EIS-TR-75-4

SOME APPLICATIONS OF BILINEAR PROGRAMMING

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June 24, 1976

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1. INTRODUCTION

This paper is addressed to some of the more important applications of bilinear programming which is a technique for solving a special class of nonconvex quadratic programming problems with the following structure:

$$\min_{x_1,x_2} \{ c_1^t x_1 + c_2^t x_2 + x_1^t Q x_2 \mid A_1 x_1 = b_1, x_1 \geq 0, A_2 x_2 = b_2, x_2 \geq 0 \} \tag{1.1}$$

where $c_i \in R^{n_i}$, $b_i \in R^{m_i}$, $A_i \in R^{m_i \times n_i}$, $i = 1, 2$, $Q \in R^{n_1 \times n_2}$ are constants and $x_i \in R^{n_i}$, $i = 1,2$ are variables. We will refer to this as a bilinear program in standard minimization form.

Corresponding to the above,

$$\min_{x_1,x_2} \{ c_1^t x_1 + c_2^t x_2 + x_1^t Q x_2 \mid A_1 x_1 \geq b_1, x_1 \geq 0, A_2 x_2 \geq b_2, x_2 \geq 0 \} \tag{1.2}$$

will be referred to as a bilinear program in canonical minimization form. Bilinear programs in standard and canonical maximization form will be defined analogously. As in the case of linear programming, bilinear program with general mixed equality and inequality constraints can be reduced to a standard form and to a canonical form as long as the linear constraints with respect to x_1 and x_2 are separable with each other.

Several papers have appeared since 1971 dealing with the algorithms to solve this class of problem or its equivalent, among which Konno [7], [10], Gallo-Ülkücü [4], Falk [3] are notable. Recently, the author implemented his algorithm on CYBER 74 to get encouraging numerical results [8]. At the same time, he established the finite convergence of his cutting plane algorithm [10] with the incorporation of facial cut introduced by Majthay and Whinston [12]. Now that there is a workable algorithm, we will pursue further to show the appliciability of bilinear programming to real world problems. In fact, the existence of many practical problems which are naturally put into the structure of bilinear program motivated the author's work in algorithm.

Before going into typical applications, we will briefly summarize the relationship of bilinear program (BLP) to other groups of mathematical programming problems.

First of all, BLP is a very straightforward extension of linear programs (LP) (see e.g. [1]).

$$\min_{x} \{ c^t x \mid Ax = b, x \geq 0 \} \tag{1.3}$$

where $c, x \in R^n$, $b \in R^m$, $A \in R^{m \times n}$ and c is a fixed cost vector. If we want to vary c as well as x in a polyhedral convex set, say,

$$C = \{ c \in R^n \mid A'c = b' \; c \geq 0 \}$$

then the problem becomes a BLP where $c_1 = c_2 = 0$ and Q is an n x n identity matrix in (1. 1). We will refer to this problem

$$\min \{ c^t x \mid Ax = b, \ x \geq 0, \ A'c = b', c \geq 0 \} \tag{1.4}$$

as an extended linear program (ELP) in standard minimization form. We will discuss several examples of ELP in this paper.

Secondly, there is a similar but entirely different class of problems called generalized linear programs (GLP) introduced by Dantzig and Wolfe [1]:

$$\min_{(c_j, a_j, x_j)} \{ \sum_{j=1}^{n} c_j x_j \mid \sum_{j=1}^{n} a_j x_j = b, x_j \geq 0, \begin{pmatrix} c_j \\ a_j \end{pmatrix} \in C_j \subset R^{m+1}, \ j = 1, ..., n \} \tag{1.5}$$

where $a_j \in R^m$ and C_j is a closed convex set, j = 1, ..., n.

Here the column vectors $\begin{pmatrix} c_j \\ a_j \end{pmatrix}$ as well as x_j's are variables and each column $\begin{pmatrix} c_j \\ a_j \end{pmatrix}$ is allowed to move in a closed convex set C_j independendly of each other. This independence property distinguishes itself from BLP and it is quite essential for GLP algorithm (decomposition algorithm) to work (see [1]). We will look into the relationship between GLP and BLP in section 4 and show that the special case of a non-standard generalized linear program, i. e., a GLP some of whose variables x_j are not restricted in sign, is essentially a BLP.

Thirdly, it is not difficult to show that the so-called linear max-min problem (LMMP):

$$\min_{x \in X} \max_{y \in Y} \{ p_1^t x + p_2^t y \mid B_1 x + B_2 y \geq b \} \tag{1.6}$$

where X and Y are polyhedral convex sets, can be converted into a BLP by taking the partial dual of (1. 6) with respect to Y under some regularity condition. This problem was treated by Falk [3] as well as by Dantzig [2] and Konno [11]. It will be shown in [11] that LMMP has several applications with game theoretic flavour.

