MAGNITUDES OF PARETO INEFFICIENCY OF

ATOMIC AND NON-ATOMIC NASH EQUILIBRIA

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Magnitudes of Pareto Inefficiency of Atomic and Non-Atomic Nash Equilibria

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Abstract

We present a measure that shows the magnitude of Pareto inefficiency, and show how to obtain it under certain conditions. We prove that the price of anarchy (PoA) presents the magnitude of Pareto inefficiency of the unique Nash equilibrium in the case where the Nash equilibrium and the social optimum are both symmetric although utilities of players may be asymmetric in other feasible states. That is, in that case, PoA is identical to the measure. Then we study the general trend of how the magnitude of Pareto inefficiency depends on the number of non-cooperative players in Nash equilibria, *i.e.,* on the degree of dispersion in decision making. We examine a symmetric economic game and a symmetric congestion game for examining the trend. We show, respectively, in the economic and congestion games, the magnitude of Pareto inefficiency of the Nash and atomic Nash equilibrium (and also PoA therein) can increase without bound.

Keywords: Pareto inefficiency, atomic and non-atomic Nash equilibrium, oligopoly, monopoly, perfect competition, congestion game, load balancing, Wardrop equilibrium, social optimum, price of anarchy.

1 Introduction

Decision makers in many systems can be regarded as players in games. Non-cooperative decisions which lead to Nash equilibria have the advantage of independence and distribution of decision making. Non-cooperative decisions, however, may not always be beneficial. Nash equilibria may be Pareto inefficient. It looks that the definition of the magnitude of Pareto inefficiency has not been settled yet. In this article, we present a measure that shows the magnitude of Pareto inefficiency. Then, we examine how the magnitude of Pareto inefficiency of the Nash equilibrium depends on the number of non-cooperative players.

The price of anarchy (Koutsoupias and Papadimitriou, 1999) is widely used for evaluating the degree of ineffectiveness of Nash equilibria. If the Nash equilibrium is unique, the price of anarchy, PoA, gives the ratio of the social cost of the Nash equilibrium to that of the social optimum. In fact, however, it does not seem to have been rigorously shown that PoA can serve as a measure that shows the magnitude of Pareto inefficiency of Nash equilibria for general cases. In section 2, we prove that PoA can do so in the case where all players have the same values of utilities in the unique Nash equilibrium and the social optimum (symmetric, homogeneous), although other feasible system states may be asymmetric (*i.e.,* the utility of every player in each state may not be identical). We note that there can exist at most one symmetric Nash equilibrium in each non-cooperative game. In the examples given in section 3, all players have the same value of utilities in the Nash equilibrium and the social optimum, and then the measure of the magnitude of Pareto inefficiency gives the same value as the price of anarchy.

The level of dispersion of decision making for systems reflects the number of non-cooperative players. In this article, we think of various levels of non-cooperative decision making, that is, of various numbers of players in non-cooperative games. In subsection 1.1, we examine the degrees in the dispersion of decision making or the number of players in non-cooperative games.

1.1 Different Degrees in the Dispersion of Decision Making

We can think of different degrees in the dispersion of decision making.

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(A)[**Completely centralized decision, monopoly**]: There is only one decision maker or only one player. The decision maker seeks to optimize a single social measure such as the overall utility or the total cost over all users (*e.g.,* the expected sojourn time). If there are multiple users or agents in a system, then the only one decision maker distributes the optimal benefit or cost among all users or agents. In the literature, the corresponding solution concept is referred to as a system optimum, overall optimum, cooperative optimum or social optimum. In economics, this situation is identical to that of *monopoly*. In this article, we call this optimum *social optimum*.

(B)[**Intermediately dispersed decision, non-cooperative games with atomic players, oligopoly**]: There are a finite number ($N(>1)$) of decision makers, agents, or players. Each decision maker optimizes non-cooperatively his/her own utility or cost (*e.g.,* the expected sojourn time). The decision of a single decision maker has a *non-negligible* impact on the performance of other groups. We refer to the situation where each player's decision has a non-negligible impact on the system state as "atomic." The situation where, in such a scheme, every player has optimized his/her decision, given the decisions of other users, and furthermore, would not unilaterally deviate from this decision is called a *Nash equilibrium*. In this optimized situation, each of a finite number of players cannot receive any further benefit by changing his/her decision. In the literature, the corresponding solution concept is referred to as a class optimum or Nash equilibrium with atomic players. This situation is identical to that of *oligopoly* in economics. We may have different levels in intermediately dispersed optimization according to the size of N. In this article, we call this equilibrium *atomic Nash equilibrium*.

(C)[**Completely dispersed decision, non-cooperative games with non-atomic players, perfect competition**]: In this situation, the decision of a single decision maker has a *negligible*, infinitesimal impact on the performance of the entire system, since, probably, the number of such players is so large. In contrast, the status of the entire system is determined from the whole decisions made by all players. We refer to the situation as "non-atomic." Each player optimizes his/her own utility or cost (*e.g.,* his/her own expected sojourn time), independently and selfishly of the others. In this optimized situation, each player cannot expect any further benefit by changing his/her own decision. The situation where every infinitesimal player has optimized his/her decision, given the entire system status, and would not unilaterally deviate from that choice, is called individual optimum, Wardrop equilibrium, or user optimum (Wardrop, 1952; Haurie and Marcotte, 1985; Patriksson, 1994) *etc.*). This situation is identical to that of *perfect competition* in economics. In this article, we call this equilibrium *non-atomic Nash equilibrium*.

Note that (B) is reduced to (A) when the number of players reduces to 1 ($N = 1$) and approaches (C) when the number of players increases without bound ($N \to \infty$) (Haurie and Marcotte, 1985). In the cases of (B) and (C), there are multiple players or independent decision makers and they can be regarded as 'non-cooperative games.'

1.2 Pareto Superiority and Efficiency

We think that, for each state of the system, the utility (or cost) of each player is determined. We recall the definition of superiority/inferiority among system states as presented in the following paragraph. The notions of Pareto superiority/inferiority and optimality/inefficiency have already been established, and we confirm the notions and their definitions.

[**Pareto superiority and inferiority**]: We consider a system consisting of a number of users or players $\boldsymbol{n}$, where $\boldsymbol{n}$ denotes the set $\{1, 2, \cdots, n\}$. Denote by S the system state $(s_1, s_2, \cdots, s_n)$ where s_i denotes the decision made by player i, $i \in \boldsymbol{n}$. Denote by $\boldsymbol{S}$ the set of feasible system states each of which presents a realizable combination of player decisions. For each state of the system, each player has his/her own utility. Denote a combination of utilities of all players in a system state $S \in \boldsymbol{S}$ by $\boldsymbol{U}(S) = (U_1(S), U_2(S), \ldots, U_n(S))$. In general, we consider the cases where $U_i(S)$ has a positive real value, $U_i(S) > 0$, for all $i \in \boldsymbol{n}$, $S \in \boldsymbol{S}$.

