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ON THE APPLICATIONS OF BILINEAR PROGRAMMING

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1. Introduction

The purpose of this paper is to discuss some of the more important applications of bilinear programming and to focus our attention on its practical value. Bilinear programming is a technique for solving a special class of nonconvex quadratic programming problem:

$$\left.\begin{array}{lll} \text{minimize} & c_1^t x_1 + c_2^t x_2 + x_1^t Q x_2 & \\ \text{subject to} & A_1 x_1 = b_1, & x_1 \geq 0 \\ & A_2 x_2 = b_2, & x_2 \geq 0 \end{array}\right\} \quad (1.1)$$

where $c_i \varepsilon R^{n_i}$, $b_i \varepsilon R^{m_i}$, $A_i \varepsilon R^{m_i \times n_i}$, $x_i \varepsilon R^{n_i}$, $i=1,2$ and $Q \varepsilon R^{n_1 \times n_2}$. We refer to (1.1) as a bilinear programming problem (BLP) in a standard form. Needless to say that a general bilinear programming problem with mixed equality and inequality constraints can be reduced to the standard form as in the case of linear programming.

The author proposed a cutting plane algorithm for solving BLP and obtained some encouraging results through numerical experiments [8]. Also Gallo and Ülkücü [6] and Falk [4] proposed algorithms of enumerative nature. More recently, Vaish and Shetty [18], Shetty and Sherali [16] extended the results of [8] and established finite convergence of the algorithm. It is hoped that these

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efforts in algorithmic development will, in the near future, enable us to solve this class of problems efficiently.

Before going into typical applications of BLP, we will briefly summarize its relationship to other classes of mathematical programming problems.

First of all, we will obtain a BLP if we allow cost coefficients c in a standard linear program:

$$\left.\begin{array}{ll} \text{minimize} & c^t x \\ \text{subject to} & Ax = b, \quad x \geq 0 \end{array}\right\} \qquad (1.2)$$

to vary in a polyhedral convex set. We will call such a problem:

$$\left.\begin{array}{ll} \text{minimize} & c^t x \\ \text{subject to} & Ax = b, \quad x \geq 0 \\ & \bar{A}c = \bar{b}, \quad c \geq 0 \end{array}\right\} \qquad (1.3)$$

an extended linear program (ELP).

Secondly, it is not difficult to show that a linear max-min problem (LMMP):

$$\underset{x \in X}{\text{minimize}} \ \max_{y \in Y} \{ p_1^t x + p_2^t y \mid B_1 x + B_2 y \geq b \} \qquad (1.4)$$

where X and Y are polyhedral convex sets, can be converted to a BLP, under some regularity condition, by taking the partial dual with respect to Y. This problem has been discussed by Falk [4] as well as by Dantzig [2] and Konno [10].

Thirdly, BLP has a close relationship with a generalized linear program to be fully discussed in section 4.

Finally, it has been proved in [9] that the minimization of a concave quadratic function subject to linear constraints (CQP):

$$\left.\begin{array}{ll} \text{minimize} & 2c^t x + x^t Q x \\ \text{subject to} & Ax = b \ , \quad x \geq 0 \end{array}\right\} \quad (1.5)$$

where Q is symmetric and negative semi-definite, is equivalent to a BLP:

$$\left.\begin{array}{ll} \text{minimize} & c^t u + c^t v + u^t Q v \\ \text{subject to} & Au = b \ , \quad u \geq 0 \\ & Av = b \ , \quad v \geq 0 \end{array}\right\} \quad (1.6)$$

The relationship between (1.5) and (1.6) is fully discussed in [9]. It is well known [13] that CQP is closely related to integer programming problems, so that BLP is indirectly related to integer programs. Figure 1.1 summarizes the relationships among the problems discussed above.