Fourthly, it is possible, at least theoretically to transform the problem with complementarity condition

$$\min \{ c_1^t x_1 + c_2^t x_2 \mid A_1 x_1 = b_1, \ x_1 \geq 0, A_2 x_2 = b_2, \ x_2 \geq 0, x_1^t x_2 = 0 \} \tag{1.7}$$

into a BLP by putting $x_1^t x_2$ term into the objective function as follows:

$$\min \{ c_1^t x_1 + c_2^t x_2 + M x_1^t x_2 \mid A_1 x_1 = b_1, \ x_1 \geq 0, \ A_2 x_2 = b_2, \ x_2 \geq 0 \}$$

where M is a large positive constant. (1. 7) was analyzed by Ibaraki [6] and by Konno [11].

Finally, it has been proved in [9] that the minimization of a concave quadratic function subject to linear constraints (CQP):

$$\min \{ 2c^t x + x^t Qx \mid Ax = b, \ x \geq 0 \} \tag{1.8}$$

where Q is a symmetric negative semi-definite matrix, can be converted into a BLP:

$$\min \{ c^t u + c^t v + u^t Qv \mid Au = b, \ u \geq 0, \ Av = b, \ v \geq 0 \} \tag{1.9}$$

The relationship between (1. 8) and (1. 9) has been fully discussed elsewhere [9], where it is shown that (i) if x* is optimal for (1. 8), then (u, v) = (x*, x*) is optimal for (1. 9) and (ii) if (u*, v*) is optimal for

(1. 9), then both u* and v* are optimal for (1. 8). Also it has been shown how to exploit the symmetric structure of (1. 9) to improve the cutting plane algorithm developed in [8]. It is well known that CQP is closely related to 0–1 integer program and therefore BLP is indirectly related to 0–1 integer program.

The following figure briefly summarizes the relationship among various problems cited above.

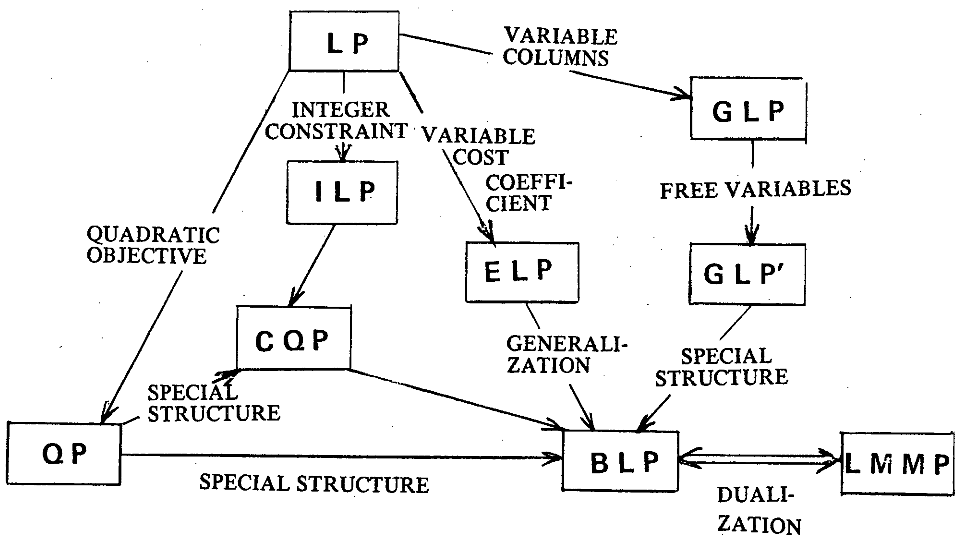

FIGURE 1. 1

In the following sections, we will choose some of the typical examples of bilinear programs and discuss them in some detail. We tried to pick up, among others, problems which are of practical, and theoretical interests.

2. LOCATION-ALLOCATION PROBLEMS

There is a large amount of literature under the title of location-allocation theory (See, for example, reference [14].)

Suppose we are given

a) a set of m points distributed in the plane

b) a vector value to be attached to each point

c) a set of indivisible centroids without predetermined locations

then the location-allocation problem in its most general form is to find locations for m centroids and an allocation of vector value associated with n points to some centroid so as to minimize the total cost. Here, we will present one original example of this type of problems which is put into the structure of BLP in a very natural way.