Consider an arbitrary pair of two (achievable) states of the system, S^a, S^b, $\in \boldsymbol{S}$. If $U_i(S^a) \leq U_i(S^b)$ for all $i \in \boldsymbol{n}$ and $U_j(S^a) < U_j(S^b)$ for some $j \in \boldsymbol{n}$, then S^a is Pareto inferior to S^b and S^b is Pareto superior to S^a. In the cases where $U_i(S^a) = U(S^a)$ and $U_i(S^b) = U(S^b)$ for all $i \in \boldsymbol{n}$, that is, the resulting utility of every player is identical (symmetric) within each state, the degree of Pareto inferiority (superiority) between them can simply defined to be, for example, $U(S^a)/U(S^b)$. In general (including asymmetric cases), however, the Pareto superiority/inferiority relations induce partial ordering in the set of system states and is not subject to total ordering or single scalar measure, like the ratio, $\sum_p U_p(S^a)/\sum_p U_p(S^b)$, of the social cost of state S^b to that of S^b. Therefore, we may rely on more complicated measures such as follows. A definition of the measure of Pareto superiority/inferiority, respectively, of state S^a to S^b has been given as (Kameda, 2009; Kameda, 2013)

$$P_/(S^a, S^b) \triangleq \min_{p \in \boldsymbol{n}} U_p(S^a)/U_p(S^b) \text{ and } Q_/(S^a, S^b) \triangleq \min_{p \in \boldsymbol{n}} U_p(S^b)/U_p(S^a) \tag{1}$$

Then, $P_{/}(S^a, S^b) <$ and ≥ 1, respectively, if S^a is Pareto inferior/indifferent and superior to S^b. $Q_{/}(S^a, S^b) <$ and ≥ 1, respectively, if S^a is Pareto superior/indifferent and inferior to S^b.

[**Pareto optimality and efficiency**]: If there exists no system state that is Pareto superior to a system state, the latter state is called a *Pareto optimum* or *efficient* state. If there exists some system state that is Pareto superior to a system state, the latter state is called a *Pareto inefficient* state. In general, multiple Pareto-optimal states may exist for a system. An overall (or social) optimum is evidently Pareto optimum. Non-atomic Nash equilibria (Wardrop equilibria) and Nash equilibria may be Pareto optimal or inefficient.

2 Magnitudes of the Pareto inefficiency of Nash Equilibria

[**The magnitude of Pareto inefficiency**]: We consider such a definition of the degree of Pareto inferiority $Q(S, S^b)$ of system state S to S^b that satisfies the following: $Q(S^a, S^b) <$ and ≥ 1, respectively, if S^a is Pareto superior/indifferent and inferior to S^b. $Q_{/}(S, S^b)$ given in relation (1) is an example of this. Then, naturally, we have the magnitude of Pareto inefficiency of a system state S^a, simply by

$$MoI(S^a) \triangleq \max_{S \in \boldsymbol{S}} Q(S^a, S) \tag{2}$$

(the larger magnitude for the greater inefficiency). It shows the maximum ratio in which the Pareto inefficient state can be improved.

Proposition 1 *$MoI(S) = 1$ if S is Pareto optimal, and $MoI(S) \geq 1$ if S is Pareto inefficient. Thus, $MoI(S)$ can serve as a measure of the magnitude of Pareto inefficiency of system state S.*

[Proof] If S^a is Pareto optimal, then $Q(S, S^a) < 1$ for all $S \neq S^a$, and $Q(S^a, S^a) = 1$, and thus $MoI(S^a) = 1$. If S^a is Pareto inefficient, then $Q(S, S^a) \geq 1$ for some $S \neq S^a$, and thus $MoI(S^a) \geq 1$. □

On the basis of the definition on Pareto inferiority given in relation (1), a definition of the magnitude of Pareto inefficiency $MoI(S^a)$ of a system state S^a is given as

$$\max_{S \in \boldsymbol{S}} \min_{p \in \boldsymbol{n}} U_p(S)/U_p(S^a) \text{ for the utility base and}$$
$$\max_{S \in \boldsymbol{S}} \min_{p \in \boldsymbol{n}} C_p(S^a)/C_p(S) \text{ for the cost base} \tag{3}$$

where $C_i(S)$ denotes the cost for player i in system state S (Legrand and Touati, 2007).

Define the weighted sum of the utilities of players for state $S \in \boldsymbol{S}$ with constant $\boldsymbol{V} > \boldsymbol{0}$:

$$O(\boldsymbol{U}(S), \boldsymbol{V}) = \sum_p U_p/V_p \quad \text{where } \boldsymbol{V} = (V_1, V_2, \cdots, V_n), V_i > 0, i \in \boldsymbol{n}. \tag{4}$$

Theorem 1 *Given a system state $\grave{S}$ with each player's utility $\boldsymbol{U}(\grave{S}) = (\grave{U}_1, \grave{U}_2, \ldots, \grave{U}_N)$, we have the weighted social optimum $\boldsymbol{U}(\bar{S}) = (\bar{U}_1, \bar{U}_2, \ldots, \bar{U}_N)$ such that $O(\boldsymbol{U}(\bar{S}), \boldsymbol{U}(\grave{S})) = \max_{S \in \boldsymbol{S}} O(\boldsymbol{U}(S), \boldsymbol{U}(\grave{S}))$. Assume that we can arrange such that $\bar{U}_i/\grave{U}_i = \bar{U}/\grave{U}$, $i \in \boldsymbol{n}$. The magnitude of Pareto inefficiency, $MoI(\grave{S})$, of $\grave{S}$ is obtained such that $MoI(\grave{S}) = \bar{U}/\grave{U}$. In this case, $MoI(\grave{S}) = 1$ and > 1, respectively, if $\grave{S}$ is Pareto optimal and inefficient. Thus, the magnitude of Pareto inefficiency, $MoI(\grave{S})$, of $\grave{S}$ determines the Pareto optimality of state $\grave{S}$.*

[Proof] Note that $O(\boldsymbol{U}((\bar{S}), \boldsymbol{U}(\grave{S})) = \max_{S \in \boldsymbol{S}} O(\boldsymbol{U}(S), \boldsymbol{U}(\grave{S}))$, and that we arrange such that $\bar{U}_i/\grave{U}_i = \bar{U}/\grave{U}$, $i \in \boldsymbol{n}$. Consider another state $S' \in \boldsymbol{S}$. Since $\bar{S}$ is the weighted social optimum, then $O(\boldsymbol{U}(S')) \leq O(\boldsymbol{U}(\bar{S}))$. Therefore, there must exist some i ($i \in \boldsymbol{n}$) such that $U'_i/\grave{U}_i \leq \bar{U}_i/\grave{U}_i = \bar{U}/\grave{U}$. Then, $Q(\grave{S}, S') = \min_p U'_p/\grave{U}_p \leq \bar{U}/\grave{U}$. Then $MoI(\grave{S}) = \max[\{\max_{S \in \boldsymbol{S}} Q(\grave{S}, S)\}, \bar{U}/\grave{U}] = \bar{U}/\grave{U}$. Thus, the magnitude of Pareto inefficiency $MoI(\grave{S})$ of $\grave{\boldsymbol{U}}$ is given by $\grave{U}/\bar{U}$. The last two statements are clear from the relation between $\boldsymbol{U}(\grave{S})$ and $\boldsymbol{U}(\bar{S})$. □

As to the existence of such a weighted social optimum $\bar{\boldsymbol{U}}$ that $\tilde{U}_i/\bar{U}_i = \tilde{U}/\bar{U}$, $i \in \boldsymbol{n}$, for a Nash equilibrium, $\tilde{\boldsymbol{U}}$, we refer to the Nash equilibrium based fair allocation (Kameda, Altman, Touti and Legrand, 2012).