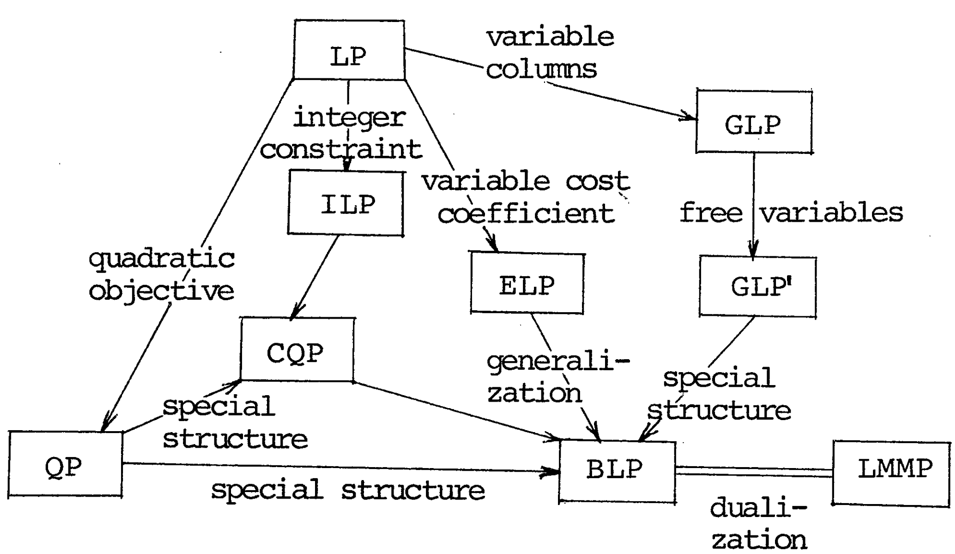

Figure 1.1

In the sections to follow, we will pick up new examples of BLP which are of practical and theoretical interests. For other examples readers are referred to [2,3,5,11].

2. Location-allocation problems

There is a large amount of literature under the title of location-allocation theory (See [15]). Suppose we are given

a) a set of m points distributed in the plane

b) a vector value to be attached to each point

c) a set of indivisible centroids without predetermined locations

then the location-allocation problem in its most general form is to find locations for m centroids and an allocation of vector value associated with n points to some centroid so as to minimize the total cost. Here, we will show that one version of this class of problems can be put into the framework of BLP in a very natural way.

(a) Single factory case

Let there be m cities P_i, $i=1,\cdots,m$ on a plane. P_i is located at (p_i,q_i) relative to some coordinate system. We are required to construct a factory F somewhere on this plane. This factory needs b_j units of n different materials M_j, $j=1,\cdots,n$. Let us assume that P_i can supply at most a_{ij} units of M_j for the unit price c_{ij} and the unit transportation cost f_j (per unit amount per unit distance). Our concern is to minimize the total expense which is represented by the sum of total purchasing cost and the total transportation cost. Let (x_0,y_0) be the location of the factory to be constructed and let u_{ij} be the amount of M_j to be purchased at P_i. Then u_{ij} has to satisfy:

$$\left.\begin{aligned}&\sum_{i=1}^{m} u_{ij} \geq b_j, && j = 1,\cdots,n, \\ &0 \leq u_{ij} \leq a_{ij}, && i = 1,\cdots,m, \quad j = 1,\cdots,n.\end{aligned}\right\} \quad (2.1)$$

Total purchasing cost C_p and total transportation cost C_T are given by

$$C_p = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}u_{ij} \quad (2.2)$$

$$C_T = \sum_{i=1}^{m}\sum_{j=1}^{n} f_j \cdot u_{ij}\, d(P_i, F) \quad (2.3)$$

where $d(P_i,F)$ is the distance between P_i and F. If we assume, in addition, that the distance $d(P_i,F)$ is given by 1 norm i.e.,

$$d(P_i,F) = d_1(P_i,F) \equiv |p_i-x_0| + |q_i-y_0| \quad (2.4)$$

then the total cost C is given by

$$C = \sum_{i=1}^{m}\sum_{j=1}^{n} [c_{ij}u_{ij}+ f_j u_{ij} (|p_i-x_0|+|q_i-y_0|)] \quad (2.5)$$

By introfucing auxiliary variables, x_{il} and y_{il} satisfying

$$\left.\begin{aligned}&x_{i1}-x_{i2} = p_i-x_0, \quad x_{i1}\geq 0, \quad x_{i2}\geq 0, \quad x_{i1}x_{i2}=0, \quad i=1,\cdots,m, \\ &y_{i1}-y_{i2} = q_i-y_0, \quad y_{i1}\geq 0, \quad y_{i2}\geq 0, \quad y_{i1}y_{i2}=0, \quad i=1,\cdots,m.\end{aligned}\right\} \quad (2.6)$$

the absolute value terms can be represented as:

$$\left.\begin{aligned}&|p_i-x_0| = x_{i1}+x_{i2} \\ &|q_i-y_0| = y_{i1}+y_{i2}\end{aligned}\right\} \quad (2.7)$$