(a) **Single Factory Case**

Let there be m cities P_i, $i = 1, \ldots, m$ on a plane. P_i is located at (p_i, q_i) relative to some coordinate system. We are going to construct a factory somewhere on this plane. This factory needs b_j units of n different materials M_j, $j = 1, \ldots, n$.

Let us assume that P_i can supply a_{ij} units of M_j at the unit price c_{ij} and the unit transportation cost (per unit amount per unit distance) of M_j is given by f_j. Our concern now is to minimize the total expense which is represented by the sum of total purchasing cost and the total transportation cost.

Let $Q(x_o, y_o)$ be the location of the factory to be constructed and let u_{ij} be the amount of M_j to be purchased at P_i. Then u_{ij} has to satisfy:

$$\sum_{i=1}^{m} u_{ij} \geq b_j, \qquad j = 1, \ldots, n,$$

$$0 \leq u_{ij} \leq a_{ij} \qquad i = 1, \ldots, m, \quad j = 1, \ldots, n \tag{2.1}$$

Total purchasing cost C_p is obviously given by:

$$C_p = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} u_{ij} \tag{2.2}$$

and total transportation cost C_T is given by

$$C_T = \sum_{i=1}^{m} \sum_{j=1}^{n} f_j \cdot u_{ij} \, d(P_i, Q) \tag{2.3}$$

where $d(P_i, Q)$ is the distance between P_i and Q.

i) Manhattan Distance

If the distance $d(P_i, Q)$ is given by 1 norm i. e.,

$$d(P_i, Q) = d_1(P_i, Q) \equiv |p_i - x_o| + |q_i - y_o| \tag{2.4}$$

then the total cost C is given by

$$C = \sum_{i=1}^{m} \sum_{j=1}^{n} [c_{ij} u_{ij} + f_j u_{ij} (|p_i - x_o| + |q_i - y_o|)] \tag{2.5}$$

By introducing auxiliary variables, x_{i1} and y_{i1} satisfying

$$x_{i1} - x_{i2} = p_i - x_o \quad x_{i1} \geq 0, \ x_{i2} \geq 0, \ x_{i1} x_{i2} = 0, \ i = 1, \ldots, m,$$

$$y_{i1} - y_{i2} = q_i - y_o \quad y_{i1} \geq 0, \ y_{i2} \geq 0, \ y_{i1} y_{i2} = 0, \ i = 1, \ldots, m. \tag{2.6}$$

the absolute value terms can be written as:

$$|p_i - x_o| = x_{i1} + x_{i2}$$

$$|q_i - y_o| = y_{i1} + y_{i2} \tag{2.7}$$

So the problem now is to

$$\text{minimize } C = \sum_{i=1}^{m} \sum_{j=1}^{n} u_{ij} \; [\, c_{ij} + f_j (x_{i_1} + x_{i2} + y_{i_1} + y_{i_2}) \,]$$

$$\begin{aligned} \text{s. t.} \quad & \sum_{i=1}^{m} u_{ij} \geq b_j \,, \quad j = 1, \ldots\ldots, n \\ & 0 \leq u_{ij} \leq a_{ij} \quad i = 1, \ldots\ldots, m, \; j = 1, \ldots\ldots, n, \\ & \left.\begin{aligned} x_{i1} - x_{i2} + x_o &= p_i \\ y_{i1} - y_{i2} + y_o &= q_i \end{aligned}\right\} \quad i = 1, \ldots\ldots, m, \\ & x_{i\ell} \geq 0, \; y_{i\ell} \geq 0, \quad i = 1, \ldots\ldots, m, \; \ell = 1, 2, \\ & x_{i1} \cdot x_{i2} = 0 \,, \; y_{i1} \cdot y_{i2} = 0 \,, \quad i = 1, \ldots\ldots m. \end{aligned} \tag{2.8}$$

It is straightforward to show that the optimal solution of the associated bilinear program without the orthogonality condition in (2. 8) automatically satisfy the orthognality property if $f_j \geq 0$, $j = 1, \ldots\ldots, n$ and hence the problem can be solved by applying the algorithm developed in [8].

ii) Euclidean Distance

If, on the other hand, the distance $d(P_i, Q)$ is given by 2 norm, i. e.,

$$d(P_i, Q) = d_2 (P_i, Q) \equiv \sqrt{(p_i - x_o)^2 + (q_i - y_o)^2} \tag{2.9}$$

then the problem becomes:

$$\text{minimize } C = \sum_{i=1}^{m} \sum_{j=1}^{n} [\, c_{ij} + f_j \sqrt{(p_i - x_o)^2 + (q_i - y_o)^2} \,] \; u_{ij}$$

$$\begin{aligned} \text{s. t.} \quad & \sum_{i=1}^{m} u_{ij} \geq b_j \quad j = 1, \ldots\ldots, n, \\ & 0 \leq u_{ij} \leq a_{ij} \quad i = 1, \ldots\ldots, m, \quad j = 1, \ldots\ldots, n. \end{aligned} \tag{2.10}$$

to which we can apply a modified version of the BLP algorithm.