Furthermore, we have the following:

Corollary 1 *Assume that we have a Nash equilibrium $\tilde{S}$ with each player's utility $\boldsymbol{U}(\tilde{S}) = (\tilde{U}_1, \tilde{U}_2, \ldots, \tilde{U}_N)$, that is symmetric, $\tilde{U}_i = \tilde{U}$, $i \in \boldsymbol{n}$. Let the social optimum $\boldsymbol{U}(\bar{S}) = (\bar{U}_1, \bar{U}_2, \ldots, \bar{U}_N)$ such that $O(\boldsymbol{U}(\bar{S}), \mathbf{1}) = \max_{S \in \boldsymbol{S}} O(\boldsymbol{U}(S), \mathbf{1})$, where $\mathbf{1} = (1, 1, \cdots, 1)$. Assume that we can arrange such that $\bar{U}_i = \bar{U}$, $i \in \boldsymbol{n}$ (the symmetric social optimum). Then, the magnitude of Pareto inefficiency $MoI(\tilde{S})$ of $\tilde{S}$ is obtained such as $\bar{U}/\tilde{U}$. $MoI(\tilde{S}) = 1$ and > 1, respectively, if $\tilde{S}$ is Pareto optimal and inefficient. That is, the magnitude of Pareto inefficiency, $MoI(\tilde{S})$, of $\tilde{S}$ determines the Pareto optimality of state $\grave{S}$.*

In subsections 3.1 and 3.2, we see examples of social optimum $\bar{S}$, that is (arranged to be) symmetric, $\bar{U}_i = \bar{U}$, $i \in \boldsymbol{n}$. By using the above corollary 1, we have $MoI(\tilde{S}) = \bar{U}/\tilde{U}$ for the utility base scheme and $MoI(\tilde{S}) = \tilde{C}/\bar{C}$ for the cost base scheme.

The price of anarchy, PoA, for the case of unique Nash equilibrium $\tilde{S}$ (the ratio of the social cost of the Nash equilibrium to that of the social optimum) is $PoA \triangleq \sum_p \tilde{C}_p / \sum_p \bar{C}_p$. If both of $\boldsymbol{U}(\tilde{S})$ and $\boldsymbol{U}(\bar{S})$ are symmetric, then $PoA = \tilde{C}/\bar{C}$, which is the same as $MoI(\tilde{S})$.

Corollary 2 *The price of anarchy presents the magnitude of Pareto inefficiency of the unique Nash equilibrium if the utility of each player is the same (symmetric) within the Nash equilibrium and if that of each player is the same (symmetric) within the social optimum, although the utilities of feasible states may not necessarily be symmetric. The price of anarchy and the magnitude of Pareto inefficiency of the unique Nash equilibrium determines the Pareto optimality of the Nash equilibrium.*

Consider the following congestion game: The network consists of an origin, a destination and paths (series of links) connecting the origin and the destination (simple examples are the Braess and Pigou networks (Braess, 1968; Pigou, 1920). Packets are forwarded from the origin to the destination. The cost for each packet is the sojourn (passage) time through the origin and the destination. By suitably assigning packets to the atomic and non-atomic players, we can obtain symmetric costs for players in the unique Nash equilibrium and in the social optimum. Then, we can apply the above corollary 2, and we see that the price of anarchy presents the magnitude of Pareto inefficiency of the congestion game. In the examples given in the subsections 3.1 and 3.2, we see the social optimum and the Nash equilibria are symmetric, and then we can apply the corollary 1 to the examples.

[Pareto inefficiency of atomic and non-atomic Nash equilibria]: It is evident that the social optima are Pareto optimal. Both atomic and non-atomic Nash equilibria may be Pareto inefficient, as exemplified in the Braess paradox on transportation and communication networks (Braess, 1968; Murchland, 1970; Frank, 1981; Cohen and Kelly, 1990; Cohen and Jeffries, 1997; Korilis, Lazar and Orda, 1995; Korilis, Lazar and Orda, 1999; Kameda, 2002; Kameda, 2009)), and in the Braess-like paradox on distributed computer systems (Kameda, Altman, Kozawa and Hosokawa, 2000; Kameda and Pourtallier, 2002), and in the well-known prisoners' dilemma, *etc.*

[Atomic Nash equilibria] It has been shown that, if the utility function of each player is twice continuously differentiable, *Nash equilibria are generally Pareto inefficient* in smooth games with the finite number of players (Smale, 1973; Dubey, 1986). In addition, *the magnitude of Pareto inefficiency of some atomic Nash equilibria can increase without bound* as seen in the congestion game given in subsection 3.2. Thus, one may think that atomic Nash equilibria are more likely Pareto inefficient.

[Non-atomic Nash equilibria] In contrast, there have been studies that seek the bounds of the magnitude of Pareto inefficiency of non-atomic Nash equilibria for congestion games as in (Roughgarden and Tardos, 2004a; Roughgarden and Tardos, 2004b). Furthermore, we have not seen the cases where the magnitude of Pareto inefficiency of non-atomic Nash equilibria can increase without bound. Moreover, we note that there exist a class of schemes given in general economic theory called Walrasian equilibria (under perfect competition) that have been shown to be Pareto optimal (well known as the first fundamental theorem of welfare economics) (Arrow, 1951; Debreu, 1951). The Walrasian schemes depend on the pricing mechanism with perfect competition where each of consumers and producers behaves as a price-taker and can have only infinitesimal effects on the prices by changing his/her decision. These schemes are regarded as examples of Pareto-optimal non-atomic Nash equilibria.

The above mentioned situations around atomic vs. non-atomic Nash equilibria may bring us such a conjecture that the magnitudes of Pareto inefficiency in atomic Nash equilibria may, on the whole, be greater than those of non-atomic Nash equilibria. This conjecture is betrayed as follows. We can see the cases where the magnitudes of Pareto inefficiency of atomic Nash equilibria, respectively, decreases and increases as the number of player N increases in the games given in the examples in the subsections 3.1 and 3.2. In addition, we see in the subsection 3.1 that the magnitude of Pareto inefficiency of non-atomic Nash equilibria for some economic games can increase without bound.