So our problem is to

$$\left.\begin{array}{ll}
\text{minimize } C = \sum_{i=1}^{m}\sum_{j=1}^{n} u_{ij}[c_{ij}+f_j(x_{i1}+x_{i2}+y_{i1}+y_{i2})] & \\
\text{subject to } \sum_{i=1}^{m} u_{ij} \geq b_j, \quad j=1,\cdots,n, & \\
0 \leq u_{ij} \leq a_{ij}, \; i=1,\cdots,m, \; j=1,\cdots,n, & \\
x_{i1}-x_{i2}+x_0=p_i, & \\
\qquad\qquad\qquad i=1,\cdots,m, & \\
y_{i1}-y_{i2}+y_0=q_i, & \\
x_{il} \geq 0, \; y_{il} \geq 0, \; i=1,\cdots,m, \; l=1,2, & \\
x_{i1}x_{i2}=0, \; y_{i1}y_{i2}=0, \; i=1,\cdots,m. &
\end{array}\right\} \quad (2.8)$$

It is straightforward to show that the optimal solution of the associated bilinear program in which the orthogonality condition in (2.8) is relaxed automatically satisfy the orthognality property if $f_j \geq 0$, $j=1,\cdots,n$ and hence the problem can be solved by applying the algorithm developed in [8].

(b) Multi-factory case

Let us consider next the multi-factory version of the problem discussed above. The basic setting of the problem is the same as before except that

(i) K (≥ 1) factories F_k, $k=1.\cdots,K$ have to be constructed

(ii) each factory produces L different types of commodities C_l, $l=1,\cdots,L$

(iii) each product has to be shipped to m cities P_i, $i=1,\cdots,m$.

Let

u_{ij}^k : the amount of M_j to be purchased at P_i and shipped to F_k

x_{il}^k : amount of C_l to be shipped to P_i from F_k

b_j^k : amount of M_j required at F_k

a_{ij} : maximum supply of M_j at P_i

c_{ij} : unit price of M_j at P_i

d_l^k : amount of C_l produced at F_k

e_{il} : demand for C_l at P_i

(p_i, q_i): location of P_i

(x_k, y_k): location of F_k

$d(P_i, F_k)$: distance between P_i and F_k

f_j : unit transportation cost of M_j

g_l : unit transportation cost of C_l

The total cost is given by

$$C = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{K} c_{ij} u_{ij}^k + \sum_{i=1}^{m} \sum_{k=1}^{K} \left(\sum_{l=1}^{L} g_l x_{il}^k + \sum_{j=1}^{n} f_j u_{ij}^k \right) d(P_i, F_k) \tag{2.9}$$

Also u_{ij}^k and x_{il}^k have to satisfy:

$$\left.\begin{aligned} &\sum_{i=1}^{m} u_{ij}^k \geq b_j^k , \quad j = 1,\cdots,n , \quad k = 1,\cdots,K , \\ &\sum_{k=1}^{K} u_{ij}^k \leq a_{ij} , \quad i = 1,\cdots,m , \quad j = 1,\cdots,n , \\ &\sum_{i=1}^{m} x_{il}^k \leq d_l^k , \quad l = 1,\cdots,L , \quad k = 1,\cdots,K , \end{aligned}\right\} \tag{2.10}$$

$$\left.\begin{array}{l} \sum_{k=1}^{K} x_{il}^{k} \geq e_{il}, \quad i = 1,\cdots,m, \quad l = 1,\cdots,L, \\ u_{ij}^{k} \geq 0, \quad x_{il}^{k} \geq 0, \quad \forall\, i,j,k,l \end{array}\right)$$

Hence the problem is to minimize (2.9) subject to (2.10) which is a BLP provided that $d(\cdot,\cdot)$ is defined by l norm as before. We assumed here that there are no inter-factory material flows. Should there be such flows , the problem can no longer be formulated in the framework of bilinear programming.

3. Applications in multi-attribute utility analysis

Suppose a decision maker is facing a problem of choosing the 'best' among m possible alternatives A_i, $i=1,\cdots,m$ in the stochastic environment where n possible events E_j, $j=1,\cdots,n$ take place with probability p_{ij} when A_i is chosen.