(b) Multi-Factory Case

Let us consider here the multi factory version of the problem treated in the previous section. The basic setting of the problem is the same as before except

(i) $K (\geq 1)$ factories F_k, $k = 1, \ldots\ldots, K$ have to be constructed

(ii) each factory is producing L different types of commodities C_ℓ, $\ell = 1, \ldots\ldots, L$

(iii) each product has to be shipped to the demand points i. e., to m cities.

Let

u_{ij}^k : the amount of M_j to be purchased at P_i and shipped to F_k

$x_{i\ell}^k$: amount of C_ℓ to be shipped to P_i from F_k

b_j^k : amount of M_j required at F_k

a_{ij} : maximum supply of M_j at P_i

c_{ij} : unit price of M_j at P_i

d_ℓ^k : amount of C_ℓ produced at F_k

$e_{i\ell}$: demand for C_ℓ at P_i

(p_i, q_i) : location of P_i

(x_k, y_k): location of F_k

$d(P_i, F_k)$: distance between P_i and F_k

f_j : unit transportation cost of M_j

g_ℓ : unit transportation cost of C_ℓ

The total cost is now given by ·

$$C = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{K} c_{ij} u_{ij}^k + \sum_{i=1}^{m} \sum_{k=1}^{K} \left(\sum_{\ell=1}^{L} g_\ell x_{i\ell}^k + \sum_{j=1}^{n} f_j u_{ij}^k \right) d(P_i, F_k) \tag{2. 11}$$

Also u_{ij}^k and $x_{i\ell}^k$ have to satisfy:

$$\sum_{i=1}^{m} u_{ij}^k \geq b_j^k, \qquad j = 1, \ldots\ldots, n, \quad k = 1, \ldots\ldots, K$$

$$\sum_{k=1}^{K} u_{ij}^k \leq a_{ij}, \qquad i = 1, \ldots\ldots, m, \quad j = 1, \ldots\ldots, n$$

$$\sum_{i=1}^{m} x_{i\ell}^k \leq d_\ell^k, \qquad \ell = 1, \ldots\ldots, L, \quad k = 1, \ldots\ldots, K$$

$$\sum_{k=1}^{K} x_{i\ell}^k \geq e_{i\ell}, \qquad i = 1, \ldots\ldots, m, \quad \ell = 1, \ldots\ldots, L$$

$$u_{ij}^k \geq 0, \qquad x_{i\ell}^k \geq 0, \qquad \forall\, i, j, k, \ell \tag{2. 12}$$

Hence now the problem is to minimize (2. 11) subject to (2. 12) which is a BLP if d (• , •) is defined by 1 norm.

We assumed here that there are no material flows between the factories to be constructed. Should there be such a flow, then the problem can no longer be formulated in the framework of bilinear programming.

3 APPLICATIONS TO DECISION ANALYSIS

Suppose a decision maker is facing a problem of choosing the 'best' among m possible alternatives A_i, $i = 1, \ldots\ldots, m$ in the stochastic environment where n possible events E_j, $j = 1, \ldots\ldots, n$ takes place with probability p_{ij} when A_i is chosen.

Let us suppose also that there are K independent attributes (objectives) T_k, $k = 1, \ldots, K$, each of which has weight (degree of importance) w_k. Also let us assume that the utility associated with the triple (A_i, E_j, T_k), is given by a_{ij}^k and that the overall utility of the decision maker is additive, i. e., the expected utility u_i obtained by choosing A_i is given by

$$u_i = \sum_{k=1}^{K} \sum_{j=1}^{n} w_k p_{ij} a_{ij}^k \tag{3. 1}$$

Given the constants w_k, p_{ij}, a_{ij}^k, we can choose the optimal alternative by simply comparing u_i, $i = 1, \ldots m$.

It sometimes happens, however, due to the lack of information that the quantities w_k, $k = 1, \ldots K$ and p_{ij}, $i = 1, \ldots, m$; $j = 1, \ldots, n$ are not known precisely. Typically, the analyst has to interview the decision maker to estimate the weight of relative importance w_k of T_k and it sometimes happens that we only have interval estimates

$$\underline{w}_k \leq w_k \leq \overline{w}_k, \qquad k = 1, \ldots, K.$$

where $\underline{w}_k$ and $\overline{w}_k$ are given constants (see [13]).