3 Examples

We present case studies on two games, economic and congestion.

3.1 Monopoly, Oligopoly and Perfect Competition — An Economic Model

Herein we investigates a system consisting of a market and multiple producers that produce a commodity of the single kind, *i.e.,* a Cournot oligopoly game wherein the players that make decisions are producers. Each producer optimizes his/her profit in producing the amount of commodity demanded in the market. In this game, consumers are not regarded as separate players. Their behaviors are reflected in the demand function that determines the price of the commodity.

3.1.1 The Model and Assumptions

The system considered here consists of n producers and a market. Producers are numbered $1, 2, \cdots, n$. Denote by $\boldsymbol{n}$ the set $\{1, 2, \cdots, n\}$. All producers produce the homogeneous commodity, and in each market, the consumers demand it. Consider the three cases as to the market.

1) Monopoly: Each producer follows the decision by only one decision maker.

2) Oligopoly: Each producer makes unilaterally his/her own decision. His/her decision may have non-negligible impact on the price.

3) Perfect competition: Each producer makes decision given the price to which he/she can exercise only infinitesimal influence. Each producer behaves as the price-taker, and the price is determined by the total amount of the commodity demanded and produced.

Let q_i ($q_i \geq 0$) denote the quantity that producer i produces, $i \in \boldsymbol{n}$. Define vector $\boldsymbol{q}$ such that $\boldsymbol{q} \triangleq (q_1, q_2, \cdots, q_n)$. q_i is the variable determined by producer i. $i \in \boldsymbol{n}$. Then, we assume an inverse demand function such as $p = a(1 - \sum_j q_j/b)$ ($a, b > 0$,), where a is the upper bound of the price in the market and b is the upper bound of the demand for the commodity in the market. For the market, if the price is p, the quantity demanded is given by $p = a\left(1 - \sum_j q_j/b\right)$.

Assume that the cost that producer i produces the amount q_i of the commodity is cq_i. c represents the marginal cost.

1) **Monopoly**: The profit P_0^i of producer i is the following: $P_0^i = a\{1 - (\sum_j q_j)/b\}q_i - cq_i$.
Then the total sum of the profits of producers is $P_0(\boldsymbol{q}) = \sum_j P_0^j = a(1 - q/b)q - cq$, where $q = \sum_j q_j$.
Therefore, the optimal decision by the monopolized producers is expressed as: $\max_{\boldsymbol{q}} P_0$.
Denote by $\bar{\boldsymbol{q}} = (\bar{q}_1, \bar{q}_2, \cdots, \bar{q}_n)$, the set of values of $q_1, q_2, \cdots, q_n$ that satisfy $P_0(\bar{\boldsymbol{q}}) = \max_{\boldsymbol{q}} P_0(\boldsymbol{q})$.

2) **Oligopoly**: The above case 2) can be regarded as a game where the producers are players and where the profit of each producer i is the utility of the player i, $i \in \boldsymbol{n}$. The profit P^i of producer i is the following: $P^i = a\left(1 - \sum_j q_j/b\right)q_i - cq_i$.
Therefore, the optimal decision by producer i is expressed as follows: $\max_{q_i} P^i$.
Denote by $\tilde{\boldsymbol{q}} = (\tilde{q}_1, \tilde{q}_2, \cdots, \tilde{q}_n)$ such values of $q_1, q_2, \cdots, q_n$ that satisfy, for $i \in \boldsymbol{n}$,

$$P^i(\tilde{\boldsymbol{q}}) = \max_{q_i} P^i(\tilde{q}_1, \cdots, \tilde{q}_{i-1}, q_i, \tilde{q}_{i+1}, \cdots, \tilde{q}_n), \text{ given } \tilde{q}_1, \cdots, \tilde{q}_{i-1}, \tilde{q}_{i+1}, \cdots, \tilde{q}_n. \tag{5}$$

If the system has such a solution $\tilde{\boldsymbol{q}}$ of $\boldsymbol{q}$ that satisfies (5) for all i at the same time, it is an *atomic Nash equilibrium*.

3) **Perfect competition**: Denote by Q be $\sum_p q_p$. The profit P_0^i of producer i is the following: $P_c^i = a(1 - Q/b)q_i - cq_i$.
Each producer increases q_i as far as $a(1 - Q/b) > c$, which finally results in $Q = \hat{Q} = b(a - c)/a$.
Then, the profit of each producer i is $P_c^i(\boldsymbol{q}) = 0$.
Denote by $\hat{\boldsymbol{q}} = (\hat{q}_1, \hat{q}_2, \cdots, \hat{q}_n)$, the set of values of $q_1, q_2, \cdots, q_n$ that satisfy $\sum_p \hat{q}_p = \hat{Q}$.
$P_c^i(\hat{\boldsymbol{q}}) = 0, i \in n$, which is unique irrespectively of the variety of $\hat{\boldsymbol{q}}$.

Consider a group of systems for which the values of parameters, a, b and c, satisfy the following constraints: $a \geq c > 0, b > 0$, so that the market can be established.

Denote the set of parameter values that satisfy the above constraints by $\boldsymbol{C}$.

Producer profit and consumer surplus: Define the following. $\tilde{R}_i = P^i(\tilde{\boldsymbol{q}})$, $\bar{R}_i = P_0^i(\bar{q}_i)$ and $\hat{R}_i = P_c^i(\hat{q}_i) = 0$. $\bar{R}_i$, $\tilde{R}_i$, and $\hat{R}_i$, respectively, denote the values of the profit for producer i in monopoly, in oligopoly and in perfect competition. $\bar{R}_i = \bar{R}$, $\tilde{R}_i = \tilde{R}$, and $\hat{R}_i = \hat{R}$ because of the symmetry. $\bar{C}$, $\tilde{C}$ and $\hat{C}$, respectively, denote the values of the consumer surplus for the market in monopoly, in oligopoly and in perfect competition. k_i^R and k^C, respectively, denote the ratios, of the profits for producer i and of the consumer surplus, of monopoly to those of oligopoly. That is, $k_i^R = \bar{R}_i/\tilde{R}_i = \bar{R}/\tilde{R} = k^R$ and $k^C = \bar{C}/\tilde{C}$.

3.1.2 The Results

The profit optimization The partial derivatives of P_0 and P^i are:

$$\frac{\partial P_0}{\partial q} = a\left(1 - \frac{2q}{b}\right) - c,\ \frac{\partial P^i}{\partial q_i} = a\left(1 - \frac{\sum_j q_j}{b}\right) - \frac{aq_i}{b} - c,\ i \in \boldsymbol{n}.$$

Therefore, the optimal decisions and the price, of monopoly and oligopoly, respectively, satisfy the following.