Let us suppose also that there are K independent attributes (objectives) T_k, $k=1,\cdots,K$ and that the utility associated with the triple (A_i, E_j, T_k), is given by a_{ij}^{k}. Also we assume that the overall utility of the decision maker is additive, i.e., the expected utility u_i obtained by choosing A_i is given by

$$u_i = \sum_{k=1}^{K} \sum_{j=1}^{n} w_k p_{ij} a_{ij}^{k} \tag{3.1}$$

where w_k is the weight representing the relative importance of T_k. Given w_k, p_{ij}, a_{ij}^{k}, the best alternative is the one corresponding to $\max_{1 \leq i \leq m} u_i$.

It sometimes happens, however, due to the lack of information that w_k's and p_{ij}'s are not known exactly. Typically, the analyst has to interview the decision maker to estimate w_k's and the best we can hope for is the interval estimates

$$\underline{w}_k \le w_k \le \overline{w}_k , \qquad k = 1,\cdots,K.$$

where $\underline{w}_k$ and $\overline{w}_k$ are given constants (see [14]).

Similar argument applies as well to the probability measure p_{ij}. Let us suppose here that

$$\underline{p}_{ij} \le p_{ij} \le \overline{p}_{ij} , \qquad i = 1,\cdots,m, \quad j = 1,\cdots,n,$$

$$\sum_{j=1}^{n} p_{ij} = 1, \qquad i = 1,\cdots,m.$$

where $\underline{p}_{ij}$ and $\overline{p}_{ij}$ are given constants.

In this case, we may not be able to identify the best alternative due to uncertainty. However, some of the alternatives may be eliminated as inefficient ones by solving BLP's.

Let

$$W = \{w=(w_1,\cdots,w_k) \mid \underline{w}_k \le w_k \le \overline{w}_k, \quad k=1,\cdots,K\} \tag{3.2}$$

$$P_i = \{p_i=(p_{i1},\cdots p_{in}) \mid \underline{p}_{ij} \le p_{ij} \le \overline{p}_{ij} , \quad j = 1,\cdots,n; \ \sum_{j=1}^{n} p_{ij} = 1\} \quad i = 1,\cdots,m. \tag{3.3}$$

which we assume to be nonempty. Let

$$\underline{u}_i = \min \{ \sum_{k=1}^{K} \sum_{j=1}^{n} w_k p_{ij} a_{ij}^k \mid w \in W, \quad p_i \in P_i \} \tag{3.4}$$

$$\overline{u}_i = \max \{ \sum_{k=1}^{K} \sum_{j=1}^{n} w_k p_{ij} a_{ij}^k \mid w \in W, \quad p_i \in P_i \} \tag{3.5}$$

It is obvious that A_s can be eliminated from the candidates of optimal alternatives if $\underline{u}_r > \bar{u}_s$.

Similarly, if

$$u_{rs} \equiv \min \{ \sum_{k=1}^{K} \sum_{j=1}^{n} w_k (p_{rj} a_{rj}^k - p_{sj} a_{sj}^k) \mid w \in W,\ p_r \in P_r,\ p_s \in P_s \} > 0 \quad (3.6)$$

then A_s can be eliminated. Problems (3.4)(3.5) and (3.6) are all bilinear programming problem with a very special structure. Let us take for example (3.4) suppressing index i:

$$\left.\begin{array}{ll} \text{minimize} & \sum_{k=1}^{K} \sum_{j=1}^{n} a_j^k p_j w_k \\ \text{subject to} & \sum_{j=1}^{n} p_j = 1, \quad \underline{p}_j \le p_j \le \bar{p}_j, \quad j=1,\cdots,n; \\ & \underline{w}_k \le w_k \le \bar{w}_k, \quad k=1,\cdots,K \end{array}\right\} \quad (3.7)$$

The next theorem characterizes the form of an optimal solution of (3.7).

<u>Theorem 3.1</u>

There exists w_k^*, $k=1,\cdots,K$; p_j^*, $j=1,\cdots,n$ which is optimal to (3.7). Also, w_k^* is equal to $\underline{w}_k$ or $\bar{w}_k$ for all k, and p_j is equal to $\underline{p}_j$ or $\bar{p}_j$ except for possibly one index j_0.