Similar situation applies as well to the probability measure p_{ij}. Let us suppose here that

$$\underline{p}_{ij} \leq p_{ij} \leq \overline{p}_{ij}, \qquad i = 1, \ldots, m, \qquad j = 1, \ldots n.$$

$$\sum_{j=1}^{n} p_{ij} = 1, \qquad i = 1, \ldots m.$$

where $\underline{p}_{ij}$ and $\overline{p}_{ij}$ are given constants.

In this case, the optimal alternative will not be uniquely determined. However, some of the alternatives may be eliminated as inefficient ones as follows:

Let

$$W = \{ (w_1, \ldots, w_k) \mid \underline{w}_k \leq w_k \leq \overline{w}_k, \quad k = 1, \ldots, K \} \tag{3. 2}$$

$$P_i = \{ (p_{i1}, \ldots, p_{in}) \mid \underline{p}_{ij} \leq p_{ij} \leq \overline{p}_{ij}, \; j = 1, \ldots n; \; \sum_{j=1}^{n} p_{ij} = 1 \}. \quad i = 1, \ldots m. \tag{3. 3}$$

which we assume to be nonempty.

Let

$$\underline{u}_i = \min \{ \sum_{k=1}^{K} \sum_{j=1}^{n} w_k p_{ij} a_{ij}^k \mid w \in W, \; p_i \in P_i \} \tag{3. 4}$$

$$\overline{u}_i = \max \{ \sum_{k=1}^{K} \sum_{j=1}^{n} w_k p_{ij} a_{ij}^k \mid w \in W, \; p_i \in P_i \} \tag{3. 5}$$

It is obvious that if $\underline{u}_r > \overline{u}_s$, then A_r is preferred to A_s and A_s can be eliminated from the candidates of optimal alternatives.

Similarly, if

$$u_{rs} \equiv \min \{ \sum_{k=1}^{K} \sum_{j=1}^{n} w_k (p_{rj} a_{rj}^k - p_{sj} a_{sj}^k) \mid w \in W, p_r \in P_r, p_s \in P_s \} > 0 \qquad (3.6)$$

then A_r is preferred to A_s and A_s can be eliminated.

Problems (3. 4) (3. 5) and (3. 6) are all bilinear programs with a very special structure. Let us take for example (3. 4) suppressing index i :

$$\min \sum_{k=1}^{K} \sum_{j=1}^{n} a_j^k p_j w_k$$

$$\text{s. t.} \sum_{j=1}^{n} p_j = 1, \quad \underline{p}_j \leq p_j \leq \bar{p}_j, \quad j = 1, \ldots, n;$$

$$\underline{w}_k \leq w_k \leq \bar{w}_k, \quad k = 1, \ldots, K \qquad (3.7)$$

The following theorem characterizes the form of an optimal solution of (3. 7) which is guaranteed to exist since W and P are non-empty compact convex sets.

Theorem 3. 1

Let $\hat{w}_k$, $k = 1, \ldots K$; $\hat{p}_j$, $j = 1, \ldots, n$ be an optimal solution of (3. 7). Then $\hat{w}_k$ is equal to $\underline{w}_k$ or $\bar{w}_k$ for all k. Also, p_j is equal to $\underline{p}_j$ or $\bar{p}_j$ except possibly for one index j_0.

Proof

W and P are bounded polyhedral convex sets. Hence by the fundamental theorem of bilinear programming [8], there exists an optimal solution $(\hat{w}, \hat{p})$ where $\hat{w}$ and $\hat{p}$ are extreme points of W and P, respectively. It is easy to see that any extreme point of W and P has the property stated in the theorem. ||

Using this theorem, we can construct a simple enumeration technique by fixing w_k equal to $\underline{w}_k$ or $\bar{w}_k$. Also it may be more appropriate in some cases to normalize w_k, $k = 1, \ldots, K$ to satisfy the condition $\sum_{k=1}^{K} w_k = 1$, as well as p_j, in which case we still have a bilinear program with somewhat more complicated structure. For the background material of decision analysis the readers are referred to Keeney-Raiffa [7] and to Sarin [13].