$a\,(1 - 2\bar{q}/b) - c = 0$ and $\bar{p} = a\,(1 - \bar{q}/b)$, where $\bar{q}_i = \bar{q}/n, i \in \boldsymbol{n}$,

$a\left\{1 - (\sum_j \tilde{q}_j + \tilde{q}_i)/b\right\} - c = 0, i \in \boldsymbol{n},\ \tilde{p} = a\left(1 - (\sum_j \tilde{q}_j)/b\right).$

Then,

$$\bar{q}_i = \frac{a-c}{2na/b} \geq 0, \bar{p} = \frac{a+c}{2},\ \ i \in \boldsymbol{n}, \tag{6}$$

$$\tilde{q}_i = \frac{a-c}{(n+1)a/b} \geq 0,\ \ i \in \boldsymbol{n}, \tag{7}$$

$$\sum_j \tilde{q}_j = \frac{n(a-c)}{(n+1)a/(b)} > 0, \tilde{p} = \frac{a+nc}{n+1}, \tag{8}$$

The inequalities in the above relations come from the constraints $\boldsymbol{C}$. In particular, that of (6) comes from the assumption that the market can be established in monopoly, that of (7) comes from the assumption that producer i may play some role in oligopoly, and that of (8) comes from the assumption that at least one producer may produce non-zero amount of products in oligopoly.

Profits of producers The profits of producer i, in monopoly, in oligopoly and in perfect competition, are, respectively, $\bar{R}_i = \bar{R} = P_0(\bar{q}_i)/n = \dfrac{(a-c)^2 b}{4na}$ $(i \in \boldsymbol{n})$, $\tilde{R}_i = \tilde{R} = P^i(\tilde{\boldsymbol{q}}) = \dfrac{a\tilde{q}_i^2}{b} = \dfrac{b(a-c)^2}{a(n+1)^2}$ $(i \in \boldsymbol{n})$ and $\hat{R}_i = \hat{R} = 0$.

Since the producer profits of the social optimum (monopoly), the atomic Nash equilibrium (oligopoly), and the non-atomic Nash equilibrium (perfect competition) are symmetric and unique, we can use the corollary 1. Therefore, the magnitude of, k^R, of Pareto inefficiency of the oligopoly is as follows: $k^R = \bar{R}/\tilde{R} = (n+1)^2/(4n)$.

Remark 1 Thus the magnitude of Pareto inefficiency of the atomic Nash equilibrium k^R decreases as the number of players n decreases and finally it reaches the Pareto optimality of the social optimum ($n = 1$). k^R increases as the number of players n increases and finally up to that, ∞, of the non-atomic Nash equilibrium ($n \to \infty$). □

Note that, in the Nash equilibria and the social optimum of this game, the profit of every player (producer) is unique and symmetric within each equilibrium and optimum. Thus, we can apply the corollaries 1 and 2 to this model.

Theorem 2 *There exist economic games where the magnitudes of Pareto inefficiency of atomic and non-atomic Nash equilibria (and the price of anarchy) can increase without bound.*

Consumer surpluses The consumer surplus in each market is as follows. Define $Z = 1 - c/a$.
In monopoly, the consumer surplus for the market is $\bar{C} = (1/2)\bar{q}(a - \bar{p}) = abZ^2/8$.
In oligopoly, the quantity, $\tilde{q}^{(c)}$ of the commodity that the consumers in the market consume is to be $b(1 - \tilde{p}/a)$. Then, the consumer surplus for the market is $\tilde{C} = (1/2)\tilde{q}^{(c)}(a - \tilde{p}) = \{a/(2b)\}\left(\sum_j \tilde{q}_j\right)^2 = [abn^2/\{2(n+1)^2\}]Z^2$.
In perfect competition, the consumer surplus for the market is $\hat{C} = (1/2)Q(a - \hat{p}) = abZ^2/2$.

Therefore, the ratio, k^C, of consumer surplus improvement for the market from that of monopoly to that of oligopoly is: $k^C = \bar{C}/\tilde{C} = (n+1)^2/(4n^2)$.

Remark 2 $(n+1)^2/(4n^2)$ decreases slowly in n, and $(n+1)^2/(4n^2) \to 1/4$ as $n \to \infty$. Therefore, as the number of producer n increases, the ratio of consumer surplus improvement $1/k^C$ from that of monopoly to that of oligopoly increases, and finally the system approaches the situation where the consumers enjoy the consumer surplus in perfect competition, that is, 4 times of that in monopoly. □

3.2 Load Balancing in a Two Symmetric Node Model — A Congestion Game

3.2.1 The Model and Assumptions

We consider a model consisting of two identical servers (nodes) and a communication means that connects both servers. Servers are numbered 1 and 2 (Fig. 1). Jobs (or customers) are classified into $2n$ classes $R_{ik}, i = 1, 2, k = 1, 2, \cdots, n$. Jobs of class R_{ik} arrive only at server i with identical rate $1/n$, Out of each class arrival, the rate x_{ik} of jobs are forwarded upon arrival through the communication means to the other server j ($i \neq j$) to be processed there. Therefore the remaining rate $1/n - x_{ik}$ of class R_{ik} jobs are processed at server i. We have $0 \leq x_{ik} \leq 1/n, \quad i = 1, 2$. We denote the vector $(x_{11}, x_{12}, \cdots, x_{1n}, x_{21}, x_{22}, \cdots, x_{2n})$ by $\boldsymbol{x}$. We denote the set of $\boldsymbol{x}$'s that satisfy the constraints by $\boldsymbol{C}$. Within these constraints, a set of values of x_{ik} ($i = 1, 2, k = 1, 2, \cdots, n$) are chosen to achieve optimization. Thus the load β_i on server i is given by $\beta_i = 1 - \sum_l x_{il} + \sum_l x_{jl}, (i \neq j)$. Then, the expected processing (including queueing) time $D_i(\beta_i)$ of a job that is processed at server i (or the cost function at server i) is $D_i(\beta_i) = 1/(\mu - \beta_i)$ for $\beta_i < \mu$ (otherwise it is infinite) (We have a simple assumption of the external time-invariant Poisson arrival for each class, and the exponentially distributed service times for each class jobs with identical service rate μ at both servers.)

As to the communication means, we consider two communication lines 1 and 2 separately for each server. One line i is used for forwarding of a job that arrives at server i. The expected communication time of a job arriving at server i and being processed at server j ($\neq i$) is expressed simply as t, *i.e.,* independent of the traffic and the job class and with no queueing delay.

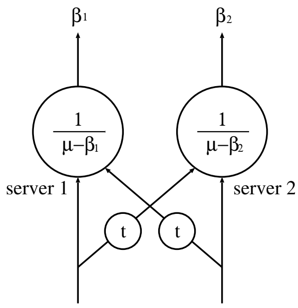

Figure 1: The system model.

