\}$, then the solution $\tilde{\boldsymbol{x}}$ is unique and given by

$$\tilde{\boldsymbol{x}}_{-(iik)} = \mathbf{0}, \textit{ i.e., } \tilde{x}_{ijk} = 0, \quad \tilde{x}_{iik} = 1, \quad \text{for all } i, j(\neq i), k.$$

The mean response time is

$$T(\tilde{\boldsymbol{x}}) = T_{ik}(\tilde{\boldsymbol{x}}) = \mu_k^{-1}D(\rho) = \frac{1}{\mu-1}, \quad i = 1, 2, \cdots, m, \quad k = 1, 2, \cdots, n.$$

In particular it is the same for the overall, individual, and global-class optima.

(b) If $t \leq 1/\{n(\mu-1)^2\}$, the solution $\tilde{\boldsymbol{x}}$ is given by

$$\tilde{x}_{ijk} = \frac{1}{m}\{1 - nt(\mu-1)^2\}, \quad \tilde{x}_{iik} = \frac{1}{m}\{1 + (m-1)nt(\mu-1)^2\}, \quad \text{for all } i, j(\neq i), k. \tag{59}$$

The mean response time is

$$\begin{aligned} T(\tilde{\boldsymbol{x}}) &= T_{ik}(\tilde{\boldsymbol{x}}) \\ &= \frac{1}{\mu-1} + \frac{m-1}{m}t\{1 - nt(\mu-1)^2\}, \quad \text{for all } i, k. \end{aligned} \tag{60}$$

For some parameters (μ, m, n), $\tilde{T} = T(\tilde{\boldsymbol{x}})$ attains its maximum in t (*i.e.*, the worst performance), that we denote $\tilde{T}_{\max}(\mu, m, n)$ for

$$t_{\max} = \frac{1}{2n(\mu-1)^2}. \quad (61)$$

We have

$$\tilde{T}_{\max}(\mu, m, n) = \frac{1}{\mu - 1}\{1 + \frac{m-1}{4mn(\mu-1)}\}. \quad (62)$$

Thus if we add the communication lines with delay $t_{\max} = 1/\{2n(\mu-1)^2\}$ to the system that has had no communication means, the mean response time $T_{ik}(\tilde{\boldsymbol{x}})$ for each class increases in the amount of $\frac{m-1}{4mn(\mu-1)^2}$ (*i.e.*, the performance degrades). This is a Braess-like paradox. We define the *worst ratio of the performance degradation* $\Delta(\mu, m, n)$ in the paradox for given μ, n to be

$$\Delta(\mu, m, n) = \frac{\tilde{T}_{\max}(\mu, m, n) - T_0(\mu)}{T_0(\mu)}, \quad (63)$$

where $T_0(\mu) = 1/(\mu - 1)$ denotes the mean response time of each class jobs for given μ when the system has no communication means. We have

$$\Delta(\mu, m, n) = \frac{m-1}{4mn(\mu-1)}. \quad (64)$$

Example 2 We have

$$\tilde{g}(x) = \frac{\theta - m(m-1)(n-1)x}{(\theta - m(m-1)nx)^2},$$

and

$$\mu^{-2}D'(\rho) - n^2\tilde{g}(0) = 1/(\mu-1)^2 - n/\theta.$$

This is the same as in the above example if we take $t = \theta^{-1}$. Therefore, we have the following:

(a) If $\theta^{-1} > 1/\{n(\mu-1)^2\}$, we obtain the same solution and mean response time as in Example 1, *i.e.*, $\tilde{\boldsymbol{x}}_{-(iik)} = \mathbf{0}$ and $T_{ik}(\tilde{\boldsymbol{x}}) = 1/(\mu - 1)$.

(b) If $\theta^{-1} \leq 1/\{n(\mu-1)^2\}$, the solution $\tilde{\boldsymbol{x}}$ is given by

$$\tilde{x}_{ijk} = \tilde{x}, \ \tilde{x}_{iik} = 1 - (m-1)\tilde{x}, \text{ for all } i, j (\neq i), k, \quad (65)$$

where $\tilde{x}$ satisfies

$$\frac{1}{n}(1 - m\tilde{x})\frac{1}{\mu - 1} = \frac{\theta - m(m-1)(n-1)\tilde{x}}{(\theta - m(m-1)n\tilde{x})^2}. \quad (66)$$

Remark 4. 1 *From the above we see that, no forwarding of jobs occurs in the overall, individual, and global-class optima and in case (a) of the class optimum. That is, in those optima, jobs arriving at each node are processed only by the node, and thereby the system has no performance improvement or degradation due to adding the communication means (which is not used).*