4. NON-STANDARD GENERALIZED LINEAR PROGRAM

Generalized linear program (GLP) introduced by Dantzig and Wolfe [1] has the following problem structure :

$$\min \left\{ \sum_{j=1}^{n} c_j x_j \mid \sum_{j=1}^{n} a_j x_j = b, \quad x_j \geq 0, \begin{pmatrix} c_j \\ a_j \end{pmatrix} \epsilon C_j, \ j = 1, \ldots, n \right\} \tag{4. 1}$$

where $a_j \epsilon R^m$, $c_j \epsilon R^1$ and $C_j \subset R^{m+1}$ is a compact convex set $j = 1, \ldots, n$ and maximization is with respect to $\begin{pmatrix} c_j \\ a_j \end{pmatrix}$ as well as x_j. The GLP algorithm by Dantzig and Wolfe proceeds roughly as follows:

Let $\begin{pmatrix} \tilde{c}_j^{\ell} \\ \tilde{a}_j^{\ell} \end{pmatrix} \epsilon C_j$, $\ell = 1, \ldots, \ell_j$, $j = 1, \ldots, n$ be given.

Then we will solve the linear program :

$$\min \left\{ \sum_{j=1}^{n} \sum_{\ell=1}^{\ell_j} \tilde{c}_j^{\ell} x_j^{\ell} \mid \sum_{j=1}^{n} \sum_{\ell=1}^{\ell_j} \tilde{a}_j^{\ell} x_j^{\ell} = b, \ x_j^{\ell} \geq 0, \ \ell = 1, \ldots, \ell_j ; \ j = 1, \ldots, n \right\} \tag{4. 2}$$

and let $\tilde{\pi} \epsilon R^m$ be an optimal multiplier vector for this linear program.

If

$$c_j - \tilde{\pi} a_j \geq 0 \quad \forall \begin{pmatrix} c_j \\ a_j \end{pmatrix} \epsilon C_j ; \quad j = 1, \ldots, n .$$

then the current solution is optimal. If, on the other hand, there is an index j and a vector $\begin{pmatrix} c_j \\ a_j \end{pmatrix} \epsilon C_j$ for which $c_j - \tilde{\pi} a_j > 0$, then the objective function will be improved by introducting this vector into the basis. To find out the vectors $\begin{pmatrix} c_j \\ a_j \end{pmatrix}$ for which $c_j - \tilde{\pi} a_j > 0$, we solve the following n sub-programs.

$$\min \left\{ c_j - \tilde{\pi} a_j \mid \begin{pmatrix} c_j \\ a_j \end{pmatrix} \epsilon C_j \right\}, \quad j = 1, \ldots, n . \tag{4. 3}$$

Let $\begin{pmatrix} c_j^* \\ a_j^* \end{pmatrix}$ be its optimal solution. If $c_j^* - \tilde{\pi} a_j^* < 0$, then we will introduce it into (4. 2) and proceed.

If C_j are all polyhedral convex sets, then this algorithm will converge to the optimal solution of (4. 1) in finitely many steps if we avoid cycling caused by degeneracy appropriately.

Now let us consider the non-standard GLP with some free variables, i. e.,

$$\begin{aligned} \min \ & \sum_{j=1}^{n} c_j x_j \\ \text{s. t.} \ & \sum_{j=1}^{n} a_j x_j = b \\ & x_j \geq 0, \quad j = 1, \ldots, \ell ; \\ & x_j \gtrless 0, \quad j = \ell + 1, \ldots, n ; \\ & \begin{pmatrix} c_j \\ a_j \end{pmatrix} \epsilon C_j, \quad j = 1, \ldots, n . \end{aligned} \tag{4. 4}$$

The usual technique of replacing a free variable by two non-negative variables destroys the structure of the problem, i. e., let

$$x_j = x_{j_1} - x_{j_2}\,,\quad x_{j_1} \geq 0\,,\ x_{j_2} \geq 0\,,\qquad j = \ell + 1\,, \ldots\ldots, n\,.$$

then the problem is

$$\min \sum_{j=1}^{\ell} c_j x_j + \sum_{j=\ell+1}^{n} c_{j_1} x_{j_1} - \sum_{j=\ell+1}^{n} c_{j_2} x_{j_2}$$

$$\text{s. t.} \quad \sum_{j=1}^{\ell} a_j x_j + \sum_{j=\ell+1}^{n} a_j x_{j_1} - \sum_{j=\ell+1}^{n} a_j x_{j_2} = b$$

$$x_j \geq 0\,,\qquad j = 1\,, \ldots\ldots, \ell$$

$$x_{j_1}\,,\ x_{j_2} \geq 0\,,\qquad j = \ell + 1\,, \ldots\ldots, n$$

$$\begin{pmatrix} c_j \\ a_j \end{pmatrix} \in C_j\,,\qquad j = 1\,, \ldots\ldots, \ell\,.\quad \begin{pmatrix} c_{j_1} \\ a_{j_2} \end{pmatrix} = \begin{pmatrix} c_{j_2} \\ a_{j_2} \end{pmatrix} \in C_j\,,\qquad j = \ell + 1\,, \ldots\ldots, n\,. \tag{4. 5}$$

Hence the columns of this problem are no longer independent and GLP algorithm in its original form would not work.