We refer to the length of time between the instant when a job arrives at a server and the instant when a job leaves one of the servers after all processing and communication, if any, are over as the *sojourn time* for the job.

Thus the expected sojourn time of a class R_{ik} job that arrives at server i is

$$T_{ik}(\boldsymbol{x}) = n\{(\frac{1}{n} - x_{ik})T_{iik}(\boldsymbol{x}) + x_{ik}T_{ijk}(\boldsymbol{x})\}, \tag{9}$$

where $T_{iik}(\boldsymbol{x}) = D_i(\beta_i)$ and $T_{ijk}(\boldsymbol{x}) = D_j(\beta_j) + t,$ for $j \neq i$. (The above expressions hold, again, only for positive values of denominators, and are otherwise infinite.)

Then, the overall expected sojourn time of a job that arrives at the system is

$$T(\boldsymbol{x}) = \frac{1}{2n} \sum_{i,k} T_{ik}(\boldsymbol{x}). \tag{10}$$

3.2.2 The Results

We have three optima, the social optimum, the non-atomic Nash equilibrium, and the atomic Nash equilibrium, as in the following.

(1) [Completely centralized optimization — social optimum] The social optimum is given by such $\bar{\boldsymbol{x}}$ as satisfies the following: $T(\bar{\boldsymbol{x}}) = \min T(\boldsymbol{x}) \quad$ with respect to $\boldsymbol{x} \in \boldsymbol{C}$.
The solution $\bar{\boldsymbol{x}}$ is unique and simply given as follows: $\bar{\boldsymbol{x}} = \boldsymbol{0}$, *i.e.,* $x_{1k} = x_{2k} = 0$ for all k and $T(\bar{\boldsymbol{x}}) = T_{ik}(\bar{\boldsymbol{x}}) = 1/(\mu - 1),\ i = 1, 2,\ k = 1, 2, \cdots, n$.
This is intuitively clear. Or, this can be easily seen if we note that, since the overall mean sojourn time $T(\boldsymbol{x})$ is expressed as follows from (9) and (10): $2T(\boldsymbol{x}) = 2\mu(\mu - 1)/\{(\mu - 1)^2 - d^2\} + st - 2,$

where $d = \sum_l (x_{1l} - x_{2l})$ and $s = \sum_l (x_{1l} + x_{2l})$, $T(\boldsymbol{x})$ is minimum if and only if $x_{1k} = x_{2k} = 0$ for $k = 1, 2, \cdots, n$.

(2) [Completely distributed optimization — non-atomic Nash equilibrium] The non-atomic Nash equilibrium (or Wardrop equilibrium) is given by such $\hat{\boldsymbol{x}}$ as satisfies the following for all i, k

$$T_{ik}(\hat{\boldsymbol{x}}) = \min\{T_{iik}(\hat{\boldsymbol{x}}), T_{ijk}(\hat{\boldsymbol{x}})\}\ (i \neq j) \quad \text{such that} \quad \hat{\boldsymbol{x}} \in \boldsymbol{C}. \tag{11}$$

The solution $\hat{\boldsymbol{x}}$ is unique and given as follows: $\hat{\boldsymbol{x}} = \boldsymbol{0}$, *i.e.,* $\hat{x}_{1k} = \hat{x}_{2k} = 0$, for all k,
And, again, $T(\hat{\boldsymbol{x}}) = T_{ik}(\hat{\boldsymbol{x}}) = 1/(\mu - 1)$, for all i, k,

[Proof] The solution $\hat{\boldsymbol{x}}$ for (11) is characterized as follows:

$$D_1(\hat{\beta}_1) > D_2(\hat{\beta}_2) + t, \qquad \hat{x}_{1k} = 1/n \tag{12}$$
$$D_1(\hat{\beta}_1) = D_2(\hat{\beta}_2) + t, \quad 0 \leq \hat{x}_{1k} \leq 1/n \tag{13}$$
$$D_1(\hat{\beta}_1) < D_2(\hat{\beta}_2) + t, \qquad \hat{x}_{1k} = 0 \tag{14}$$
$$D_2(\hat{\beta}_2) > D_1(\hat{\beta}_1) + t, \qquad \hat{x}_{2k} = 1/n \tag{15}$$
$$D_2(\hat{\beta}_2) = D_1(\hat{\beta}_1) + t, \quad 0 \leq \hat{x}_{2k} \leq 1/n \tag{16}$$
$$D_2(\hat{\beta}_2) < D_1(\hat{\beta}_1) + t, \qquad \hat{x}_{2k} = 0 \tag{17}$$

for all k. We can easily see that these are satisfied if and only if $\hat{x}_{1k} = \hat{x}_{2k} = 0$ for all k, by noting that, for example, (12) and (13) contradict with any of (15), (16), and (17), and thus that only (14) and (17) can hold together. □

(3) [Intermediately distributed optimization — atomic Nash equilibrium] The atomic Nash equilibrium is given by such $\tilde{\boldsymbol{x}}$ as satisfies the following for all i, k, $T_{ik}(\tilde{\boldsymbol{x}}) = \min_{x_{ik}} T_{ik}(\tilde{\boldsymbol{x}}_{-(ik)}; x_{ik})$, such that $(\tilde{\boldsymbol{x}}_{-(ik)}; x_{ik}) \in \boldsymbol{C}$.
where $(\tilde{\boldsymbol{x}}_{-(ik)}; x_{ik})$ denotes the $2n$ vector in which the element corresponding to $\tilde{x}_{ik}$ has been replaced by x_{ik}.

(A) The case where $t > 1/\{n(\mu - 1)^2\}$: The solution $\tilde{\boldsymbol{x}}$ is unique and given as follows: $\tilde{\boldsymbol{x}} = \boldsymbol{0}$, *i.e.,* $\tilde{x}_{1k} = \tilde{x}_{2k} = 0$, for all k.
And, again, $T(\tilde{\boldsymbol{x}}) = T_{ik}(\tilde{\boldsymbol{x}}) = 1/(\mu - 1)$, $i = 1, 2$, $k = 1, 2, \cdots, n$.