On the other hand, in case (b) of the class optimum, each class forwards a part of its jobs through the communication means to other nodes for remote processing, and thereby has degradation in its mean response time. The ratio of such degradation can become unlimitedly large *as the total arrival rate approaches the processing capacity of each node, i.e., as* $\mu \simeq 1$. *In Example 1 of the model, we see that, as* n *and thus the number of classes* $(m \times n)$ *increase up to infinity with the number of nodes,* m, *fixed, the ratio of degradation,* $\Delta(\mu, m, n)$, *and the chances of the paradox decrease and finally disappear.*

The effect of m *is not so large in the example, wherein, if we increase* m *from 2 unlimitedly, the worst ratio of performance degradation increases only up to twice.*

5 Numerical Examples

We examine Example 1 with $m = 5$, *i.e.*, the system with five nodes, and consider the case: $\mu = 1.01$. The mean response time is $T_0(\mu) = 1/(\mu - 1) = 100$ in the overall optimum, in the individual optimum (Wardrop equilibrium), and in the case of no communication line and no forwarding of jobs.

Firstly, we consider the case where $n = 1$, *i.e.*, the total number of classes R_{ik} is 5. $T = T_{ik}$ takes its maximum value

$$\tilde{T}(1.01, 5, 1) = 2100 \quad \text{(see (62))},$$

and the worst ratio of the performance degradation $\Delta(\mu, m, n)$ in the paradox is

$$\Delta(1.01, 5, 1) = 20 \ (i.e.,\ \ 2000\% \text{ degradation}) \quad \text{(see (63))},$$

when $t = 1/\{2(\mu - 1)^2\} = 5000$ (see (61)). Then

$$\tilde{x}_{ijk} = (1/5)\{1 - t(\mu - 1)^2\} = 1/10 \ (k = 1) \quad \text{(see (59))}.$$

In this case, $\tilde{x}_{ijk}$ $(= \tilde{x})$ decrease from 1/5 down to 0 as t increases from 0 to 10000 $(= 1/(\mu - 1)^2)$, and for $t > 10000$, no forwarding of jobs occurs.

It is amazing that each class keeps to forward a part of its jobs equally to the other nodes even though the communication delay for forwarding is much greater than the processing delay at the node at which its jobs arrive.

Then we consider the case where $n = 100$, *i.e.*, the total number of classes is 500.
$T = T_{ik}$ takes its maximum value

$$\tilde{T}(1.01, 5, 100) = 120 \text{ (see (62))},$$

and the worst ratio of the performance degradation $\Delta(\mu, m, n)$ in the paradox is

$$\Delta(1.01, 5, 100) = 0.2 \ (i.e.,\ \ 20\% \text{ degradation}) \quad \text{(see (63))},$$

when $t = 1/\{2n(\mu - 1)^2\} = 50$ (see (61)). Then

$$\tilde{x}_{ik} = (1/5)\{1/n - t(\mu - 1)^2\} = 1/1000, \qquad \text{for all } k \text{ (see (59))}.$$

In this case, $\tilde{x}_{ik}$ decrease from 1/500 down to 0 as t increases from 0 to 100 $(= 1/(\mu - 1)^2)$, and for $t > 100$, no forwarding of jobs occurs.

Thus we see that the chances of paradoxes and the magnitude in the degradation of the performance in the paradox are greatly reduced from the case $n = 1$.

Furthermore we consider other values of μ with $n = 1$.
For $\mu = 1.001$, $\Delta(1.001, 5, 1) = 200$ (*i.e.*, 20000% degradation), and
for $\mu = 1.00001$, $\Delta(1.00001, 2, 1) = 20000$ (*i.e.*, 2000000% degradation), *etc.* In this way, we see that the worst ratio of the performance degradation $\Delta(\mu, m, n)$ in the paradox becomes unlimitedly large as μ approaches 1 with $n = 1$.

6 Concluding Remarks

In this paper, we have examined the model consisting of symmetrical nodes with identical arrivals to all nodes where forwarding of jobs to the other nodes through communication means with nonzero delays may clearly lead to performance degradation. We have confirmed that in the overall optimization and in the individual optimization (Wardrop equilibrium) such forwarding never occurs. We have shown that, in some parameter setting of the class optimization (Nash equilibrium for one set of decision makers), mutual forwarding of jobs for remote processing through communication means definitely occurs and the ratio of the performance degradation may become unlimitedly large. We have also shown that in the global-class optimization (Nash equilibrium for another set of decision makers) such forwarding never occurs.

That is, such a paradoxical behavior may occur only in the class optimum and does never occur for the overall, Wardrop, and global-class optima, in the same setting of this symmetrical node model.

We have obtained the uniqueness of class and global-class optima on the basis of only the special assumptions on the communication means (*i.e.,* dedicated lines and bus-type connections). It has been quite hard to extend the proofs to more general assumptions. It is not certain whether in some cases of the communication means the optima may still be unique. It has been also difficult for us to analyze asymmetrical models. These are open future problems.

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