Proof: W and P are bounded polyhedral convex sets. Hence by the fundamental theorem of bilinear programming [8], there exists an optimal solution (w*,p*) where w* and p* are extreme points of W and P, respectively. It is easy to see that any extreme point of W and P has the property stated in the theorem.||

Also it may be more appropriate in some cases to normalize w_k,

$k=1,\cdots,K$ so that they satisfy the condition $\sum_{k=1}^{K} w_k=1$, in which case we still have a bilinear programming problem with somewhat more complicated structure. For the background material of utility analysis the readers are referred to Keeney and Raiffa [7] and to Sarin [14].

4. Non-standard generalized linear program

Let us consider the generalized linear program (GLP) introduced by Dantzig and Wolfe [1]

$$\left.\begin{array}{ll} \text{minimize} & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} & \sum_{j=1}^{n} a_j x_j = b \\ & x_j \geq 0, \quad \begin{pmatrix} c_j \\ a_j \end{pmatrix} \varepsilon C_j, \qquad j=1,\cdots,n \end{array}\right\} \quad (4.1)$$

where $a_j \varepsilon R^m$, $c_j \varepsilon R^1$ and $C_j \varepsilon R^{m+1}$ is a compact convex set for $j=1,\cdots,n$. Note that minimization is taken over $\begin{pmatrix} c_j \\ a_j \end{pmatrix}$ as well as x_j. The algorithm for solving GLP proceeds roughly as follows:

Given $\begin{pmatrix} c_j^l \\ a_j^l \end{pmatrix} \varepsilon\ C_j$, $l=1,\cdots,l_j$, $j=1,\cdots,n$, solve the linear program:

$$\left.\begin{array}{ll} \text{minimize} & \sum_{j=1}^{n} \sum_{l=1}^{l_j} c_j^l x_j^l \\ \text{subject to} & \sum_{j=1}^{n} \sum_{l=1}^{l_j} a_j^l x_j^l = b, \\ & x_j^l \geq 0, \quad l=1,\cdots,l_j; \quad j=1,\cdots,n \end{array}\right\} \quad (4.2)$$

and let $\pi \varepsilon R^m$ be an optimal multiplier vector for this linear program. It can be shown that if

$$c_j - \pi a_j \geq 0 \qquad \forall \begin{pmatrix} c_j \\ a_j \end{pmatrix} \varepsilon C_j; \qquad j = 1,\cdots,n.$$

then the current solution is optimal. If, on the other hand, there exists an index j and a vector $\begin{pmatrix} c_j \\ a_j \end{pmatrix} \varepsilon C_j$ for which $c_j - \pi a_j < 0$, then the objective function will be improved by introducing this vector into the basis. To find the vectors $\begin{pmatrix} c_j \\ a_j \end{pmatrix}$ for which $c_j - \pi a_j < 0$, we solve the following n subproblems.

$$\text{minimize } \{ c_j - \pi a_j \mid \begin{pmatrix} c_j \\ a_j \end{pmatrix} \varepsilon C_j \}, \qquad j = 1,\cdots,n \qquad (4.3)$$

Let $\begin{pmatrix} c_j^* \\ a_j^* \end{pmatrix}$ satisfy $c_j^* - \pi a_j^* < 0$, then we will introduce this column vector into (4.2) and proceed. It has been shown that this algorithm converges to an optimal solution of (4.1) in finitely many steps if C_j are polyhedral for all j.

Now let us consider the non-standard GLP with some free variables, i.e.,

$$\left.\begin{array}{lll} \text{minimize} & \sum_{j=1}^{n} c_j x_j & \\ \text{subject to} & \sum_{j=1}^{n} a_j x_j = b & \\ & x_j \geq 0, & j = 1,\cdots,l; \\ & x_j \gtrless 0, & j = l+1,\cdots,n; \\ & \begin{pmatrix} c_j \\ a_j \end{pmatrix} \varepsilon C_j, & j = 1,\cdots,n. \end{array}\right\} \qquad (4.4)$$