(B) The case where $t \leq 1/\{n(\mu - 1)^2\}$: The solution $\tilde{\boldsymbol{x}}$ is unique and given as follows:

$$\tilde{x}_{1k} = \tilde{x}_{2k} = \frac{1}{2}\{\frac{1}{n} - t(\mu - 1)^2\}, \text{ for all } k. \tag{18}$$

And in that case, we have

$$\begin{aligned} T(\tilde{\boldsymbol{x}}) &= T_{1k}(\tilde{\boldsymbol{x}}) = T_{2k}(\tilde{\boldsymbol{x}}) \\ &= \frac{1}{\mu - 1} + \frac{t}{2}\{1 - nt(\mu - 1)^2\}, \text{ for all } k. \end{aligned} \tag{19}$$

[Proof] From the definition (9) we have

$$\begin{aligned} (1/n)\frac{\partial T_{ik}}{\partial x_{ik}} = &-\frac{\mu - \frac{n-1}{n} + \sum_{l \neq k} x_{il} - \sum_l x_{jl}}{(\mu - 1 + \sum_l x_{il} - \sum_l x_{jl})^2} \\ &+ \frac{\mu - 1 - \sum_{l \neq k} x_{il} + \sum_l x_{jl}}{(\mu - 1 - \sum_l x_{il} + \sum_l x_{jl})^2} + t \quad (i \neq j). \end{aligned} \tag{20}$$

By simple inspection of (20), we see that $\frac{\partial T_{ik}}{\partial x_{ik}}$ is monotonically increasing with the increase in x_{ik} with feasible $\boldsymbol{x} \in \boldsymbol{C}$. Thus if we can find a set of such values of $\tilde{\boldsymbol{x}}$ that satisfies

$$\frac{\partial T_{ik}}{\partial x_{ik}}(\tilde{\boldsymbol{x}}) = 0, \text{ for all } i, k, \tag{21}$$

then the set of values is a solution of the atomic Nash equilibrium. We have from (20) and defining $d = \sum_l (x_{1l} - x_{2l})$

$$\begin{aligned}
&\sum_l (1/n)\{\frac{\partial T_{1l}}{\partial x_{1l}} - \frac{\partial T_{2l}}{\partial x_{2l}}\} \\
&\qquad = \frac{2n\mu - (2n-1)(1+d)}{(\mu - 1 - d)^2} - \frac{2n\mu - (2n-1)(1-d)}{(\mu - 1 + d)^2} \\
&\qquad = (\frac{2d}{(\mu-1)^2 - d^2})(\frac{2\mu(\mu-1)}{(\mu-1)^2 - d^2} + 2n - 1),
\end{aligned} \tag{22}$$

If condition (21) holds, then from (22), we have $d = 0$. Then from (20) we have

$$(1/n)\frac{\partial T_{ik}}{\partial x_{ik}} = \frac{2x_{ik} - 1/n}{(\mu - 1)^2} + t = 0, \ (i \neq j) \text{ for all } i, k. \tag{23}$$

Therefore $x_{ik} = (1/2)(1/n - t(\mu-1)^2)$ for all i, k if $t \leq 1/\{n(\mu-1)^2\}$.
From the above derivation, it is clear that this is a unique solution (in case (B)).

If $t > 1/\{n(\mu - 1)^2\}$ (in case (A)), we have from (23) when $x_{ik} = 0$, for all i, k, $(\frac{1}{n})\frac{\partial T_{ik}}{\partial x_{ik}} = t - \frac{1}{n(\mu-1)^2} > 0$, for all i, k.
Considering that $\frac{\partial T_{ik}}{\partial x_{ik}}$ is monotonically increasing with x_{ik}, we have that $\tilde{x}_{ik} = 0$, for every i, k, is an atomic Nash equilibrium solution.

We can easily see the uniqueness as in the following. Suppose $\tilde{x}_{1k} > 0$ for some k. From definitions on d and by (20) we have then

$$(1/n)\frac{\partial T_{1k}}{\partial x_{1k}} = -\frac{\mu - \dfrac{n-1}{n} + d - \tilde{x}_{1k}}{(\mu - 1 + d)^2} + \frac{\mu - 1 - d + \tilde{x}_{1k}}{(\mu - 1 - d)^2} + t = 0. \tag{24}$$

Then from the above and condition on t we have

$$\begin{aligned}
&\{\frac{1}{(\mu - 1 + d)^2} + \frac{1}{(\mu - 1 - d)^2})\}\tilde{x}_{1k} \\
&\qquad = -t - \frac{2d}{(\mu-1)^2 - d^2} + \frac{1}{n(\mu - 1 + d)^2} \\
&\qquad < \frac{1}{n(\mu - 1 + d)^2} - \frac{1}{n(\mu-1)^2} - \frac{2d}{(\mu-1)^2 - d^2}.
\end{aligned} \tag{25}$$

This implies $d < 0$ for which there must exist some nonzeo $x_{2k'}$. Then by using the argument similar to the above on $x_{2k'}$ we have $d > 0$, which is a contradiction. Thus $\tilde{\boldsymbol{x}} = \boldsymbol{0}$ is the unique atomic Nash equilibrium solution.

For the proofs of the existence and uniqueness of those optima for more general setting, see (Altman and Kameda, 2005; Altman, Kameda and Hosokawa, 2002; Orda, Rom and Shimkin, 1993). □

A proof for the model more general than the one presented here has been given, but it covers more than 5 journal pages and will take much time to follow (Kameda and Pourtallier, 2002). Then, herein, we show a proof specific to this special model and much simpler than the general proof.

Remark 3 Consider the case (B) in the atomic Nash equilibrium. In this case, the atomic Nash equilibrium is Pareto inefficient. In contrast, the solutions of the social optimum, the non-atomic Nash equilibrium and the atomic Nash equilibrium in case (A) are identical and all Pareto optimal. As n increases in the atomic Nash equilibrium with case (B) (see eq. (19)), $T(\tilde{\boldsymbol{x}})$ decreases as far as $t \leq 1/\{n(\mu-1)^2\}$ holds. Then, as n increases further, $t > 1/\{n(\mu-1)^2\}$ (case (A)) holds and $T(\tilde{\boldsymbol{x}})$ becomes the same as those of the social optimum and the non-atomic Nash equilibrium. Since the solutions of the social optimum and the Nash equilibrium are symmetric, we can use the corollary 1. Thus the magnitude of Pareto inefficiency of the atomic Nash equilibrium decreases as the number of players $2n$ increases and finally it reaches the non-atomic Nash equilibrium that is Pareto optimal. On the other hand, we cannot let the atomic Nash equilibrium be down to the social optimum since we cannot reduce the number of atomic players $2n$ down to 1. □

Furthermore, we can easily see that $T_{ik}(\tilde{\boldsymbol{x}})(= T(\tilde{\boldsymbol{x}}))$, for every i, k, has its maximum $\tilde{T}(\mu, n)$ (*i.e.,* the worst-case performance) for given μ, n.

$$\tilde{T}(\mu, n) = \frac{1}{\mu - 1}\{1 + \frac{1}{8n(\mu - 1)}\}, \tag{26}$$

when

$$t = \frac{1}{2n(\mu - 1)^2}. \tag{27}$$

Thus if we add the communication lines with delay t $(= 1/\{2n(\mu - 1)^2\})$ to the system that has had no communication means, the expected sojourn time of each class (player) $T_{ik}(\tilde{\boldsymbol{x}})$ increases in the amount of $\frac{1}{8n(\mu - 1)^2}$ (*i.e.,* the performance degrades).