Now let us consider the simplest case of the above in which a_j 's are constant and only c_j 's are allowed to move in compact convex sets, i. e., closed interval in this case:

$$\min \sum_{j=1}^{\ell} c_j x_j + \sum_{j=\ell+1}^{n} c_j x_j$$

$$\text{s. t.} \quad \sum_{j=1}^{\ell} a_j x_j + \sum_{j=\ell+1}^{n} a_j x_j = b$$

$$x_j \geq 0\,,\quad j = 1\,, \ldots\ldots, \ell\,;$$

$$\underline{c}_j \leq c_j \leq \overline{c}_j\,,\quad j = 1\,, \ldots\ldots, n\,. \tag{4. 6}$$

Since $x_j \geq 0$, $j = 1\,, \ldots\ldots, \ell$, it is obvious that optimal c_j 's are $\overline{c}_j$'s for $j = 1\,, \ldots\ldots, \ell$. Hence the problem simplifies somewhat to

$$\min \sum_{j=1}^{\ell} \overline{c}_j x_j + \sum_{j=\ell+1}^{n} y_j x_j$$

$$\text{s. t.} \quad \sum_{j=1}^{\ell} a_j x_j + \sum_{j=\ell+1}^{n} a_j x_j = b$$

$$x_j \geq 0\,,\qquad j = 1\,, \ldots\ldots, \ell\,;$$

$$\underline{c}_j \leq y_j \leq \overline{c}_j\,,\qquad j = \ell + 1\,, \ldots\ldots, n\,. \tag{4. 7}$$

We will use the standard elimination technique to obtain an expression of x_k, $k = \ell+1$, ... n with respect to x_j , $j = 1\,, \ldots\ldots, \ell$.

Let

$$x_j = d_{jo} + \sum_{k=1}^{\ell} d_{jk} x_k , \qquad j = \ell + 1 , \ldots\ldots , n . \tag{4.8}$$

Substituting these into (4. 7), we obtain

$$\min \sum_{j=1}^{\ell} [\, \bar{c}_j + \sum_{k=\ell+1}^{n} d_{kj} y_k \,] \; x_j + \sum_{j=\ell+1}^{n} d_{jo} y_j$$

$$\text{s. t.} \quad \sum_{j=1}^{\ell} a_j' x_j = b'$$

$$x_j \geq 0, \qquad j = 1 , \ldots\ldots , \ell$$

$$\underline{c}_j \leq y_j \leq \bar{c}_j \qquad j = \ell + 1 , \ldots\ldots , n . \tag{4.9}$$

which is a BLP with a special structure. The following theorem characterizes the form of the optimal solution.

Theorem 4. 1

Let c_j^* , x_j^* , $j = 1, \ldots\ldots , n$ be an optimal solution (it is exists at all) of (4. 4). Then $c_j^* = \underline{c}_j$, $j = 1, \ldots\ldots \ell$ and c_j^* is either $\underline{c}_j$ or $\bar{c}_j$ for $j = \ell + 1 , \ldots\ldots , n$.

Proof

By the fundamental theorem of BLP [8] , there is an optimal solution $y^* = (y_{\ell+1}^* , \ldots\ldots , y_n^*)$ where y^* is an extreme point of the constraint set $\{ (y_{\ell+1} , \ldots\ldots , y_n) \mid \underline{c}_j \leq y_j \leq \bar{c}_j , \; j = \ell + 1 , \ldots\ldots n \}$. | |

We have shown that bilinear programming technique gives a way to solve (4. 4). This need not, of course, be the best way to solve this class of problems. Typically, the modified version of generalized linear programming algorithm might be able to solve them more efficiently. We will not, however, go into this subject in more detail here.

5. COMPLEMENTARY PLANNING PROBLEMS

Let us consider the following class of problems

$$\text{minimize} \quad c_1^t x_1 + d_1^t y_1 + c_2^t x_2 + d_2^t y_2$$

$$\text{s. t.} \quad A_1 x_1 + B_1 y_1 \geq b_1$$

$$A_2 x_2 + B_2 y_2 \geq b_2$$

$$x_1 \geq 0 , \quad y_1 \geq 0 , \quad x_2 \geq 0 , \quad y_2 \geq 0$$

$$x_1^t x_2 = 0 \tag{5.1}$$

where c_1 , $c_2 \in R^{\ell}$, $d_i \in R^{n_i}$, $A_i \in R^{m_i \times \ell}$, $B_i \in R^{m_i \times n_i}$, $b_i \in R^{m_i}$, $i = 1, 2$ and x_i , y_i are variable vectors of appropriate dimensions. The last constraint $x_1^t x_2 = 0$ will be called complementary constraints in the sequel.