The standard technique of replacing a free variable by two non-negative variables destroys the structure of the problem, as we shall see. Let

$$x_j = x_{j1} - x_{j2}, \qquad x_{j1} \geq 0, \quad x_{j2} \geq 0, \qquad j = l+1,\cdots,n.$$

then the problem is

$$\left.\begin{array}{ll}
\text{minimize} & \sum_{j=1}^{l} c_j x_j + \sum_{j=l+1}^{n} c_{j1} x_{j1} - \sum_{j=l+1}^{n} c_{j2} x_{j2} \\
\text{subject to} & \sum_{j=1}^{l} a_j x_j + \sum_{j=l+1}^{n} a_j x_{j1} - \sum_{j=l+1}^{n} a_j x_{j2} = b \\
& x_j \geq 0, \qquad j = 1,\cdots,l \\
& x_{j1}, x_{j2} \geq 0, \qquad j = l+1,\cdots,n \\
& \begin{pmatrix} c_j \\ a_j \end{pmatrix} \in C_j, \qquad j = 1,\cdots,l \\
& \begin{pmatrix} c_{j1} \\ a_{j2} \end{pmatrix} = \begin{pmatrix} c_{j2} \\ a_{j2} \end{pmatrix} \in C_j, \qquad j = l+1,\cdots,n.
\end{array}\right\} \quad (4.5)$$

Hence the columns of this problem are no longer independent and GLP algorithm would not work.

Now let us consider the simplest case of the above in which a_j's are constant and only c_j's are allowed to move in compact convex sets, i.e., closed interval in this case:

$$\left.\begin{array}{ll}
\text{minimize} & \sum_{j=1}^{l} c_j x_j + \sum_{j=l+1}^{n} c_j x_j \\
\text{subject to} & \sum_{j=1}^{l} a_j x_j + \sum_{j=l+1}^{n} a_j x_j = b \\
& x_j \geq 0, \qquad j = 1,\cdots,l; \\
& \underline{c}_j \leq c_j \leq \overline{c}_j, \qquad j = 1,\cdots,n.
\end{array}\right\} \quad (4.6)$$

Since $x_j \geq 0$, $j=1,\cdots,l$, it is obvious that optimal c_j's are $\underline{c}_j$'s for $j=1,\cdots,l$. Hence the problem simplifies somewhat to

$$\left.\begin{array}{ll}\text{minimize} & \sum_{j=1}^{l} \underline{c}_j x_j + \sum_{j=l+1}^{n} y_j x_j \\ \text{subject to} & \sum_{j=1}^{l} a_j x_j + \sum_{j=l+1}^{n} a_j x_j = b \\ & x_j \geq 0, \qquad j = 1,\cdots,l; \\ & \underline{c}_j \leq y_j \leq \overline{c}_j, \qquad j = l+1,\cdots,n. \end{array}\right\} \quad (4.7)$$

Apply the standard elimination technique to obtain an expression of x_k, $k=l+1,\cdots,n$ with respect to x_j, $j=1,\cdots,l$, i.e., let

$$x_j = d_{j0} + \sum_{k=1}^{l} d_{jk} x_k, \qquad j = l+1,\cdots,n. \quad (4.8)$$

Substituting (4.8) into (4.7), we obtain

$$\left.\begin{array}{ll}\text{minimize} & \sum_{j=1}^{l} [\underline{c}_j + \sum_{k=l+1}^{n} d_{kj} y_k]\, x_j + \sum_{j=l+1}^{n} d_{j0} y_j \\ \text{subject to} & \sum_{j=1}^{l} a'_j x_j = b' \\ & x_j \geq 0, \qquad j = 1,\cdots,l \\ & \underline{c}_j \leq y_j \leq \overline{c}_j\,, \qquad j = l+1,\cdots,n. \end{array}\right\} \quad (4.9)$$

which is a BLP. The following theorem characterizes the form of an optimal solution.

Theorem 4.1

Suppose (4.9) has an optimal solution. Then there exists y_j^*, x_j^*, $j=1,\cdots,n$ which is optimal to (4.9) such that $y_j^*=\underline{c}_j$, $j=1,\cdots,l$ and y_j^* is either $\underline{c}_j$ or $\overline{c}_j$ for $j=l+1,\cdots,n$.

Proof: By the fundamental theorem of BLP [8], ther exists an optimal solution $y^*=(y_{l+1}^*,\cdots,y_n^*)$ where y^* is an extreme point of the constraint set $\{(y_{l+1},\cdots,y_n) \mid \underline{c}_j \leq y_j \leq \overline{c}_j,\ j=l+1,\cdots,n\}$.||

We have shown that bilinear programming technique gives a way to solve (4.4). Our analysis have shown, at the same time, that a GLP without nonnegativity condition on x_j's are essentially different from the standard GLP which belongs to a class of nice convex problems.