The magnitude of Pareto inefficiency of the atomic Nash equilibrium for given μ, n is to be

$$\Delta(\mu, n) = \frac{\tilde{T}(\mu, n)}{T_0(\mu)}, \tag{28}$$

where $T_0(\mu) = 1/(\mu - 1)$ denotes the mean sojourn time of the social optimum for given μ. Then we have

$$\Delta(\mu, n) = 1 + \frac{1}{8n(\mu - 1)}. \tag{29}$$

Remark 4 Consider the atomic Nash equilibrium with the case (B). In this case, each player forwards a part of his/her jobs through the communication means to the other server for remote processing, and thereby has degradation in his/her mean sojourn time. The ratio of such degradation, and thus *the magnitude of Pareto inefficiency can increase without bound* as the total arrival rate approaches the processing capacity of each server. Differently from the economic game presented in subsection 3.1, herein the increase without bound of MoI may occur for all $i > 1$ but with the approach of the job processing capacity μ of the nodes to the job arrival rate 1. □

Note that, in the Nash equilibria and social optimum of this game, the cost of every player is unique and symmetric within each equilibrium and optimum. Thus, we can apply the corollaries 1 and 2 to this model.

Theorem 3 *There exist congestion games where the degrees of Pareto inefficiency of atomic Nash equilibria (and the price of anarchy) can increase without bound.*

3.2.3 Numerical Examples

For example, we examine the following case: $\mu = 1.01$. Then the mean sojourn time is $T_0(\mu) = 1/(\mu - 1) = 100$ in the social optimum, in the non-atomic Nash equilibrium (Wardrop equilibrium), and in the case of no communication line and no forwarding of jobs.

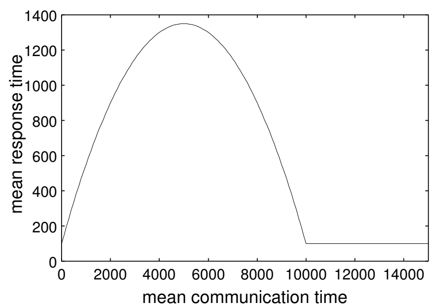

Figure 2: The mean response (sojourn) times for each class jobs (player) in the atomic Nash equilibria (or Nash equilibria) with $\mu = 1.01$ and $n = 1$ for the various values of mean communication time t. We see that, in the paradoxically worst case, adding the communication means with $t = 5000$ to the system increases the mean response (sojourn) time up to 1350 from 100, that of no communication means.

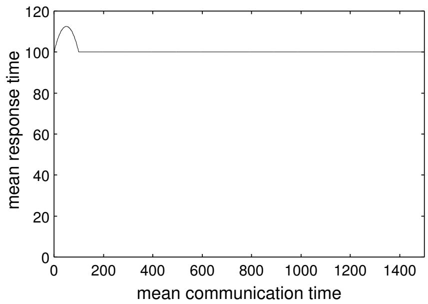

Figure 3: The mean response (sojourn) times for each class jobs (player) in the atomic Nash equilibria (or Nash equilibria) with $\mu = 1.01$ and $n = 100$ for the various values of mean communication time t. We see that, in the paradoxically worst case, adding the communication means with $t = 50$ to the system increases the mean response (sojourn) time up to only 112.5 from 100, that of no communication means.

Firstly, we consider the case where $n = 1$, *i.e.,* the number of classes is 2. The mean sojourn time of the atomic Nash equilibria (Nash equilibrium) for various values of t is shown in Fig. 2.

As we can see from the figure, $T = T_{ik}$ takes its maximum value

$$\tilde{T}(\mu, n) = 1350 \text{ (see(26))}$$

and the magnitude of Pareto inefficiency (the worst-case ratio of the performance degradation) $\Delta(\mu, n)$ is

$$\Delta(\mu, n) = 13.50 \text{ (\textit{i.e.}, 1350\% degradation) (see (28))}$$

when $t = 1/\{2(\mu - 1)^2\} = 5000$ (see (27)). Then

$\tilde{x}_{1k} = \tilde{x}_{2k} = (1/2)\{1 - t(\mu - 1)^2\} = 1/4$ ($k = 1$) (see (18)). In this case, $\tilde{x}_{1k} = \tilde{x}_{2k}$ decrease from 1/2 down to 0 as t increases from 0 to 10000 ($= 1/(\mu - 1)^2$), and for $t > 10000$, no forwarding of jobs occurs.

It is amazing that, in the Nash equilibrium, each class (player) keeps to forward a part of his/her jobs to the other server even though the communication delay for forwarding is much greater than the processing delay at the server at which his/her jobs arrive.

Then we consider the case where $n = 100$, *i.e.,* the number of classes is 200.

The mean sojourn time of the atomic Nash equilibria (Nash equilibrium) for various values of t is shown in Fig. 3.

As we can see from the figure, $T = T_{ik}$ takes its maximum value $\tilde{T}(\mu, n) = 112.5$ (see(26)) and the magnitude of Pareto inefficiency $\Delta(\mu, n)$ is $\Delta(\mu, n) = 1.125$ (*i.e.*, 12.50% degradation) (see (28)) when $t = 1/\{2n(\mu - 1)^2\} = 50$ (see (27)). Then

$$\tilde{x}_{1k} = \tilde{x}_{2k} = (1/2)\{1/n - t(\mu - 1)^2\} = 1/400 \quad \text{for all } k \text{ (see (18)).}$$

In this case, $\tilde{x}_{1k} = \tilde{x}_{2k}$ decrease from 1/200 down to 0 as t increases from 0 to 100 ($= 1/(\mu - 1)^2$), and for $t > 100$, no forwarding of jobs occurs.

Thus we see that the magnitude of Pareto inefficiency is greatly reduced from the case of $n = 1$.

Furthermore we consider other values of μ with $n = 1$.
For $\mu = 1.001$, $\Delta(\mu, n) = 126$ (*i.e.*, 12600% degradation), and
for $\mu = 1.00001$, $\Delta(\mu, n) = 12501$ (*i.e.*, 1250100% degradation), *etc.*
In this way, we see that the magnitude of Pareto inefficiency $\Delta(\mu, n)$ can increase without bound as μ approaches 1 with $n = 1$.

4 Concluding Remarks

We have tried to make clear the definition of the magnitude of Pareto inefficiency of Nash equilibria. We have proven that the price of anarchy (PoA) presents the magnitude of Pareto inefficiency of the unique Nash equilibrium in the case where the Nash equilibrium and the social optimum are both symmetric although utilities of players may be asymmetric in other feasible states. Then we have observed the way how the magnitude of Pareto inefficiency depends on the degree of dispersion of decision making, *i.e.,* the number of non-cooperative players in Nash equilibria. We have examined a simple economic game and a simple congestion game for examining the trends on the magnitude of Pareto inefficiency of atomic and non-atomic Nash equilibria. We have shown, respectively, in the economic and congestion games, the magnitude of Pareto inefficiency of the Nash and atomic Nash equilibrium (and also PoA therein) can increase without bound.

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