More general problems with complementary constraints

$$\begin{aligned}
\text{minimize} \quad & c_1^t x_1 + c_2^t x_2 + d^t y \\
\text{s. t.} \quad & A_1 x_1 + A_2 x_2 + By \geq b \\
& x_1 \geq 0 , \quad x_2 \geq 0 , \quad y \geq 0 \\
& x_1^t x_2 = 0 .
\end{aligned} \tag{5. 2}$$

has been discussed by Ibaraki [6], who proved the following theorem and porposed an enumeration type of algorithm.

Theorem 5. 1

If the constraint set of (5. 2) is bounded, then (5. 2) has an optimal solution among basic feasible solutions.

There are many real world applications of (5. 1) and (5. 2) such as complementary flow problems, orthogonal scheduling problems to name only a few.

The best known technique to solve (5. 1) is to introduce an ℓ-dimensional vector u of 0–1 components and replace the constraints $x_1^t x_2 = 0$, $x_1 \geq 0$, $x_2 \geq 0$ by:

$$\begin{aligned}
& x_1 \leq M_O u \\
& x_2 \leq M_O (e_\ell - u) \\
& x_1 \geq 0 , \quad x_2 \geq 0 .
\end{aligned} \tag{5. 3}$$

Here e_ℓ is the ℓ dimensional vector all of whose components are 1's and M_O is a constant satisfying

$$M_O \geq \max \left\{ e_\ell^t x_i \mid A_i x_i + B_i y_i \geq b_i, \ x_i \geq 0, \ y_i \geq 0 \right\}, \ i = 1, 2$$

Hence (5. 1) is equivalent to the following mixed 0–1 integer programming problem:

$$\begin{aligned}
\text{minimize} \quad & c_1^t x_1 + d_1^t y_1 + c_2^t x_2 + d_2^t y_2 . \\
\text{s. t.} \quad & A_1 x_1 + B_1 y_1 \geq b_1 \\
& A_2 x_2 + B_2 y_2 \geq b_2 \\
& x_1 - M_O u \leq 0 \\
& x_2 + M_O u \leq M_O e_n \\
& x_1 \geq 0 , \quad y_1 \geq 0 , \quad x_2 \geq 0 , \quad y_2 \geq 0 \\
& u = (u_1 , u_2 , \ldots\ldots , u_\ell) \\
& u_j = 0 \text{ or } 1 , \ j = 1 , \ldots\ldots , \ell .
\end{aligned} \tag{5. 4}$$

This can be solved by a usual branch and bound technique if ℓ is not too large. Instead, we will propose another classical approach, i. e., penalty function approach by putting $x_1^t x_2 = 0$ term into the objective function:

$$\begin{aligned} \text{maximize} \quad & c_1^t x_1 + d_1^t y_1 + c_2^t x_2 + d_2^t y_2 - M x_1^t x_2 \\ \text{s. t.} \quad & A_1 x_1 + B_1 y_1 \geq b_1 \\ & A_2 x_2 + B_2 y_2 \geq b_2 \\ & x_1 \geq , \quad y_1 \geq 0, \quad x_2 \geq 0, \quad y_2 \geq 0. \end{aligned} \tag{5.5}$$

which is a BLP in canonical maximization form.

Theorem 5. 2

If the constraint set of (5. 1) is bounded, then there exists a constant M_O such that (5. 1) is equivalent to (5. 5) for $M > M_O$.

Proof

This can be proved by a standard technique and will be omitted here.

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REPORT DOCUMENTATION PAGE	REPORT NUMBER EIS-TR-76- 4

TITLE

SOME APPLICATIONS OF BILINEAR PROGRAMMING

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REPORT DATE	NUMBER OF PAGES
June 24, 1976	14
MAIN CATEGORY	CR CATEGORIES
Mathematical Programming	5.4

KEY WORDS

nonconvex quadratic programming, bilinear objective function, location-allocation problem, decision analysis, generalized linear program, complementary problem.

ABSTRACT

Some of the more important applications of bilinear programming, which is a technique for solving a special class of nonconvex quadratic programs, are discussed. Topics included are location-allocation problems, decision analytic problems, nonstandard generalized linear programs and complementary planning problems. The relationship of bilinear programs to other classes of classical mathematical programming problems are also discussed in some detail.

SUPPLEMENTARY NOTES