5. Complementary planning problems

Let us consider the problem

$$\left.\begin{aligned} &\text{minimize} && c_1^t x_1 + d_1^t y_1 + c_2^t x_2 + d_2^t y_2 \\ &\text{subject to} && A_1 x_1 + B_1 y_1 \geq b_1 \\ & && A_2 x_2 + B_2 y_2 \geq b_2 \\ & && x_1 \geq 0,\quad y_1 \geq 0,\quad x_2 \geq 0,\quad y_2 \geq 0 \\ & && x_1^t x_2 = 0 \end{aligned}\right\} \qquad (5.1)$$

where $c_1, c_2 \in R^l$, $d_i \in R^{n_i}$, $A_i \in R^{m_i \times l}$, $B_i \in R^{m_i \times n_i}$, $b_i \in R^{m_i}$, $i=1,2$ and x_i, y_i are variable vectors of appropriate dimensions.

There are many real world applications of (5.1) and (5.2) such as complementary flow problems, orthogonal scheduling problems to name only a few [10].

The classical technique to solve this problem [19] is to introduce an l-dimensional vector u of 0-1 components and replace the constraints $x_1^t x_2 = 0$, $x_1 \geq 0$, $x_2 \geq 0$ by:

$$x_1 \leq M_0 u$$

$$x_2 \leq M_0(e_l - u)$$

$$x_1 \geq 0, \quad x_2 \geq 0.$$

where e_l is the l dimensional vector all of whose components are 1's and M_0 is a constant satisfying

$$M_0 \geq \max \{e^t x_i \mid A_i x_i + B_i y_i \geq b_i, \quad x_i \geq 0, \ y_i \geq 0\}, \quad i = 1,2$$

Hence (5.1) is equivalent to the following mixed 0-1 integer programming problem:

$$\left.\begin{array}{ll} \text{minimize} & c_1^t x_1 + d_1^t y_1 + c_2^t x_2 + d_2^t y_2. \\ \text{subject to} & A_1 x_1 + B_1 y_1 \geq b_1 \\ & A_2 x_2 + B_2 y_2 \geq b_2 \\ & x_1 - M_0 u \leq 0 \\ & x_2 + M_0 u \leq M_0 e_n \\ & x_1 \geq 0, \quad y_1 \geq 0, \quad x_2 \geq 0, \quad y_2 \geq 0 \\ & u = (u_1, u_2, \cdots, u_l) \\ & u_j = 0 \text{ or } 1, \quad j = 1, \cdots, l. \end{array}\right\} \quad (5.2)$$

This can be solved by a usual branch and bound technique if l is small. Instead, we will propose another classical approach, i.e., penalty function approach by putting the constraint $x_1^t x_2 = 0$ into the objective function:

$$\left.\begin{array}{ll} \text{minimize} & c_1^t x_1 + d_1^t y_1 + c_2^t y_2 + M x_1^t x_2 \\ \text{subject to} & A_1 x_1 + B_1 y_1 \geq b_1 \\ & A_2 x_2 + B_2 y_2 \geq b_2 \\ & x_1 \geq 0, \quad y_1 \geq 0, \quad x_2 \geq 0, \quad y_2 \geq 0. \end{array}\right\} \quad (5.3)$$

which is a BLP. Note that BLP formulation (5.3) have fewer variables and constraints than its counterpart (5.2).

Theorem 5.1

If the constraint set of (5.1) is bounded, then there exists a constant M_0 such that (5.1) is equivalent to (5.3) for $M>M_0$.

Proof: This can be proved by a standard technique and will be omitted. ||

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KEY WORDS

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ABSTRACT

Some of the more important applications of bilinear programming, a technique for solving a special class of nonconvex quadratic programming problem:

$$\begin{array}{lll} \text{minimize} & c_1^t x_1 + c_2^t x_2 + x_1^t Q x & \\ \text{subject to} & A_1 x_1 = b_1, & x_1 \geq 0 \\ & A_2 x_2 = b_2, & x_2 \geq 0 \end{array}$$

are discussed. Applications included are (i) location-allocation problems, (ii) multi-attribute utility analysis, (iii) non-standard generalized linear program, and (iv) complementary planning problems. The relationship of bilinear programming problems to other classes of classical mathematical programming problems are also discussed.

SUPPLEMENTARY NOTES